Math 3280
Differential Equations with Linear Algebra

Test 1
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Name___A.K.____

SHOW ALL WORK.

Calculators may be used for algebra and graphing only. You may NOT use calculators to solve differential equations (using commands like DSolve) or to take integrals or derivatives (using commands like D or Integrate).

p2._________/28+3

p3._________/24+3

p4._________/34

p5._________/16

Total________/102+6

Directions: Do all problems. Show all work. Make no mistakes. Calculators may be used only for algebra and graphing, but not for symbolic tasks like integrating, differentiating, or solving differential equations. Implicit solutions will receive full credit only if labelled as **implicit**. For all soutions to differential equations, write the answer with "full notation" (for example, $y(x) = ...$, rather than just $y = ...$). Label the axes of all graphs. Include the variable of integration for all integrals ($\int f(x)dx$ rather than $\int f(x)$).

1. (4pts) Are the following differential equations separable, linear, both, or neither?

   (a) $\frac{dy}{dx} = x + y^2$ not sep, not linear

   (b) $\frac{dy}{dx} = y + x^2$ linear, not sep.

2. (6pts) For what (constant) values of r and A is $P(t) = Ae^{rt}$ a solution to $P''(t) + 3P'(t) + 2P(t) = 0$?

$P = Ae^{rt} \Rightarrow P' = Are^{rt}, P'' = Ar^2e^{rt}$

$LHS = Ar^2e^{rt} + 3Are^{rt} + 2Ae^{rt}$

$= Ae^{rt}(r^2 + 3r + 2) = 0$ if $A = 0$, $e^{rt} = 0$ (never), or $r^2 + 3r + 2 = (r+1)(r+2) = 0$

ie, Ae^{rt} is a solution if $A = 0$ (and r is any real #) or $r = -1$ or -2 (and A is any real #)

3. (6pts) What is the analytic explicit solution for $\frac{1}{2}\frac{dx}{dt} = x$, $x(0) = 3$. Solve by inspection or any other technique you know.

By inspection: $\dot{x} = 2x$ has solution $x(t) = Ce^{2t}$.

$x(0) = 3 \Rightarrow C = 3$, so $x(t) = 3e^{2t}$

4. Find both the general solution to the given differential equations, and the specific solution corresponding to the given initial conditions. Shobothw your work. Express all answers explicitly.

   (a) (6 pts) $\frac{dy}{ds} = s^2$, $y(2) = -1$

Integrate w.r.t. s: $\int \frac{dy}{ds} ds = \int s^2 ds + C$

ie, $y(s) = \frac{s^3}{3} + C$

$y(2) = -1 \Rightarrow -1 = \frac{2^3}{3} + C$

or $C = -1 - \frac{8}{3} = -\frac{11}{3}$

So $y(s) = \frac{s^3}{3} - \frac{11}{3}$

   (b) (6 pts) $\frac{dy}{ds} = y^2$, $y(2) = -1$. Express your solutions explicitly.

$\frac{1}{y^2} dy = 1 \cdot ds \Rightarrow \int \frac{1}{y^2} dy = \int 1 ds + C$, ie, $\frac{y^{-1}}{-1} = s + C$, ie $\frac{1}{y} = -(s+c)$,

So $y(s) = \frac{1}{-(s+c)}$

$y(2) = -1 \Rightarrow -1 = \frac{1}{-(2+c)} \Leftrightarrow 2 + c = 1 \Rightarrow c = -1$, So $\boxed{y(s) = \frac{1}{-(s-1)} = 1/(1-s)}$

(Extra Credit +3pts) What is the interval of existence for the solution to this initial value problem?

$y(s) = \frac{1}{1-s} \Rightarrow$ "problem" @ $s = 1$. $\therefore$ The interval of existence starts at $s = 2$ and goes back toward $s = 1$, and forward toward ∞. That is the interval of existence is

$s \in (1, \infty)$.

(c) (10 pts) $\dot{y} - 4t^3 y = 3e^{t^4}$, $y(0) = 2$. Show all steps; don't just "plug in" to a formula. Express your solutions explicitly.

Linear, so determine integrating factor $f(t) = e^{-\int 4t^3 dt} = e^{-\frac{4t^4}{4}} = e^{-t^4}$

Multiply through: $\dot{y}e^{-t^4} - 4t^3 e^{-t^4} = 3e^{t^4}\left(e^{-t^4}\right) = 3$

i.e. $\dfrac{d}{dt}\left(y \cdot e^{-t^4}\right) = 3$

Integrate w.r.t. t: $\int \frac{d}{dt}\left(y e^{-t^4}\right) dt = \int 3\,dt + C$

i.e. $y e^{-t^4} = 3t + C$

or $y(t) = (3t + C)\left(e^{t^4}\right)$

$= 3te^{t^4} + Ce^{t^4}$

$y(0) = 2 \Rightarrow C = 2$, so $y(t) = 3te^{t^4} + 2e^{t^4}$

(d) (6 pts) $\dfrac{dR(t)}{dt} = \dfrac{3t^2 + 1}{R^2}$, $R(1) = 2$.

Separable!

$R^2\,dR = (3t^2 + 1)\,dt$

$\dfrac{R^3}{3} = \dfrac{3t^3}{3} + t + C$ (Implicit)

i.e. $\boxed{\dfrac{R(t)^3}{3} = t^3 + t + C}$

$R(1) = 2 \Rightarrow$

$\dfrac{2^3}{3} = 1^3 + 1 + C \Rightarrow C = \frac{8}{3} - 2 = \frac{2}{3}$

$\therefore \boxed{\dfrac{R(t)^3}{3} = t^3 + t + \frac{2}{3} \text{ (Implicit)}}$

Explicit: $R^3 = 3t^3 + 3t + 3C$ or $R(t) = \left(3t^3 + 3t + 3C\right)^{\frac{1}{3}}$

$R(1) = 2 \Rightarrow R(t) = \left(3t^3 + 3t + 2\right)^{\frac{1}{3}}$

5. Consider the initial value problem (differential equation and initial condition):

$$y' - 4xy = \frac{2}{x}\, y \ln(y),\ y(2) = 1$$

(a) (8 pts) Obtain a differential equation for v by using the substitution $y = e^v$ to eliminate y and obtain a new differential equation and new initial condition. Do **not** solve!

$y = e^v \Rightarrow y' = e^v \cdot v'$ (chain rule)

Plug in: $e^v v' - 4xe^v = \frac{2}{x} e^v \ln e^v = \frac{2}{x} e^v \cdot v$

Factor out e^v: $\boxed{v' - 4x = \frac{2}{x} \cdot v}$

equivalently: $v' - \frac{2}{x} v = 4x$

IC, $y(2) = 1$

$\Rightarrow 1 = y(2) = e^{v(2)}$

$\Rightarrow \ln 1 = v(2)$

or $v(2) = 0$

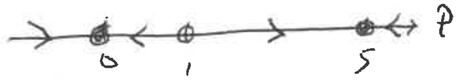
(b) (Extra Credit +3 pts) Is the new differential equation any easier to solve than the original differential equation? Explain.

In this form, the "new" d.e. is seen to be linear.

The original was not separable, not linear, so yes, the new d.e. is easier to solve.

6. Consider the differential equation $\dot{P} = P(1-P)(P-5)$. $= 0$ if $P = 0, 1, 5$

(a) (5 pts) Sketch a phase line for the differential equation. Label your axis, indicate any equilibrium points with dots, and indicate the direction of growth/decline of the population P (as time t increases) with arrows.

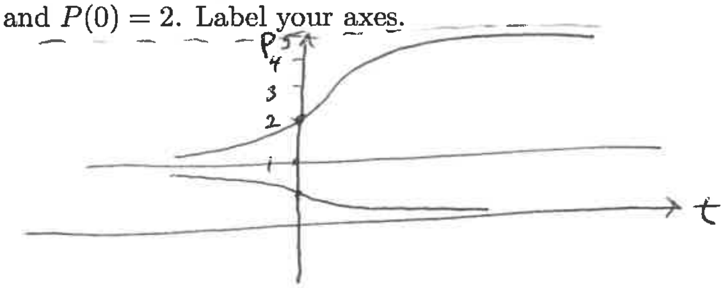


For arrows: Let RHS $= f(P)$.

$f(-1) = (-1)\cdot 2 \cdot (-6) > 0$

$f(\tfrac{1}{2}) < 0$, $f(2) > 0$, $f(6) < 0$

(b) (6 pts) Using only the information from the phase line, sketch possible solutions (both forward and backward in time) corresponding to the initial conditions $P(0) = 0.5$, $P(0) = 1$ and $P(0) = 2$. Label your axes.



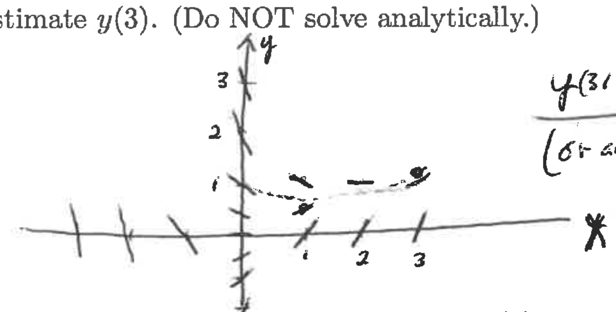
(c) (3 pts) Describe the long-term (forward) fate of the population P for all three initial conditions. (Do not solve the differential equation analytically.)

$P(0) = .5$: $P(t) \to 0$ $\qquad$ $P(0) = 2 \Rightarrow P(t) \to 5$

$P(0) = 1$, $P(t) = 1$

7. (10 pts) Consider the initial value problem: Sketch the slope field for $y'(x) = \dfrac{x}{1+(y(x))^2} - y(x)$

(a) Sketch the slope field. Include slope marks at $(x, y) = (-1, 1), (0, 1), (1, 1), (2, 2)$ and several slope marks along each axis. Include enough additional slope marks to allow you to sketch the solution corresponding to $y(0) = 1$. Show your work. Use this sketch to estimate $y(3)$. (Do NOT solve analytically.)



$y(3) \approx 1.2$?

(or anything close to 1)

x, y	slope
−1 1	$-3/2$
0 1	-1
1 1	$-\frac{1}{2}$
2 2	0
1 1/2	$4/5 - 1/2 = 3/5$
0 0	0
0 y	$-y$
x 0	x

(b) (10 pts) Assuming initial conditions, $y(0) = 1$, numerically approximate $y(3)$ using Euler's method with a step size of 1.0. Show your work. Do not solve analytically. Graph your Euler estimates at $x = 0, 1, 2, 3$.

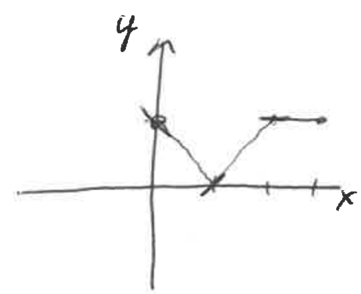
n	x_n	y_n	$f(x_n, y_n)$
0	0	1	−1
1	1	0	1
2	2	1	0
3	3	1	

$y_1 = y_0 + h \cdot f(x_0, y_0) = 1 + 1\cdot(-1) = 1 - 1 = 0$

$y_2 = y_1 + h\, f(x_1, y_1) = 0 + 1 \cdot 1 = 1$

$y_3 = y_2 + h\, f(x_2, y_2) = 1 + 1 \cdot 0 = 1$

$\therefore \boxed{y(3) \approx 1}$



8. Assume that an object travels along a straight line with constant acceleration a.

(a) (5pts) Write an appropriate differential equation and solve it to show that the position can be expressed as $x(t) = \frac{1}{2}at^2 + v_0 t + x_0$ for some constants v_0 and x_0. Show your work.

d.e. $\ddot{x} = a \Rightarrow$ (Integrate w.r.t.) $\dot{x} = at + v_0$

$\Rightarrow$ (integrate again) $x(t) = \underline{\frac{1}{2}at^2 + v_0 t + x_0}$

(b) (5pts) Assume a car is travelling at 100 ft per second (about 68 mph), and that it has a constant deceleration of 10 $\frac{ft}{sec^2}$ when the brakes are applied. How far does the car travel before it stops? Show all work.

$a = -10 \Rightarrow \dot{x}(t) = v(t) = -10t + 100$, since $v(0) = 100$,

$\Rightarrow x(t) = -10\frac{t^2}{2} + 100t + x_0 = -\frac{10t^2}{2} + 100t + 0$ (assuming we choose $x=0$ at the point on the road where the brakes are first applied.)

Stop $\Leftrightarrow v(t) = 0$, i.e. $-10t + 100 = 0$, so $t_{stop} = 10$.

$x(10) = -10\frac{(10)^2}{2} + 100 \cdot 10$

$= -500 + 1000 = \underline{500 \text{ ft}}$

9. (6pts) Assume that a pond in northern Minnesota is stocked with 300 trout at the beginning of May. (None existed before stocking.) Assume the fish grow at a rate proportional to the number of fish in the lake at any instant in time, but, due to competition for food, they die at a rate proportional to the cube of the current population. Assume that the DNR allows fishermen (and fisherwomen) to catch a total of 20 trout per month. Define appropriate varibles to be write an initial value problem (differential equation and initial conditions) to model this population so that any parameters in the differential equation are positive. (Do **not** solve.)

Let $F(t)$ = number of fish at time t (measured in months)

$$\boxed{\dot{F}(t) = k_1 F(t) - k_2 F(t)^3 - 20, \quad F(0) = 300}$$