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Diff. Equations and Lin. Alg.
Math 3280
Quiz 3, Spring 2020
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Consider initial value problem $\frac{dx}{dt} = -2x$, $x(0) = -1$. Find or estimate $x(1)$ using the follwing techniques INDEPENDENTLY. For the analytic solutions, give an exact answer for $P(1)$ and a numerical approximation (using a calculator). Implicit solutions are acceptable if labelled.

1. Find an analytic solution "by inspection."

$x(t) = Ce^{-2t}$, $x(0)=1 \Rightarrow -1 = Ce^{-2\cdot 0} \Rightarrow C=1$, so $\underline{x(t)=(-1)e^{-2t}}$ and $\underline{x(1) = -e^{-2}}$ $\underline{\approx -.135}$

2. Find an analytic solution by separation of variables.

$\frac{dx}{dt} = -2x \Rightarrow \frac{1}{x}\frac{dx}{dt} = -2 \Rightarrow \int \frac{1}{x}\frac{dx}{dt}\,dt = \int -2\,dt + C$ ie, $\int \frac{1}{x}\,dx = -2\int dt + C$

$\Rightarrow \ln|x| = -2t + C$ (implicit) ie, $\ln|x(t)| = -2t + C$

$\Rightarrow |x(t)| = e^{-2t+C} = e^{C}\cdot e^{-2t}$. $x(0)=-1 \Rightarrow |x(t)| = -x(t)$, so $-x(t) = e^{C}\cdot e^{-2t} = ke^{-2t}$

or $x(t) = -ke^{-2t}$. $x(0) = -1 \Rightarrow k=1$, so $\underline{x(t) = (-1)e^{-2t}}$ and $\underline{x(1) = -e^{-2}}$

3. Find an analytic solution using the first order linear technique.

Rewrite: $\frac{dx}{dt} + 2x = 0$. Integrating factor $\rho(t) = e^{\int 2\,dt} = e^{2t}$

Multiply through: $e^{-2t}\frac{dx}{dt} + e^{-2t}\cdot 2\cdot x = 0\cdot e^{-2t}$

ie, $\frac{d}{dt}\left(e^{-2t}x\right) = 0 \Rightarrow \int \frac{d}{dt}\left(e^{-2t}x\right)dt = \int 0\,dt + C \Rightarrow e^{2t}x = 0 + C$

or $x(t) = Ce^{-2t}$. $x(0)=-1 \Rightarrow C=-1$

So $x(t) = -e^{-2t}$

and $\underline{x(1) = -e^{-2}}$

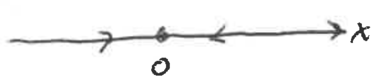
4. Find a sketch of a solution by first drawing a phase line, then sketching a solution which is "consistent with the phase line."

$\dot{x} = -2x = f(x)$ $= 0$ if $x=0$

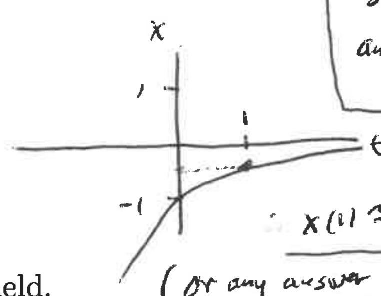
> 0 if $x<0$

< 0 if $x>0$

$\therefore$ Phase line:



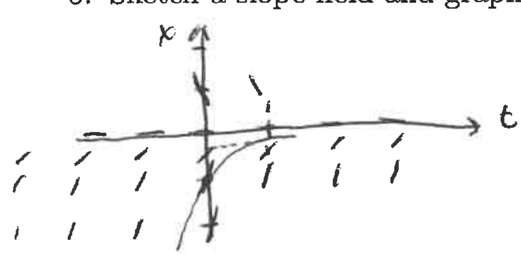
$\Rightarrow$



$\therefore$ $x(1) \approx -.4$

(or any answer between -1 and 0)

5. Sketch a slope field and graph the solution on the slope field.



If $x=-1$, slope $= (-2)(-1) = 2$

$x = -\frac{1}{2}$ slope $= (-2)(-\frac{1}{2}) = 1$

$\vdots$

$\Rightarrow \underline{x(1) \approx -.2}$ (or any answer between $-.5$ and 0)

6. Use Euler's (numerical) method with a step size of $h = .5$.

n	t_n	x_n	$f(t_n, x_n)$
0	0	-1	$(-2)\cdot(-1) = 2$
1	.5	0	$x_0 + h\,f(t_0, x_0) = -1 + .5(-2)\cdot(-1) = -1+1 = 0$
2	1	0	$x_1 + h\cdot(-2\cdot 0) = 0 + .5\cdot 0 = 0 \Rightarrow$ $\underline{x(1) \approx 0.}$