1. Use the tangent line approximation to estimate \((4.01)^2\). Is this estimation an over or under approximation? Justify your answer with a sketch.

\[
\hat{f}(x) = f(a) + f'(a)(x-a)
\]

\[
\hat{f}(x) = f(4) + f'(4)(x-4) = 16 + 8(x-4)
\]

\[
\hat{f}(4.01) = 16 + 8(4.01-4) = 16 + 8(0.01)
\]

\[
= 16.08
\]

2. Evaluate the following: (5 pts each)

a) \[\int x\sqrt{x^2-4}dx = \int u^{\frac{1}{2}}du = \frac{2}{3}u^{\frac{3}{2}} + C\]

b) \[\int \sin(3x) - 3x + 3 - \frac{3}{x}dx = \frac{-1}{3}\cos(3x) - \frac{3}{2}x^2 + 3x - 3\ln|x| + C\]

c) \[\int xe^{-2x}dx = x\left(-\frac{1}{2}e^{-2x}\right) - \int \frac{-1}{2}e^{-2x}dx = \frac{-1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx = \frac{-1}{2}xe^{-2x} + \frac{1}{4}e^{-2x} + C\]

d) \[\lim_{t \to \infty} \int_0^t 2x^{-2}dx = \lim_{t \to \infty} \frac{2x^{-2}}{-2} \bigg|_0^t = \lim_{t \to \infty} \left[-\frac{1}{x^2}\right]_0^t = \lim_{t \to \infty} \left[-\frac{1}{t^2} - \frac{-1}{0^2}\right] = \frac{1}{4}\]
3. Standing on the ground, I toss a ball upward with a velocity of 15 meters per second. When does it hit the ground? Given that the acceleration due to gravity $a(t) = -9.8 \frac{m}{s^2}$. Hint: $a(t) = \frac{dv}{dt}$ where $v(t) = \text{velocity}$.

$p(t) = 0 \quad v(t) = 15$

$v(t) = \int a(t) \, dt = \int -9.8 \, dt = -9.8 \, t + C$

$v(0) = -9.8 \cdot 0 + C = 15 \implies C = 15$

$v(t) = -9.8 \, t + 15$

$p(t) = \int v(t) \, dt = \int (-9.8 \, t + 15) \, dt = -9.8 \frac{t^2}{2} + 15 \, t + C$

$p(0) = -9.8 \cdot 0 + 15 \cdot 0 + C = 0 \implies C = 0$

$p(t) = -9.8 \frac{t^2}{2} + 15 \, t$

$\frac{dx}{dt} = -9.8 \frac{t^2}{2} + 15 \, t$

$x = \frac{-9.8 \frac{t^2}{2} + 15 \, t}{-9.8} + C$

When the ball hits the ground, $p(t) = 0$

$-4.9 \frac{t^2}{2} + 15 \, t = 0$

$t = 0, \quad \frac{-15}{4.9} = 3.061\text{ s}$

4. The rate of growth of a sunflower is defined by $\frac{dS}{dt} = 2 - 0.5t^2$ cm per week. If time is measured in weeks and length is measured in centimeters, how much does the sunflower grow during the second month (weeks 5-8)? Please label your solution with the correct units.

$\frac{8}{3} \left[ 2 - t + 0.5t^2 \right] dt = 2 \, t - \frac{1}{2} + \frac{0.5}{3} \, t^3 \right\rvert_{5}^{8}$

$= \left( 2 \cdot 8 - \frac{1}{2} \cdot 8^2 + \frac{0.5}{3} \cdot 8^3 \right) - \left( 2 \cdot 5 - \frac{1}{2} \cdot 5^2 + \frac{0.5}{3} \cdot 5^3 \right)$

$= -51 \text{ cm}$

5. Verify $y(x) = 1 - e^{-2x}$ is the solution to the differential equation $\frac{dy}{dx} = 2(1 - y)$.

$LHS: \quad \frac{dy}{dx} = -e^{-2x}(-2) = 2e^{-2x}$

$LHS: \quad \frac{dy}{dx} = 2(1 - (1 - e^{-2x})) = 2(x + e^{-2x}) = 2e^{-2x}$

$LHS: \quad \frac{dy}{dx} = 2(1 - (1 - e^{-2x})) = 2(x + e^{-2x}) = 2e^{-2x}$
6. Find the area of the region bounded by the curves \( y = x^2 \) and \( y = \sqrt{x} \).

\[
\begin{align*}
\int_0^3 \left( \sqrt{x} - x^2 \right) \, dx &= \int_0^3 x^{3/2} - x^2 \, dx \\
&= \left. \frac{2}{3} x^{3/2} - \frac{1}{3} x^3 \right|_0^3 \\
&= \frac{2}{3} (3)^{3/2} - \frac{1}{3} (3)^3 - (0) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}
\end{align*}
\]

7. Use Euler's method with \( \Delta t = 1 \) to approximate \( P(4) \), given \( \frac{dP}{dt} = 2P^2 + 1 \) and \( P(0) = 2 \).

\[
\begin{align*}
P(0) &= P(0) + P'(0) \Delta t = 2 + (2 (2)^2 + 1) = 11 \Rightarrow P(1) \\
P(1) &= P(1) + P'(1) \Delta t = 11 + (2 (11)^2 + 1) = 259 \\
P(3) &= P(2) + P'(2) \Delta t = 259 + (2 (259)^2 + 1) = 129887
\end{align*}
\]

8. Suppose the differential equation \( \frac{dG}{dt} = \frac{2G}{e^{-t}} \) describes the rate of growth a population of geese. If the population is 10 on day 3, is the population growing or shrinking at this time?

\[
\begin{align*}
\frac{dG}{dt} &= \frac{2 (10)}{e^{-12}} = \frac{20}{e^{-12}} = 3.855 \times 10^{-8} > 0
\end{align*}
\]

Population is growing.

9. From the graph of the differential equation provided sketch the phase line diagram and a plot of the solution if the initial condition is \( P(0) = 2 \). Please label your axes correctly.