Math 1297 – Spring 2008 - Exam 3 Name:____________________ Sec:__________

1. Determine whether the **sequence** $a_n = \dfrac{n^2-4}{3n^2+n}$ converges or diverges.

$$\underbrace{\lim_{n\to\infty} a_n}_{+3} = \lim_{n\to\infty} \frac{n^2-4}{3n^2+n} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2}-\frac{4}{n^2}}{\frac{3n^2}{n^2}+\frac{n}{n^2}} = \lim_{n\to\infty} \frac{1-\frac{4}{n^2}}{3+\frac{1}{n}} = \frac{1}{3}\ \lrcorner +3$$

$\therefore$ converges to $\frac{1}{3}\ \lrcorner +4$

−3 using test for divergence

2. Determine if $\sum_{k=2}^{\infty} 6^{-k}$ converges or diverges. If convergent, find the sum of the series.

(+3)

$$\sum_{k=2}^{\infty} 6^{-k} = \sum_{k=2}^{\infty} \frac{1}{6^k} = \frac{1}{6^2} + \frac{1}{6^3} + \frac{1}{6^4} + \cdots$$ geometric series $a = \frac{1}{6^2}$ $r = \frac{1}{6}\ \lrcorner +2$

So, yes convergent $_{+2}$ $\sum_{k=2}^{\infty} 6^{-k} = \dfrac{\frac{1}{6^2}}{1-\frac{1}{6}} = \dfrac{\frac{1}{6^2}}{\frac{5}{6}} = \frac{1}{6^2}\cdot\frac{6}{5} = \boxed{\frac{1}{30}}\ \lrcorner +5$

−or− $\sum_{k=2}^{\infty} 6^{-k} = \underbrace{\sum_{k=0}^{\infty}\left(\frac{1}{6}\right)^k - a_0 - a_1}_{+5} = \sum_{k=0}^{\infty}\left(\frac{1}{6}\right)^k - 1 - \frac{1}{6} = \frac{1}{1-\frac{1}{6}} - 1 - \frac{1}{6} = \frac{1}{\frac{5}{6}} - 1 - \frac{1}{6}$

$= \frac{6}{5} - 1 - \frac{1}{6} = \frac{36}{30} - \frac{30}{30} - \frac{5}{30} = \boxed{\frac{1}{30}}\ \lrcorner +5$

3. Determine the convergence or divergence of ~~$\sum_{k=1}^{\infty} \frac{9+\cos k}{k}$~~. State the test used for the determination.

$\sum_{k=1}^{\infty} \frac{1}{k^3+1}$

Consider $\sum_{k=1}^{\infty} \frac{1}{k^3}$ (Convergent p series $p=3$) $\lrcorner +3$

Since $\frac{1}{k^3+1} \le \frac{1}{k^3}\ \lrcorner +4$

By the Direct Comparison Test $\sum_{k=1}^{\infty} \frac{1}{k^3+1}$, is also convergent $\lrcorner +3$

Or Limit Comparison test. The ratio test is inconclusive

4. Determine the convergence or divergence of $\sum_{k=1}^{\infty}(-1)^k \frac{4}{\sqrt{k+5}}$. State the test used for the determination.

A.S.T $a_k = \frac{4}{\sqrt{k+5}}$

i) $\lim_{k\to\infty} \frac{4}{\sqrt{k+5}} = 0\ \lrcorner +4$

ii) $f(x) = \frac{4}{\sqrt{x+5}} = 4(x+5)^{-\frac{1}{2}}$

$f'(x) = -2(x+5)^{-\frac{3}{2}} < 0\ \lrcorner +4$

$\therefore$ a_k is decreasing

$\therefore \sum_{k=1}^{\infty}(-1)^k \frac{4}{\sqrt{k+5}}$ is convergent $\lrcorner +2$

5. Is the series $\sum_{k=1}^{\infty} \frac{(-1)^k 4}{k^2-2}$ divergent, conditionally convergent, or absolutely convergent?

$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{4}{k^2-2}$ +4

Limit comparison test $\sum_{k=1}^{\infty} \frac{4}{k^2}$, $\lim_{k\to\infty} \frac{4}{k^2-2} \cdot \frac{k^2}{4} = 1$ +2 $\therefore$ Convergent +2

So, $\sum_{k=1}^{\infty} \frac{(-1)^k 4}{k^2-2}$ is $\boxed{\text{absolutely convergent.}}$ +2

-or-

First use AST to show $\sum_{k=1}^{\infty} \frac{(-1)^k 4}{k^2-2}$ is convergent +4

and then show $\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{4}{k^2-2}$ is convergent using limit comparison test +4

to conclude the series is absolutely convergent +2

* The ratio test is inconclusive for this series (-4)

6. Estimate the error in using S_{110} to approximate the sum of $\sum_{k=1}^{\infty} \frac{4}{k^2}$.

$$R_{110} \le \int_{110}^{\infty} \frac{4}{x^2}\,dx = \lim_{R\to\infty} \int_{110}^{R} 4x^{-2}\,dx = \lim_{R\to\infty} -4x^{-1}\Big]_{110}^{R}$$

+4 +2 +2

$$= \lim_{R\to\infty} \frac{-4}{R} - \frac{-4}{110} = \frac{4}{110} = \boxed{0.0\overline{36}}$$

+2

7. Approximate the sum of the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$ with an error less than 0.001.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} = \frac{1}{1} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} \cdots$$

Alternating series: need $b_{k+1} \le 0.001$ +3

$b_5 = 0.00833$ $b_6 = 0.00139$ $b_7 = 0.0002 < 0.001$

$n = 6$ 3

$$S_6 = \frac{1}{1} + \frac{-1}{2!} + \frac{1}{3!} + \frac{-1}{4!} + \frac{1}{5!} + \frac{-1}{6!}$$ +3

$$= 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} = \boxed{0.6319\overline{4}}$$ +1

8. Find the interval of convergence for the power series $\sum_{k=1}^{\infty} \frac{x^k}{3k+1}$. → not getting $|x|$ as limit "-7"

Ratio test:

$$\lim_{k\to\infty} \left| \frac{x^{k+1}}{3(k+1)+1} \cdot \frac{3k+1}{x^k} \right| = \lim_{k\to\infty} \frac{\cancel{x^k} x \; 3k+1}{(3k+4)\cancel{x^k}} = \underline{|x| < 1} \Rightarrow -1 < x < 1 \quad +3$$

Testing endpoints:

$x = -1$

$\sum_{k=1}^{\infty} \frac{(-1)^k}{3k+1}$ Alt. Series i) $\lim_{k\to\infty} \frac{1}{3k+1} = 0$ ii) $f(x) = \frac{1}{3x+1} = (3x+1)^{-1}$ $f'(x) = -1(3x+1)^{-2}(3) = \frac{-3}{(3x+1)^2} < 0$

$\therefore$ Convergent +3

$x = 1$

$\sum_{k=1}^{\infty} \frac{1^k}{3k+1} = \sum_{k=1}^{\infty} \frac{1}{3k+1}$ Limit Comparison test w/ $\sum_{k=1}^{\infty} \frac{1}{k}$ (divergent)

$\lim_{k\to\infty} \frac{1}{3k+1} \cdot \frac{k}{1} = \frac{1}{3}$ $\therefore$ Divergent +3

$$\boxed{I = [-1, 1)} \quad +1$$

9. Given that the geometric series $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ for |x|<1, find a power series representation for ln$(1+x)$.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k \quad +2$$

Wrong: $\sum_{k=0}^{\infty} \frac{(-x)^{k+1}}{k+1}$ −1

Correct answer but no "+C" −1

Without evaluating "C" : −1

$$\int \frac{1}{1+x}\,dx = \ln(1+x) + C \quad +3$$

$$\int \sum_{k=0}^{\infty} (-1)^k x^k\,dx = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} + C = \ln(1+x) \quad +3$$

at $x = 0$ $\sum_{k=0}^{\infty} \frac{(-1)^k (0)^{k+1}}{k+1} + C = \ln(1+0)$

$0 + C = 0 \Rightarrow C = 0$ +1

$$\boxed{\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1}} \quad +1$$

10. Use a Taylor polynomial of degree 4 to approximate $\sqrt{1.15}$ centered about 1. Estimate the error in the approximation using Taylor's Theorem.

n	$f^n(x)$	$f^n(1)$
0	$\sqrt{x} = x^{\frac{1}{2}}$	1
1	$\frac{1}{2}x^{-\frac{1}{2}}$	$\frac{1}{2}$
2	$-\frac{1}{4}x^{-\frac{3}{2}}$	$-\frac{1}{4}$
3	$\frac{3}{8}x^{-\frac{5}{2}}$	$\frac{3}{8}$
4	$-\frac{15}{16}x^{-\frac{7}{2}}$	$-\frac{15}{16}$
5	$\frac{105}{32}x^{-\frac{9}{2}}$	

+2

$$\sqrt{x} = \sum_{k=0}^{\infty} \frac{f^k(1)}{k!}(x-1)^k = \frac{f(1)(x-1)^0}{0!} + \frac{f'(1)(x-1)^1}{1!} + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} + \frac{f^4(1)(x-1)^4}{4!} + \ldots$$

$$P_4(x) = 1 + \frac{1}{2}(x-1) - \frac{\frac{1}{4}}{2!}(x-1)^2 + \frac{\frac{3}{8}}{3!}(x-1)^3 - \frac{\frac{15}{16}}{4!}(x-1)^4 \quad +4$$

$$P_4(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{15}{384}(x-1)^4$$

$$P_4(1.15) = 1 + \frac{1}{2}(1.15-1) - \frac{1}{8}(1.15-1)^2 + \frac{1}{16}(1.15-1)^3 - \frac{15}{384}(1.15-1)^4 = \boxed{1.0723} \quad +2$$

$$|R_3(1.15)| = \left| \frac{f^5(z)}{5!}(1.15-1)^5 \right| \le \left| \frac{\frac{105}{32}}{5!}(0.15)^5 \right| = 2.076 \times 10^{-6}$$

for $1 < z < 1.15$ +2 +1 +1

$$f^5(x) = \frac{105}{32}x^{-\frac{9}{2}} = \frac{105}{32x^{\frac{9}{2}}}$$

is a decreasing function on $[1, 1.15]$

so it is largest at 1.