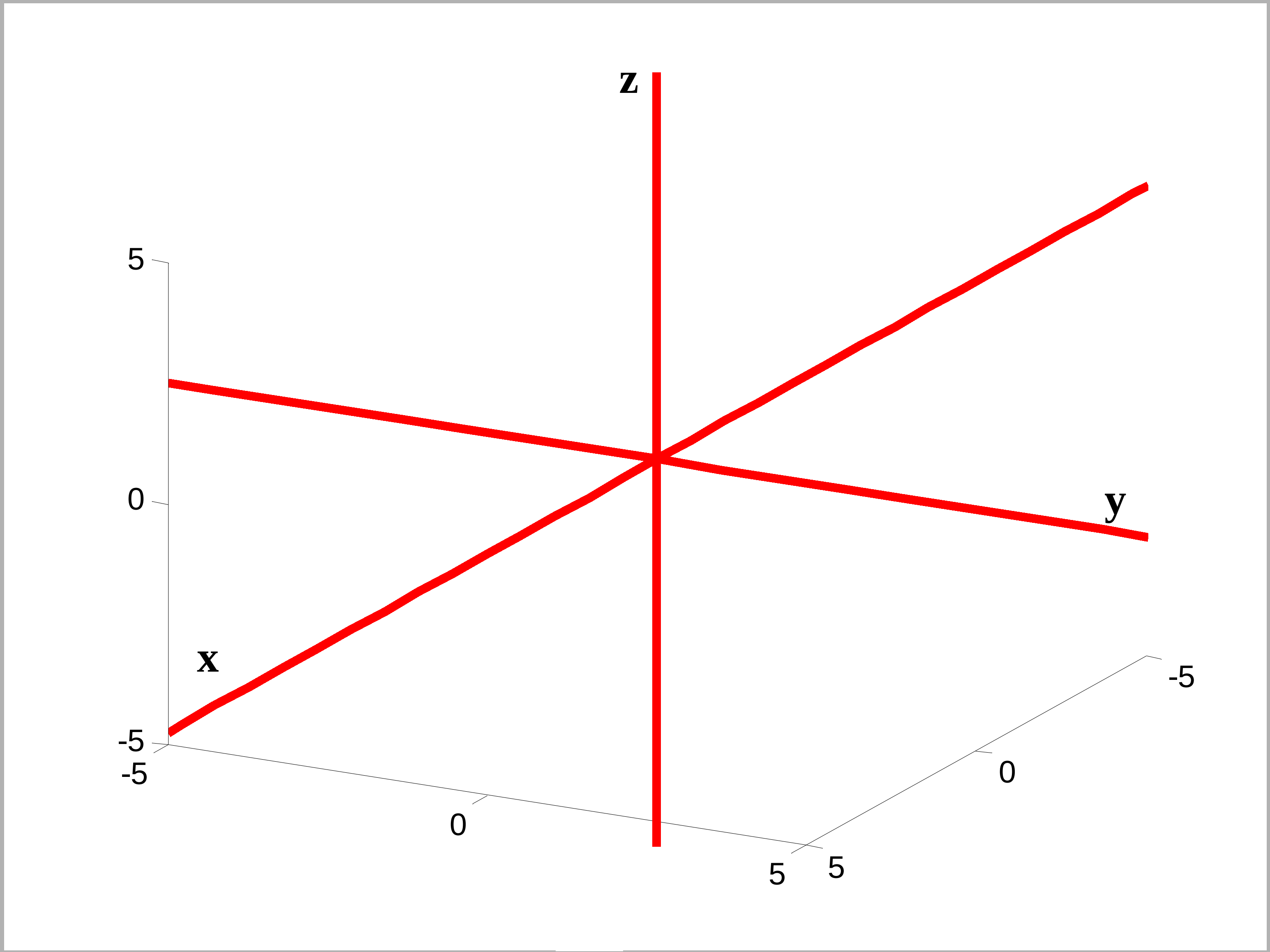
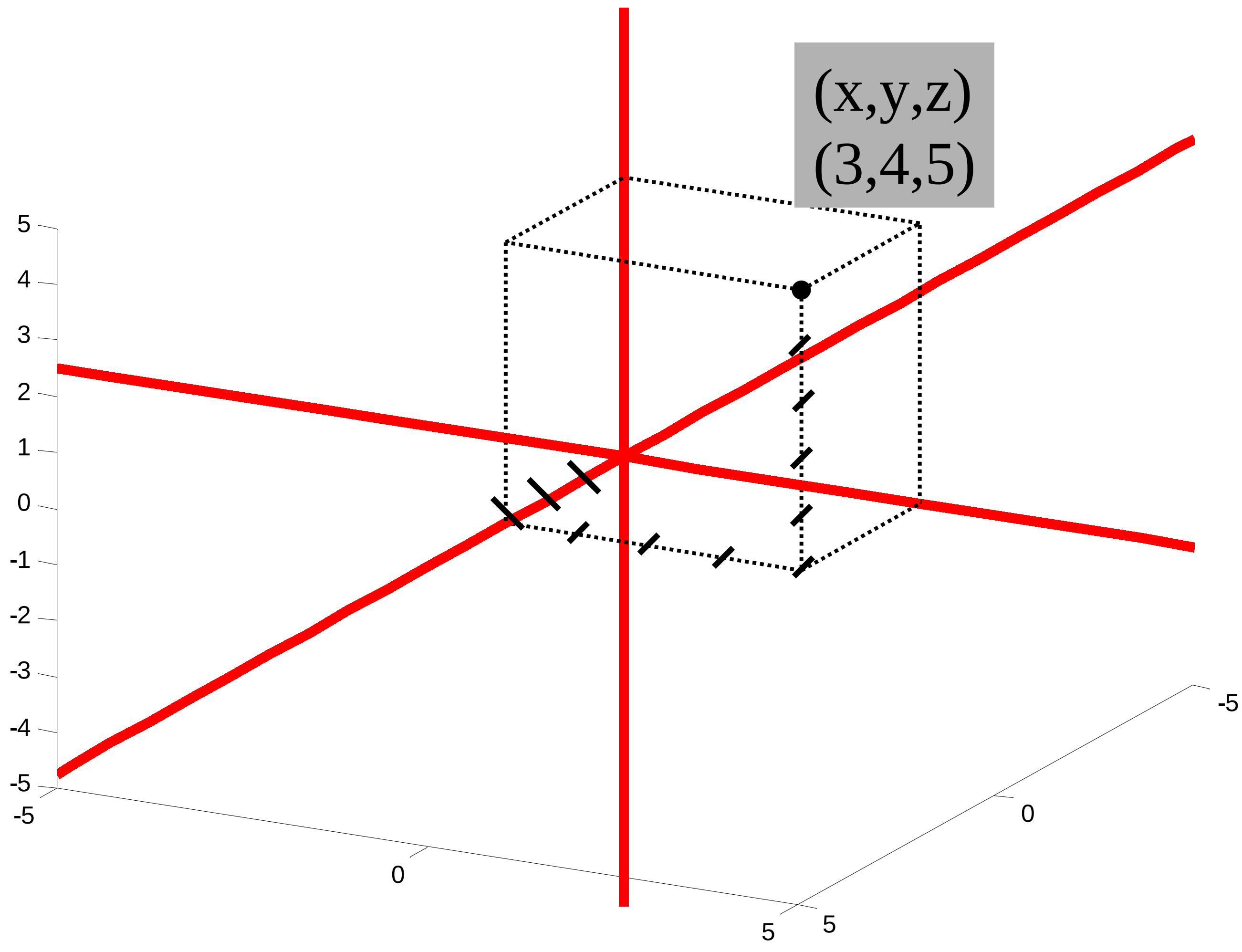
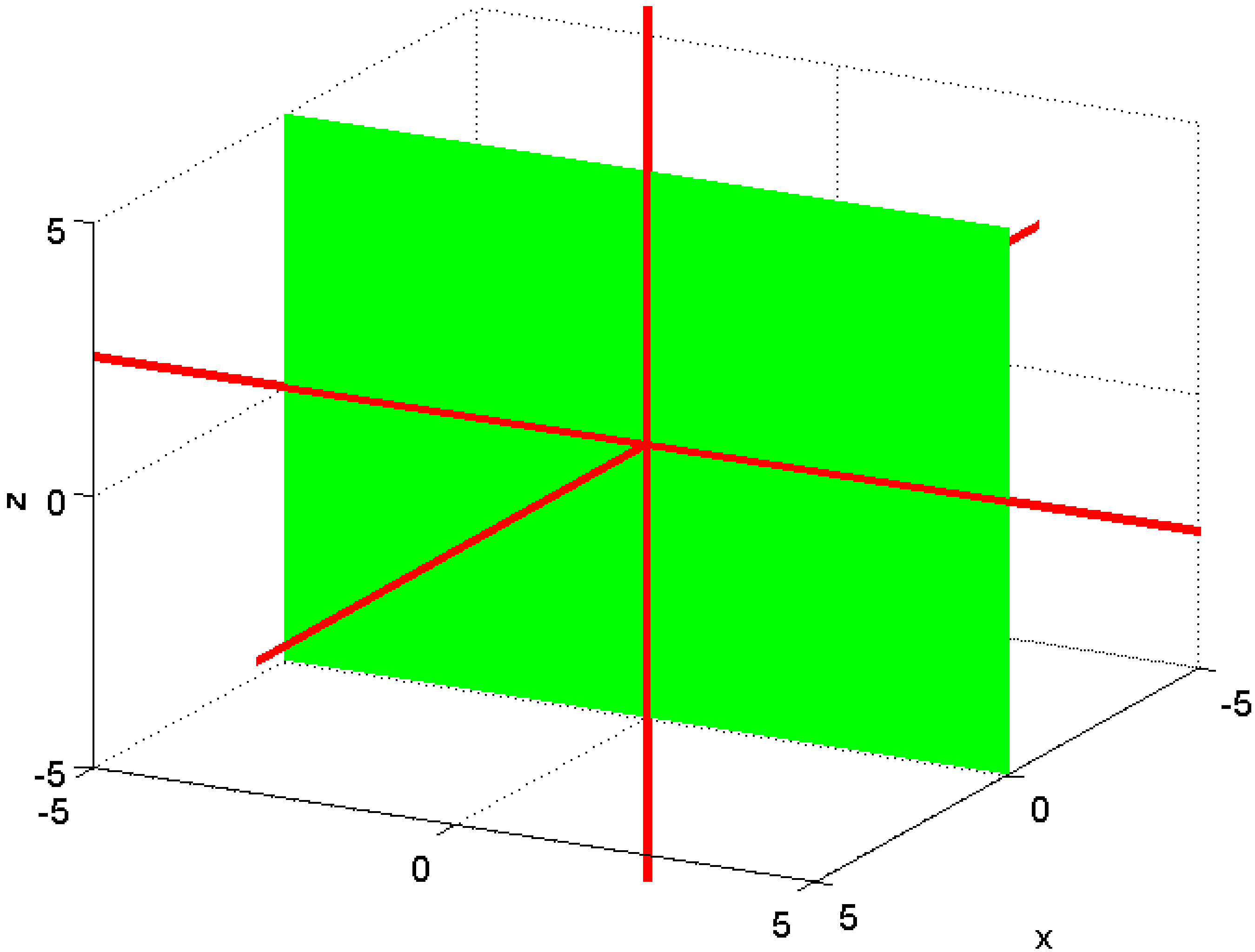
3D Coordinate System



Point



Coordinate Planes



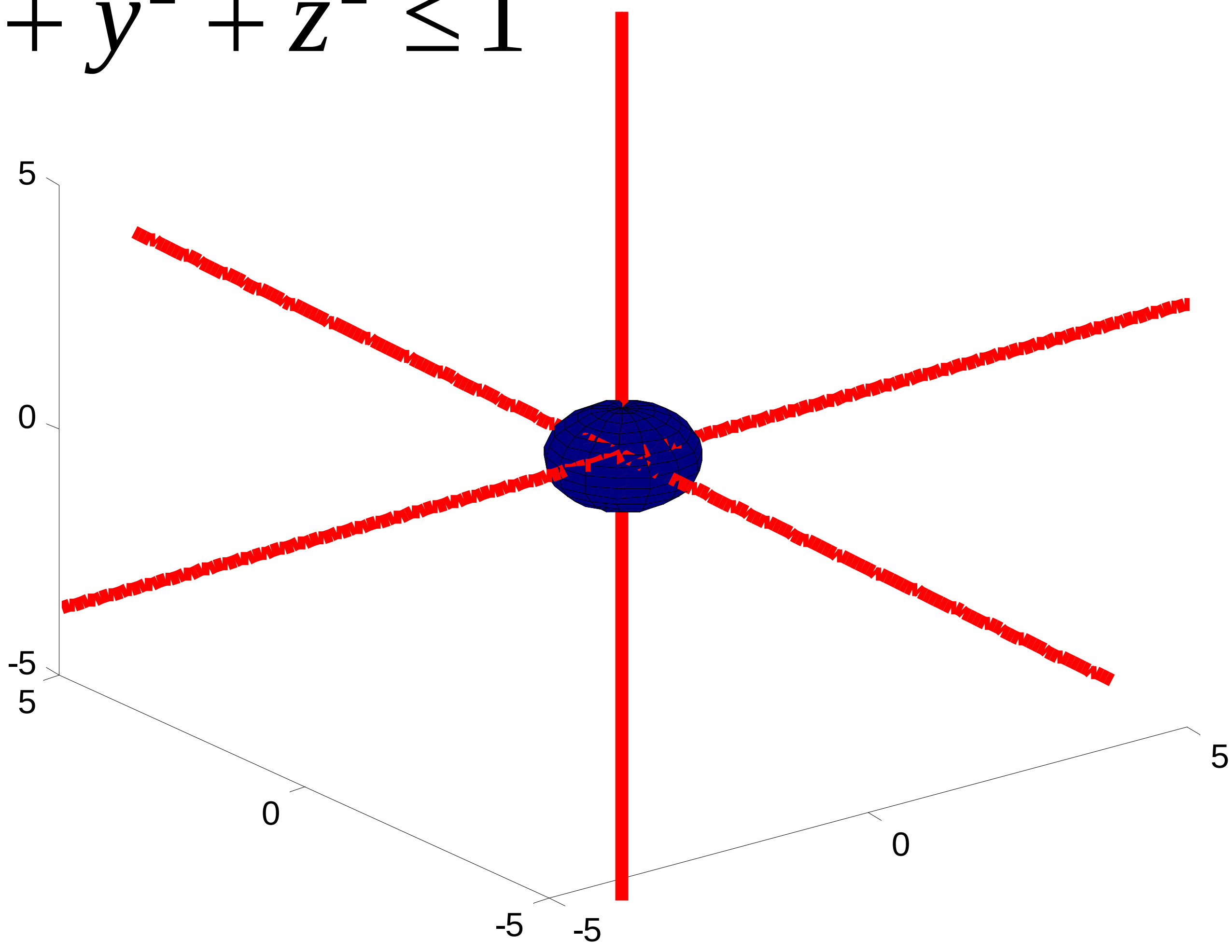
xy plane:
(x,y,0)

xz plane:
(x,0,z)

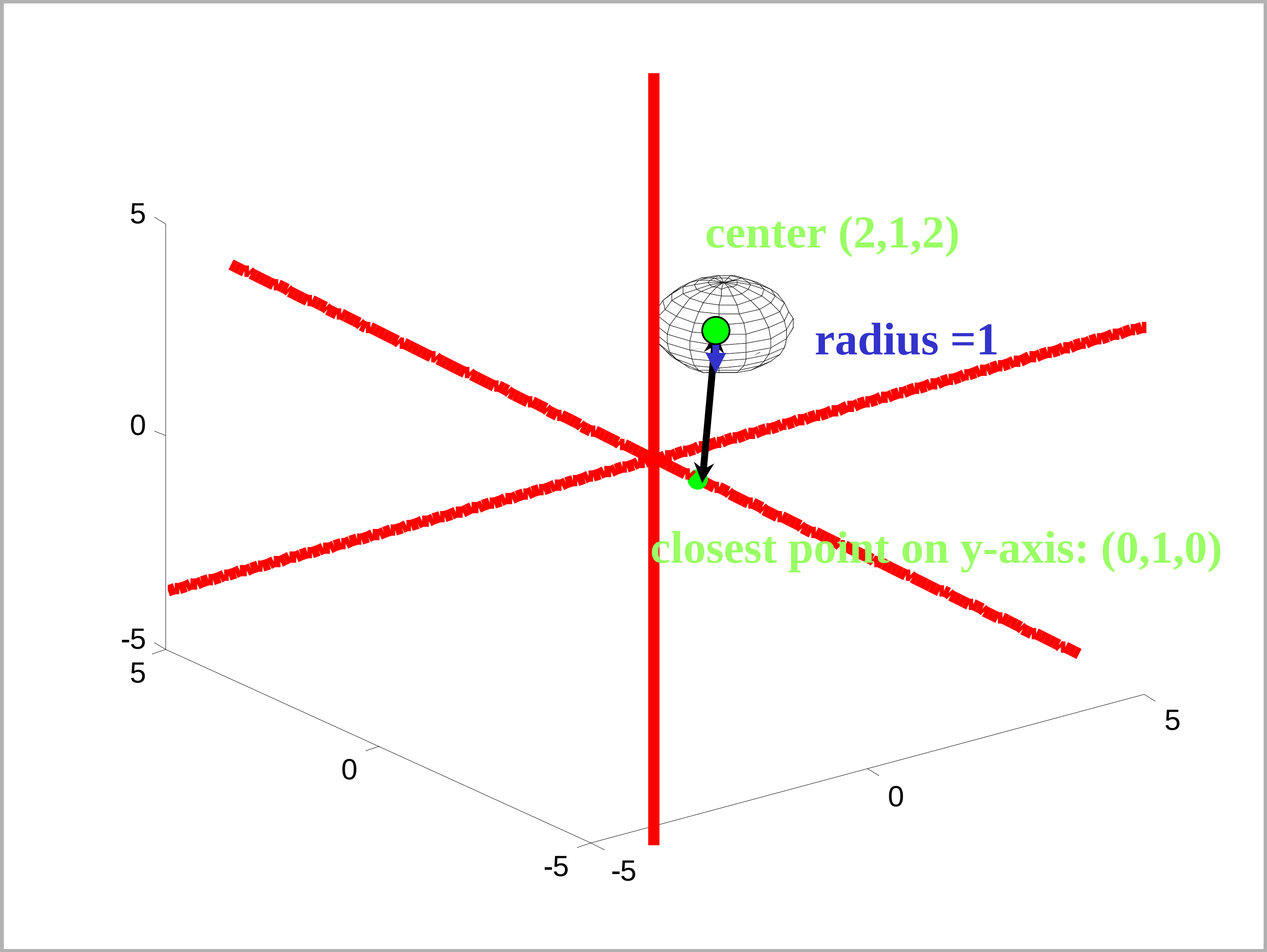
yz plane:
(0,y,z)

Spheres

$$x^2 + y^2 + z^2 \leq 1$$



Spheres $(x-2)^2+(y-1)^2+(z-2)^2=1$



How far is this sphere from the y axis?

How far is this sphere from the y axis?

$$(x-2)^2+(y-1)^2+(z-2)^2=1$$

Note: It's center is at C(2,1,2) and it's radius is 1.
Also, the closest point along the y-axis is P(0,1,0)

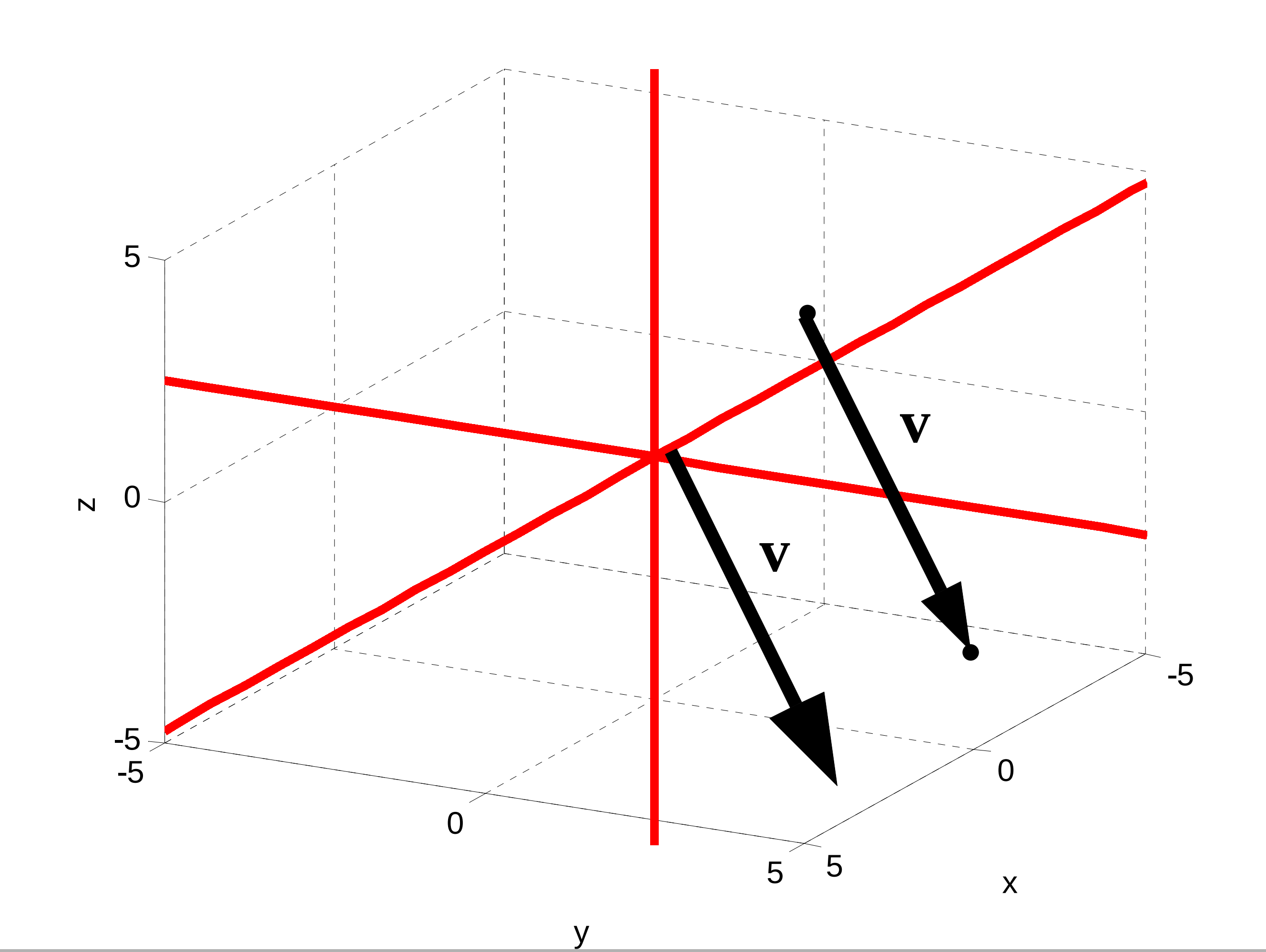
Distance from the center of the sphere to the closest point on the y-axis:

$$|CP|=\sqrt{(2-0)^2+(1-1)^2+(2-0)^2}=\sqrt{8}$$

Reduce this distance by the length of the radius and the distance from the sphere to the y axis is:

$$\sqrt{8}-1\approx 1.83$$

Vectors



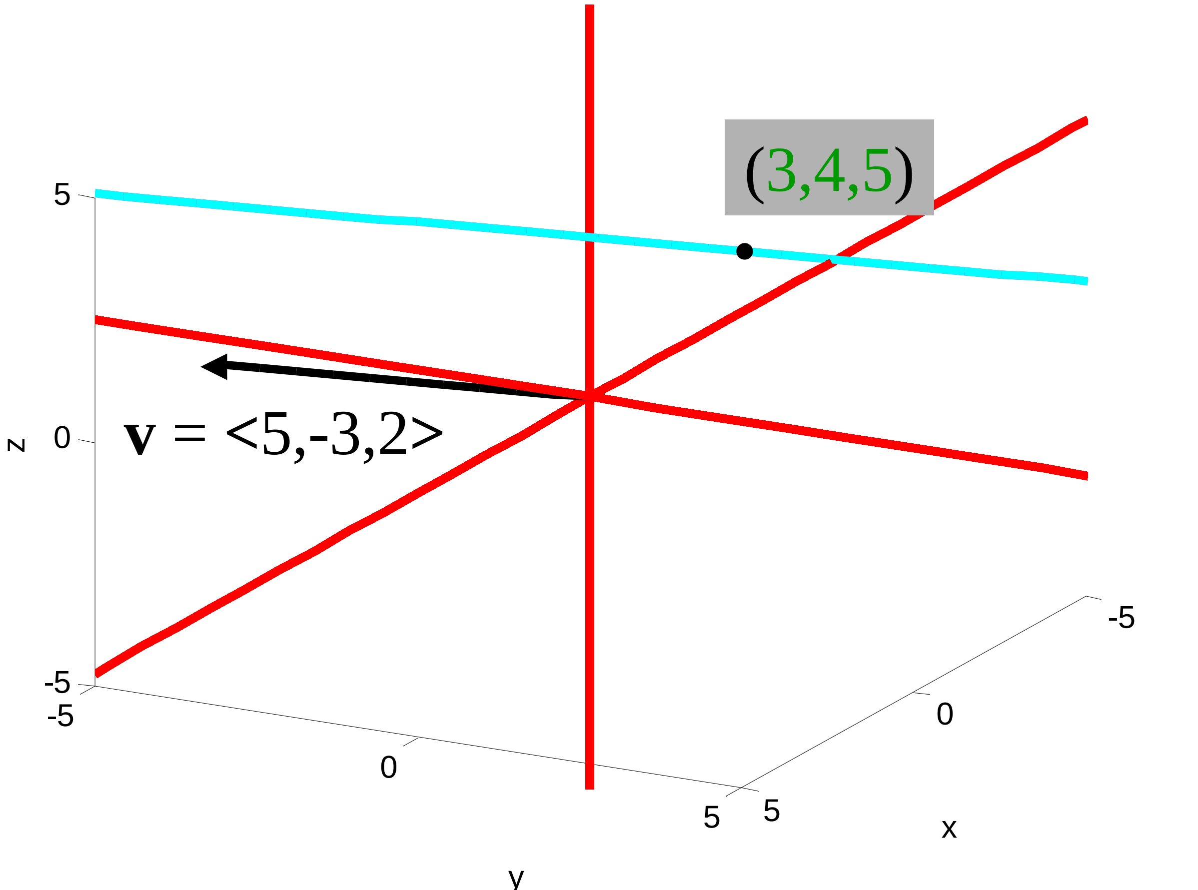
Two points are required to describe a vector

v = <2-3,6-4,-2-5>

v = <-1,2,-7>

Position vector – representation starts at the origin

Line



Parametric Equations

x = 3+5t

y = 4- 3t

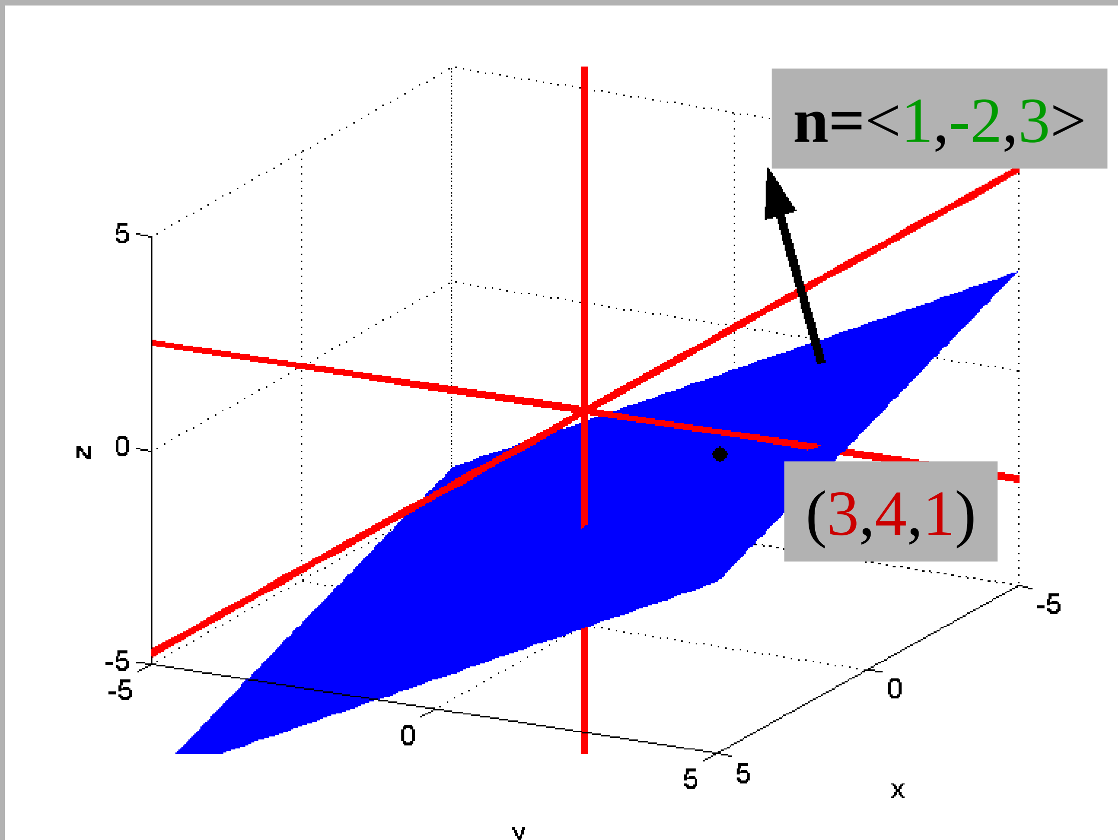
z = 5+2t

Symmetric Equations

$$\frac{x-3}{5}=\frac{y-4}{-3}=\frac{z-5}{2}$$

The direction vector, **v** is parallel to the line.

Planes



$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

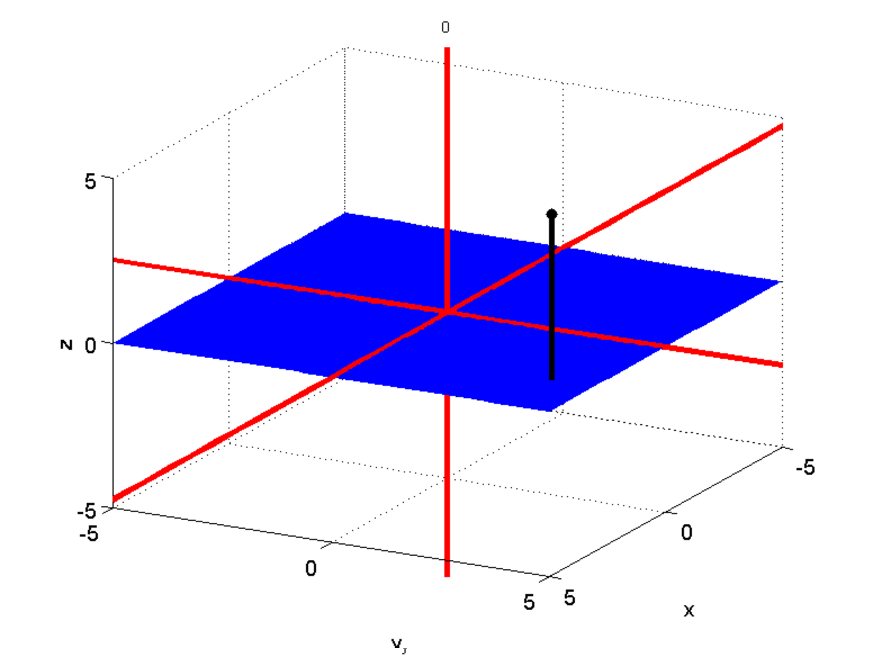
$$ax - ax_1 + by - by_1 + cz - cz_1 = 0$$

$$ax + by + cz + d = 0 \quad \text{where } d = -ax_1 - by_1 - cz_1$$

Typical ? 1) What is the distance from a point to a plane?

1. If the plane is one of the coordinate planes (xy, xz, yz), then the answer is trivial.

Ex. What is the distance from the point (3,4,5) to the xy plane?



Since the xy plane contains all the points of the form (x,y,0), one point on the plane is the point (3,4,0).

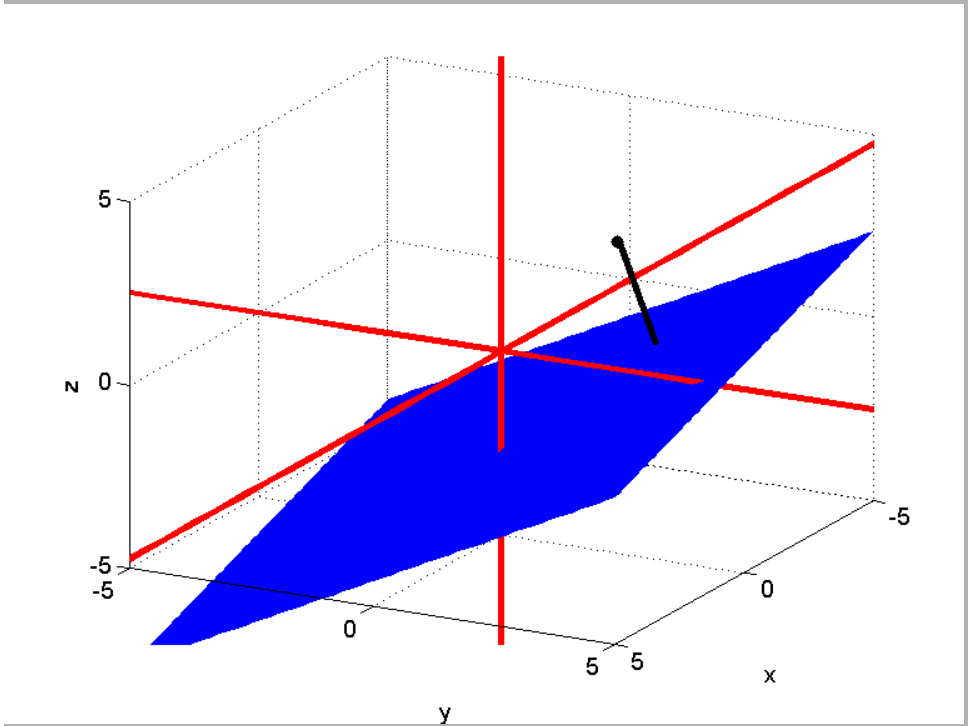
The distance between (3,4,5) and (3,4,0) is 5

or

$$D=\sqrt{(3-3)^2+(4-4)^2+(5-0)^2}=\sqrt{25}=5$$

However, if the plane is not a coordinate plane, the problem is more complex.

Ex. What is the distance from the point (3,4,5) to the plane x-2y+3z+8=0?



$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

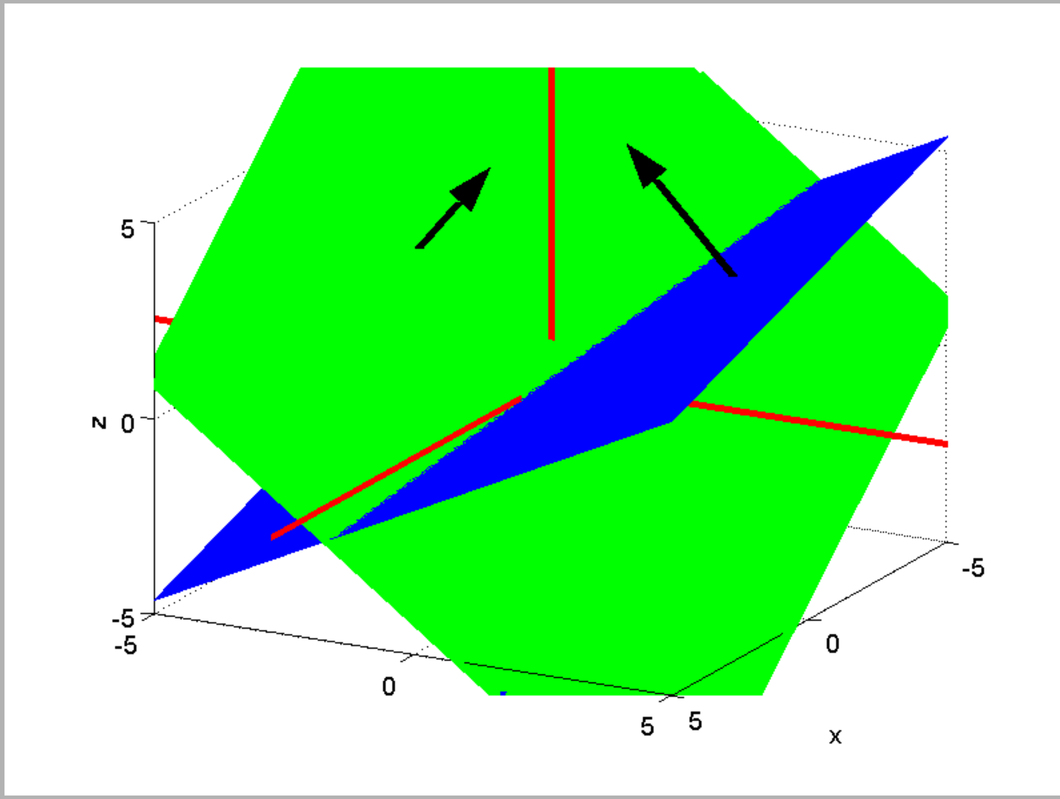
$$D = \frac{|1*3 + (-2)*4 + 3*5 + 8|}{\sqrt{(1)^2 + (-2)^2 + (3)^2}}$$

$$D = \frac{18}{\sqrt{14}} \approx 4.8$$

Typical ?

What is the angle between 2 planes (or lines)?

Ex. Find the angle between the planes x-2y+3z = 1 and x+y+z=1.



Consider the normal vectors:

n_1=<1,-2,3> and n_2=<1,1,1>

$$\cos\theta = \frac{n_1 \bullet n_2}{|n_1||n_2|}$$

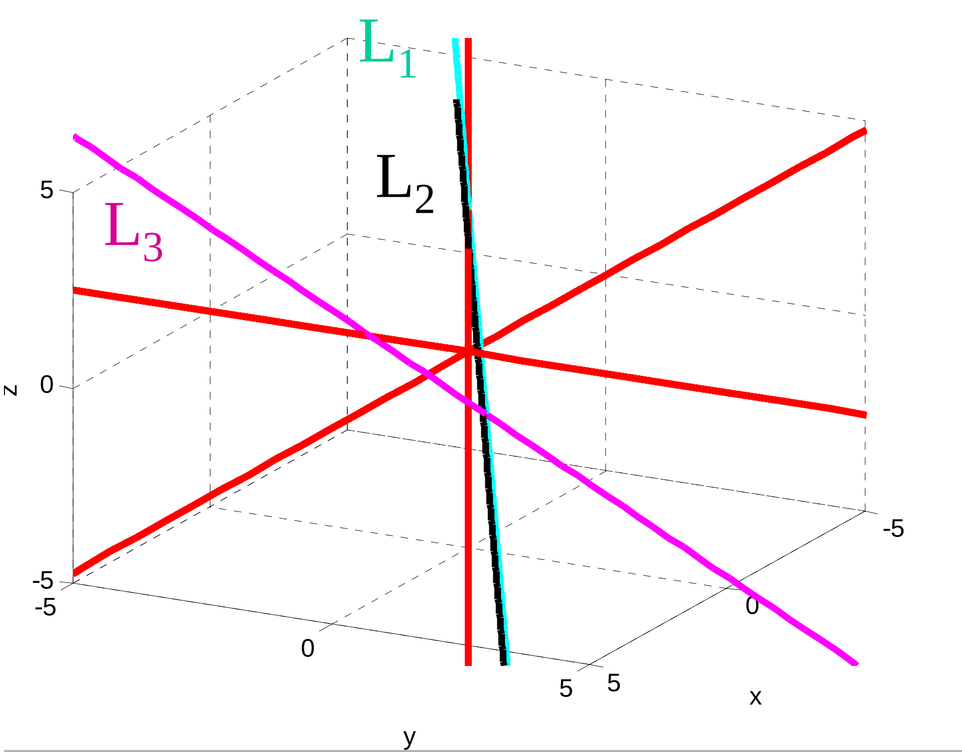
θ = 1.26 radians

Typical ? 3) Are two lines parallel, skew or intersecting?

Ex. Determine the relationship between L_1: $x = 3+2t$ $y = 2+t$ $z = -2+2t$

and L_2: $x = 1+4s$ $y = 1+2s$ $z = -3+4s$

and L_3: $x = 3+2r$ $y = -1 +4r$ $z = 2-r$



Direction vectors:

$v_1=<2,1,2>$

$v_2=<4,2,4>$

$v_3=<2,4,-1>$

Two lines are parallel if their direction vectors are scalar multiples.

Since $v_2= 2v_1$ we know L_2 is parallel to L_1 and neither are parallel to L_3

Now consider L_2: $x = 1+4s$ $y = 1+2s$ $z = -3+4s$ and L_3: $x = 3+2r$ $y = -1 +4r$ $z = 2-r$

L_2 intersects L_3 if $x = x$ when $y = y$ when $z = z$

Is there a solution to the set of equations?

$1 + 4s = 3 + 2r$ $\quad$ $\boxed{1+2s = -1 + 4r \qquad -3+4s = 2-r}$ $\quad$ $r = 2+3-4s = \boxed{5-4s}$

$$1+2s = -1 + 4(5-4s)$$

$$18s = 18$$

$s = 1$ and $r = 5 - 4(1) = 1$

Now check to see that the third equation is also satisfied

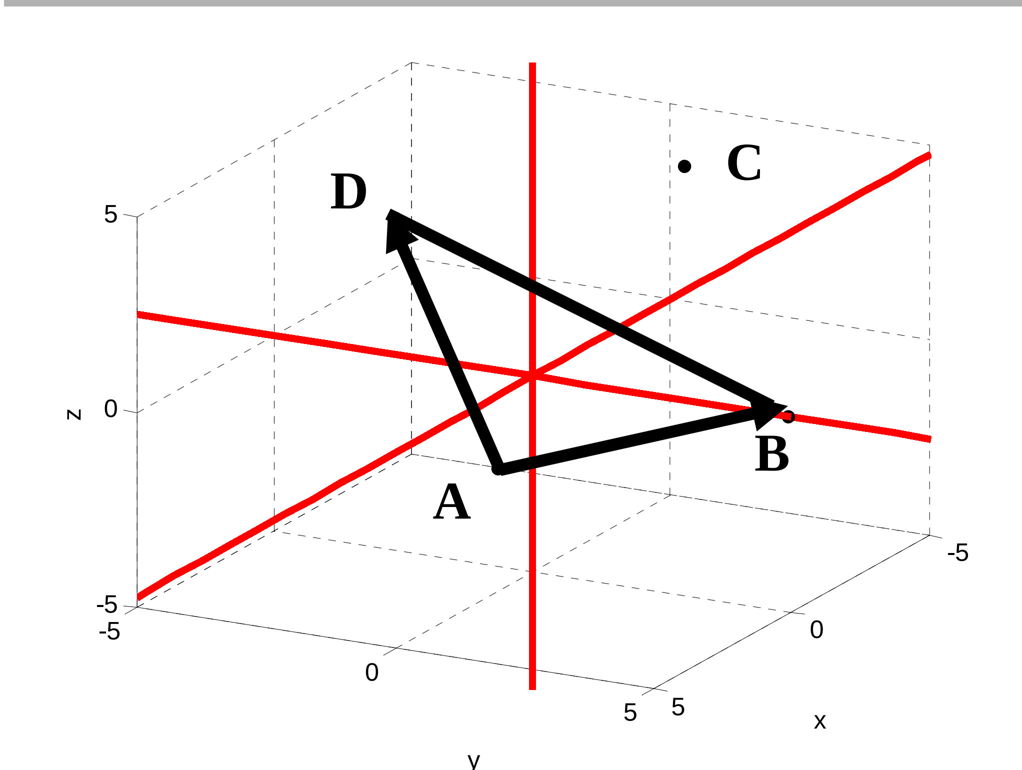
$1 + 4(1) \stackrel{?}{=} 3 + 2(1)$ **Yes, it's satisfied!!!**

So, L_2 intersects L_3 when $s = r = 1$ at the point $(x,y,z) = (5,3,1)$

Checking if L_1 intersects L_3 also we find that all 3 sets of equations are not satisfied. Thus, L_1 and L_3 are skew.

Typical ? 4) What is the area of the parallelogram (or triangle)?

Ex. Find the area of the parallelogram with vertices at the points A(5,2,0) B(2,6,1) C(2,4,7) and D(5,0,6)



Pick a corner and construct 2 vectors

AB = -3i +4j +k

AD = -2j +6k

$$AB \times AD = \begin{vmatrix} i & j & k \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix}$$

$$= \boldsymbol{i}\begin{vmatrix} 4 & 1 \\ -2 & 6 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} -3 & 1 \\ 0 & 6 \end{vmatrix} + k\begin{vmatrix} -3 & 4 \\ 0 & -2 \end{vmatrix}$$

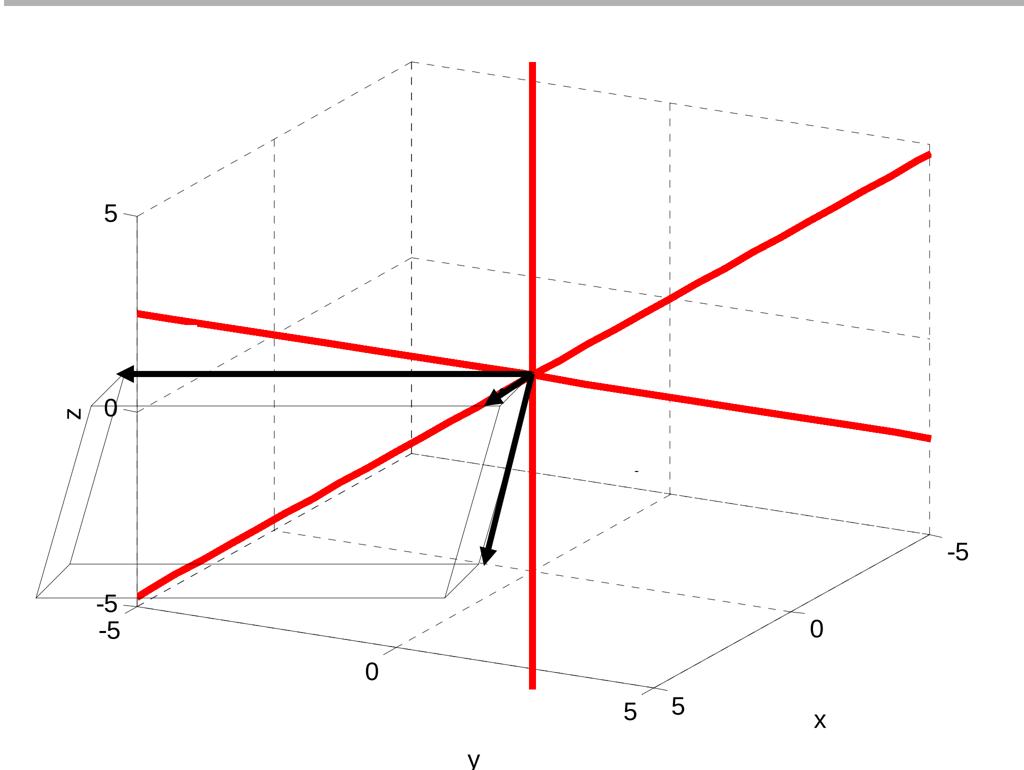
= 26i+18j+6k

Area of ▰ABCD = $|AB \times AD| = \sqrt{26^2 + 18^2 + 6^2} = \sqrt{1036} \approx 32.19$

Furthermore, the area of ▲DAB = ▰ABCD / 2

Typical ? 5) What’s the volume of the parallelepiped?

Ex. Find the volume of the parallelepiped with a=<3,-5,1>, b=<0,2,-2> and c=<3,1,1> as adjacent edges.



$$V = |\, a \cdot (b \times c) \,|$$ **(triple scalar product)**

$$= \begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix}$$

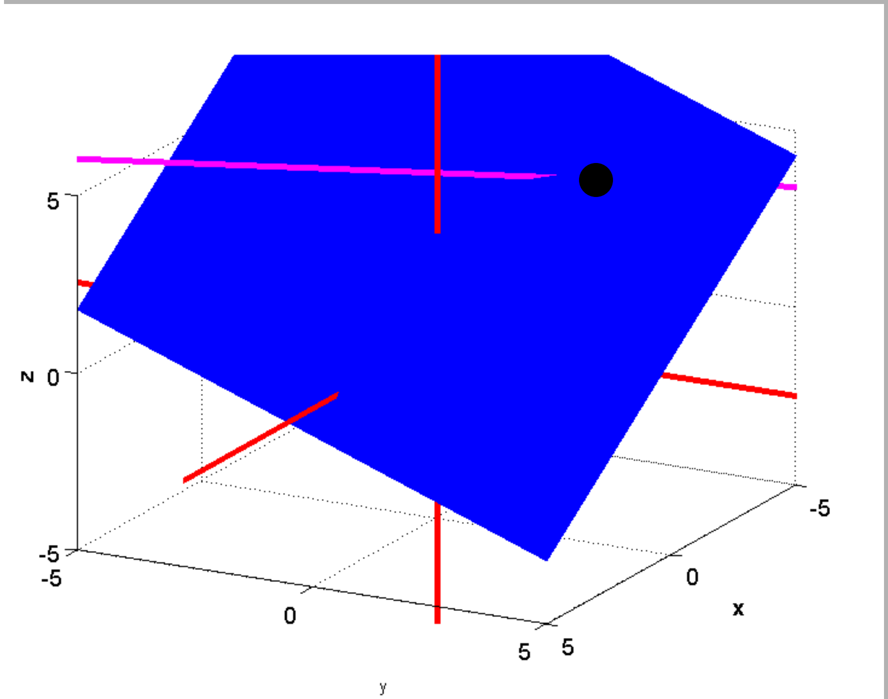
$$= 3\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - (-5)\begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}$$

=3(4) +5(6) +1(-6) = 36

*if the V = 0 then the points lie in the same plane (coplanar)

Typical ? 6) Where does the line intersect the plane?

Ex. Find the intersection point of $3x+2y+4z=12$ and $\frac{x-3}{5} = \frac{y+6}{-7} = \frac{z-5}{1}$



Convert the symmetric equation for the line into the corresponding parametric equations.

$t = \frac{x-3}{5} \Rightarrow x = 5t+3$

Likewise… $y = -7t-6$ and $z = t+5$

Substitute these into for the x,yand z in the equation of the plane: $3x+2y+4z=12$

$$3(5t+3) + 2(-7t-6)+4(t+5) = 12$$

$$15t+9 - 14t - 12+4t+20 = 12$$

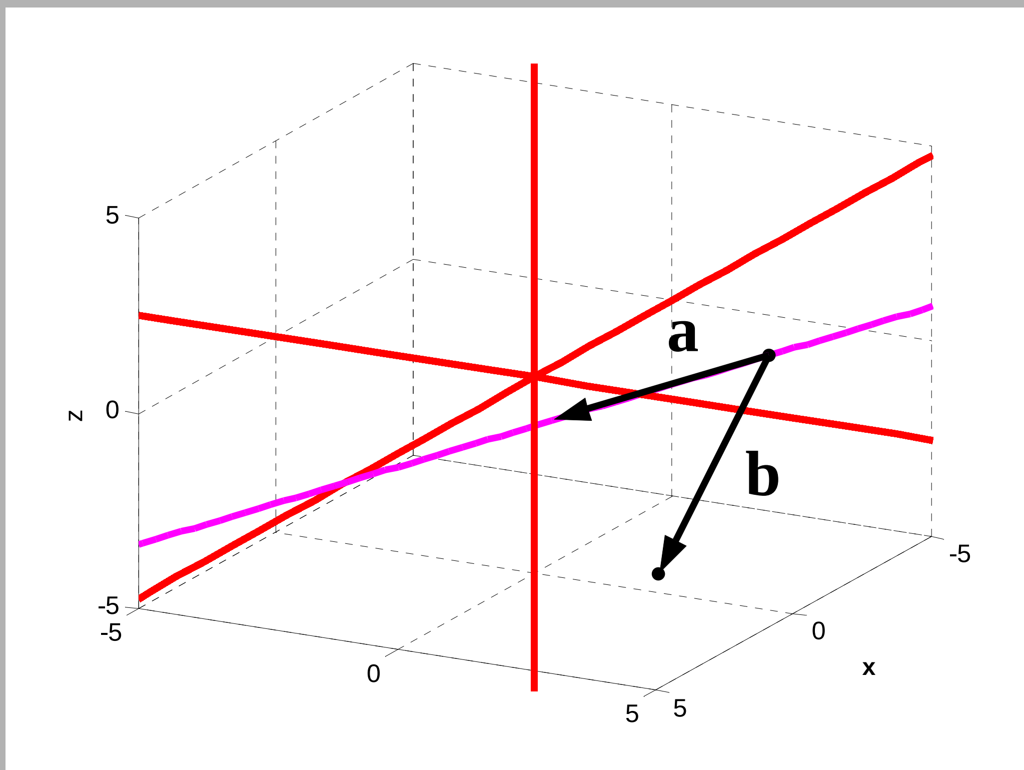
$$5t = -5 \quad \text{or } t = -1$$

$x = 5(-1)+3 = -2$
$y = -7(-1)-6 = 1$
$z = (-1) + 5 = 4$

$\boxed{P(-2,1,4)}$

Typical ? 7) What is the distance between a point and a line?

Ex. What is the distance between P(3,4,-3) and $\frac{x+2}{1} = \frac{y-1}{3} = \frac{z+1}{2}$



Convert line to parametric equations:

x = -2+t y = 1+3t z = -1+2t

Find a point on the line (let t = -1)

x=-2+(-1)= **-3**

y=1+3(-1)= **-2**

z= -1+2(-1)= **-3**

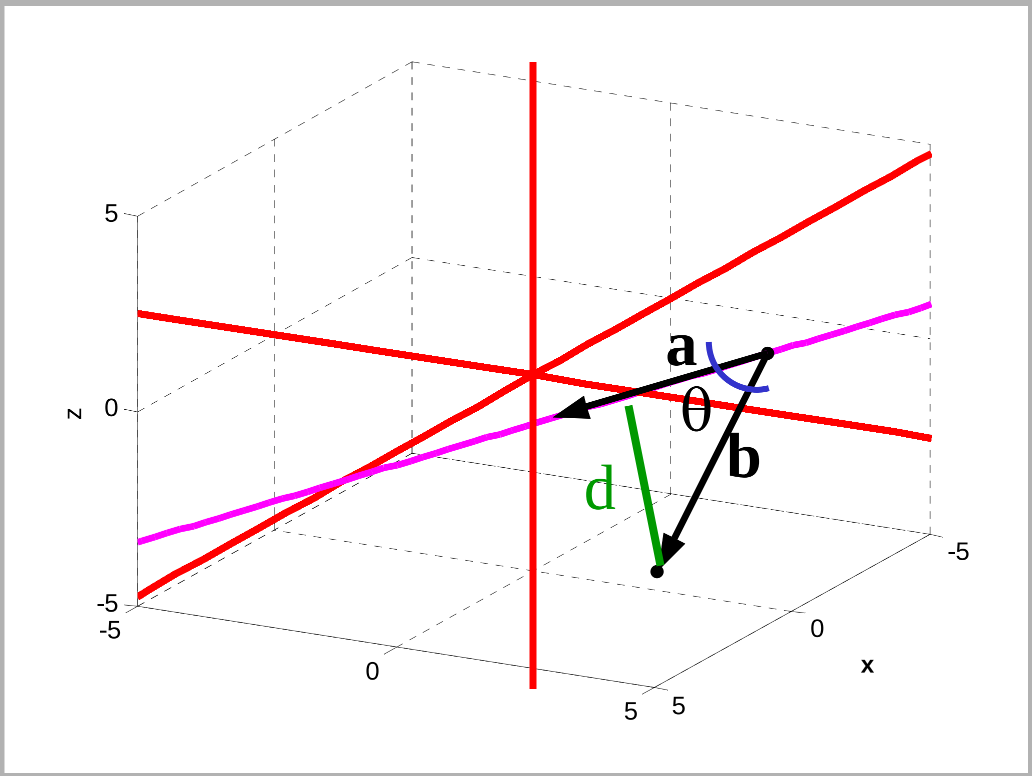
Create two vectors **a** and **b**

a = <1,3,2>

b = <3-(-3),4-(-2),-3-(-3)> =<6,6,0>

Two methods for solving the rest!

Method 1:



Compute the angle between the vectors

a = <1,3,2>

b = <6,6,0>

$$\cos\theta = \frac{a \bullet b}{|a||b|}$$

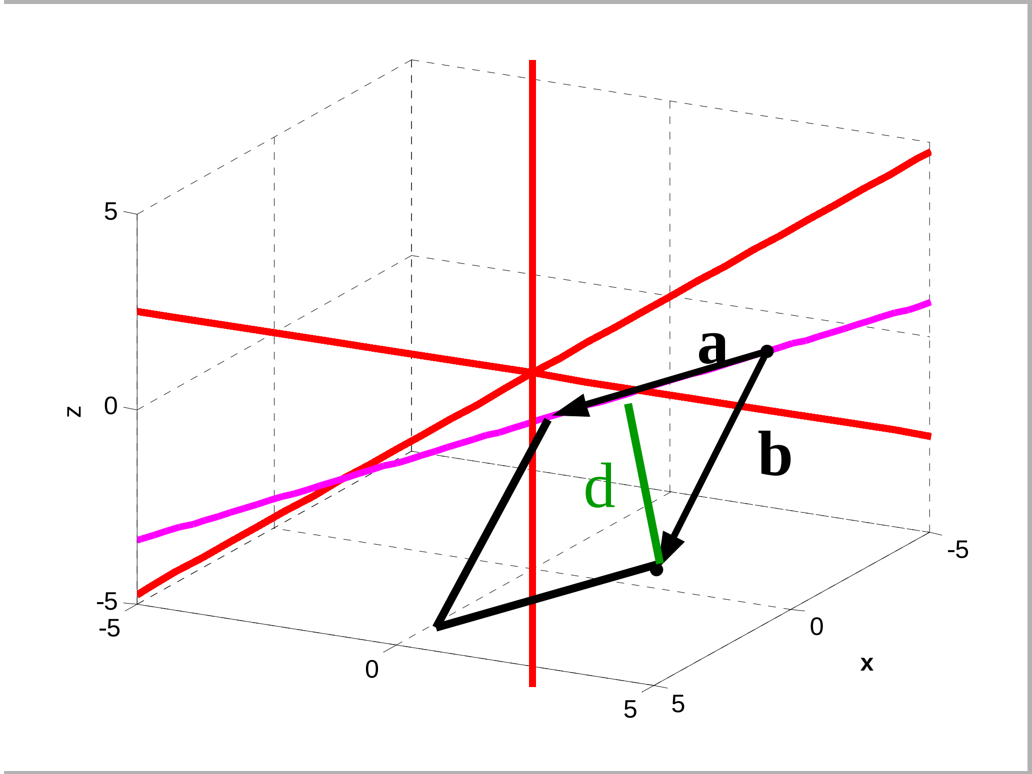
$$\theta = 0.714 \text{ radians}$$

$$\sin\theta = \frac{d}{|b|}$$

$$d = |\mathbf{b}|\sin\theta$$

$$d = \sqrt{72}\sin(0.714) = 5.56$$

Method 2:



Compute the area of the parallelogram

A = | a x b |

$$a \times b = \begin{vmatrix} i & j & k \\ 6 & 6 & 0 \\ 1 & 3 & 2 \end{vmatrix}$$ **= 12i -12j +12k**

$$|a \times b| = \sqrt{(12)^2 + (12)^2 + (12)^2}$$

d = $$\frac{Area\ of\ \ par.}{base\ length} = \frac{|a \times b|}{|a|} = \frac{\sqrt{432}}{\sqrt{14}} = 5.56$$