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Diff. Equations and Lin. Alg.
Math 3280, B. Peckham
Quiz 6, Fall 2020

1. (3 pts) Determine whether $\vec{w}$ is a linear combination of $\vec{v_1}$ and $\vec{v_2}$, where $\vec{w} = \begin{bmatrix}1\\0\\7\end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix}1\\3\\4\end{bmatrix}$, and $\vec{v}_2 = \begin{bmatrix}3\\3\\6\end{bmatrix}$. Justify fully starting from the system of equations you need to solve.

Solve $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{w}$, ie, $c_1\begin{bmatrix}1\\3\\4\end{bmatrix} + c_2\begin{bmatrix}3\\3\\6\end{bmatrix} = \begin{bmatrix}1\\0\\7\end{bmatrix}$

ie, $\begin{bmatrix}1 & 3\\3 & 3\\4 & 6\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1\\0\\7\end{bmatrix}$. Using row reduction: $\left[\begin{array}{cc:c}1 & 3 & 1\\3 & 3 & 0\\4 & 6 & 7\end{array}\right] \xrightarrow{R2-3R1} \left[\begin{array}{ccc}1 & 3 & 1\\0 & -6 & -3\\4 & 6 & 7\end{array}\right]$

$\xrightarrow{R3-4R1} \left[\begin{array}{ccc}1 & 3 & 1\\0 & -6 & -3\\0 & -6 & 3\end{array}\right] \xrightarrow{R3-R2} \left[\begin{array}{cc:c}1 & 3 & 1\\0 & -6 & -3\\0 & 0 & 6\end{array}\right]$. Last row $\Rightarrow$ inconsistent.
$\Rightarrow \vec{w}$ is not a lin comb. of $\vec{v}_1$ and $\vec{v}_2$.

2. Let $S = \left\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$. Show directly from the definition of linearly independent and span (not just computing a determinant) that

(a) (2pts) $S$ is a linearly independent set

Solve $c_1\begin{bmatrix}1\\-1\end{bmatrix} + c_2\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$. ie, $\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

Row reduce: $\left[\begin{array}{cc:c}1 & 1 & 0\\-1 & 1 & 0\end{array}\right] \xrightarrow{R2+R1} \left[\begin{array}{cc:c}1 & 1 & 0\\0 & 2 & 0\end{array}\right]$. Last row $\Rightarrow c_2 = 0$.
1st rows (back solving) $\Rightarrow c_1 = 0$.
$\therefore$ $S$ is a lin. indep. set.

(b) (2pts) the span of $S$ is all of $\Re^2$.

Solve $c_1\begin{bmatrix}1\\-1\end{bmatrix} + c_2\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$ ie, $\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$

Row reduce: $\left[\begin{array}{cc:c}1 & 1 & x\\-1 & 1 & y\end{array}\right] \xrightarrow{R2+R1} \left[\begin{array}{cc:c}1 & 1 & x\\0 & 2 & x+y\end{array}\right]$

Last row $\Rightarrow 2c_2 = x+y$
or $c_2 = \frac{x+y}{2}$.
1st row $\Rightarrow c_1 + c_2 = x$
$\Rightarrow c_1 = x - c_2 = x - \frac{x+y}{2}$
$= \frac{x-y}{2}$

(turn over)

Since there is a solution for $c_1, c_2$ for any $x, y \in \mathbb{R}$, then Span $S = \mathbb{R}^2$

3. (3 pts) Find a basis for the set of solutions to the following vector equation. Hint: first write down the solution to this system and put in vector form. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Matrix is already in echelon form, so just read off the solution:

Leading variables: $x_1$, $x_2$. Free variables: $x_3, x_4$. So let $x_3 = t$, $x_4 = s$,

Eqn 2 $\Rightarrow$ $x_2 + 4x_3 = 0 \Rightarrow x_2 = -4x_3 = -4t$

Eqn 1 $\Rightarrow$ $x_1 + 2x_2 + 3x_3 \Rightarrow x_1 = -2x_2 - 3x_3 = -2(-4t) - 3t$

$= 8t - 3t = 5t$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5t \\ -4t \\ t \\ s \end{bmatrix} = t \begin{bmatrix} 5 \\ -4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Basis for sln: $\left\{ \begin{bmatrix} 5 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

4. (2 pts EC) Let $W = \{(a, b, c, c) \in \Re^4 : a + b + c = 1\}$. Define a matrix $A$, and a vector $\vec{b}$ such that $W = \{\vec{x} \in \Re^4 : A\vec{x} = \vec{b}\}$. Is $W$ a subspace (of $\Re^4$)? Justify briefly without doing a full proof.

Let $\vec{x} = (a, b, c, d)$. Then $\vec{x} \in W \Leftrightarrow$

$a + b + c = 1$ and $c = d$ ie, $1a + 1b + 1c + 0d = 1$

$0a + 0b + 1c + (-1)d = 0$

ie, Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

then $\vec{x} \in W \Leftrightarrow A\vec{x} = \vec{b}$ ✓

$W$ is not a subspace since $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in W$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in W$, but $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \notin W$

$\therefore$ $W$ is not closed under vector addition.

(Similarly $2\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin W$, so $W$ is not closed under scalar multiplication. Only one of these 2 "failures" is needed.)