Math 3280
Differential Equations with Linear Algebra

Test 2

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Directions: Do all problems. Make no mistakes. SHOW ALL WORK. Closed book. Calculators may be used for algebraic computations, but not for solving differential equations or doing row reduction.

2.________/24

3.________/34 29

4.________/26+1 25

5.________/16

EC________/+6

Total________/100+6

1. Consider the differential equation $y'' + y' - 6y = 0$.

   (a) (8 pts) Find the general solution by guessing solutions of the form $y = e^{rx}$. Show your work from this guess.

   $y = e^{rx} \Rightarrow y' = re^{rx} \Rightarrow y'' = r^2e^{rx}$

   Plug in: $r^2e^{rx} + re^{rx} - 6e^{rx} = 0$

   factor $e^{rx}(r^2+r-6) = 0$

   $\Rightarrow r^2+r-6 = 0$ (e^{rx} never $= 0$)

   $\Rightarrow (r+3)(r-2) = 0 \Rightarrow r = -3, 2$

   $\therefore e^{-3x}, e^{2x}$ are 2 solutions

   $\Rightarrow c_1e^{-3x} + c_2e^{2x}$ is the general solution

   (b) (6 pts) Find one solution to the related nonhomogeneous differential equation: $y'' + y' - 6y = 3x$ by guessing a soution of the form $y = Ax + B$.

   $y = Ax + B \Rightarrow y' = A \Rightarrow y'' = 0$

   Plug in: $0 + A - 6(Ax+B) = 3x$

   ie, $A - 6B - 6Ax = 0 + 3x$

   $\therefore -6A = 3,\ A - 6B = 0 \Rightarrow A = -\frac{3}{6} = -\frac{1}{2}$

   $\Rightarrow -\frac{1}{2} - 6B = 0$

   $\Rightarrow 6B = -\frac{1}{2}$

   $\Rightarrow B = -\frac{1}{12}$

   $\therefore y = -\frac{1}{2}x - \frac{1}{12}$ is one solution

   (c) (2pts) Use (a) and (b) to determine the general solution to $y'' + y' - 6y = 3x$? If you did not answer (a) or (b), indicate how you would use those answers to determine the answer to this problem.

   $y(x) = c_1e^{-3x} + c_2e^{2x} + \left(-\frac{1}{2}x - \frac{1}{12}\right)$

2. Consider the differential equation $y'' + 4y = 32e^{2x}$. One solution to this differential equation is $y_p(x) = 3e^{2x}$. The complementary solution, to $y'' + 4y = 0$, is $y_c(x) = c_1\cos(2x) + c_2\sin(2x)$.

   (a) (2 pts) What is the general solution to $y'' + 4y = 32e^{2x}$?

   $y(x) = c_1\cos(2x) + c_2\sin(2x) + 3e^{2x}$

   (b) (6 pts) What is the solution to $y'' + 4y = 32e^{2x}$ that also satisfies the initial conditions $y(0) = 2, y'(0) = 0$?

   (a) $\Rightarrow y'(x) = -2c_1\sin(2x) + 2c_2\cos(2x) + 6e^{2x}$

   So $y(0) = c_1 + c_2 \cdot 0 + 3 \cdot 1 = 2$

   $y'(0) = 0c_1 + 2c_2 + 6 \cdot 1 = 0$

   ie, $1c_1 + 0c_2 = -1$

   $0c_1 + 2c_2 = -6$

   $\Rightarrow c_2 = -3$

   $c_1 = -1$

   $\therefore y(x) = -\cos(2x) - 3\sin 2x + 3e^{2x}$

3. (6 pts) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Define a matrix B so that $BA = \begin{bmatrix} a_{11} - 2a_{21} & a_{12} - 2a_{22} & a_{13} - 2a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$BA = \underbrace{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A}$$

4. (3 pts) If A is a 3×3 matrix, and $\det(A) = 5$, what is $\det(2A)$? Explain briefly.

$$\det(2A) = 2^3 \cdot \det(A)$$

5. (8 pts) Solve the following linear system USING GAUSSIAN ELIMINATION (row reduction to echelon or reduced echelon form). Leave your answers as exact fractions - not calculator approximations.

$$\begin{bmatrix} 2 & -1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 2 & -1 & 1 \\ 4 & 5 & 3 \end{array}\right] \to \left[\begin{array}{cc:c} 2 & -1 & 1 \\ 0 & 7 & 1 \end{array}\right] \to \left[\begin{array}{cc:c} 2 & -1 & 1 \\ 0 & 1 & \frac{1}{7} \end{array}\right]$$

Last row $\Rightarrow x_2 = \frac{1}{7}$

1^{st} row $\Rightarrow 2x_1 - \frac{1}{7} = 1 \Rightarrow x_1 = +\frac{8}{14}$

So $\vec{x} = \begin{bmatrix} +\frac{4}{7} \\ \frac{1}{7} \end{bmatrix}$

6. (a) (8 pts) Find all solutions to $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 2 & 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Write your answer in vector form.

Row reduce: $\left[\begin{array}{cccc:c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc:c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$

Interpret: x_3 is free: $x_3 = t$.

Last eqn $\Rightarrow -2x_4 = 0 \Rightarrow x_4 = 0$

2^{nd} row $\Rightarrow 1 \cdot x_2 + 2x_3 + 0x_4 = 0$

ie $x_2 = -2x_3 = -2t$

1^{st} row $\Rightarrow x_1 + 2x_2 + 0x_3 + 1 \cdot x_4 = 0 \Rightarrow x_1 = 4t + 0 = 4t$

$\therefore \vec{x} = t \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

(b) (2 pts) What is the dimension of the set of solutions to part (a)?

1

7. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2-R1} \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

(a) (6 pts) Find A^{-1} using the Gauss-Jordan (row reduction) technique.

$$\to \left[\begin{array}{cccccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right] \xrightarrow{R2+2R3} \left[\begin{array}{cccccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & \frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right] \xrightarrow{R1-2R3} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & -1 & 1 & \frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

(b) (2 pts) Check your answer by multiplying AA^{-1}.

$$A \cdot A^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{2}{3} \\ -1 & 1 & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

8. (3 pts) Write the vector equation $c_1 \begin{bmatrix}1\\2\\3\end{bmatrix} + c_2 \begin{bmatrix}3\\1\\0\end{bmatrix} + c_3 \begin{bmatrix}3\\1\\-2\end{bmatrix} = \begin{bmatrix}4\\2\\0\end{bmatrix}$ in the form $A\vec{x} = \vec{b}$.

That is, identify A, $\vec{x}$ and $\vec{b}$. Do not solve.

$$A = \begin{bmatrix}1 & 3 & 3\\2 & 1 & 1\\3 & 0 & -2\end{bmatrix}, \quad \vec{x} = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} \quad \vec{b} = \begin{bmatrix}4\\2\\0\end{bmatrix}$$

9. (6 pts) Evaluate the following determinant. Show your work.

Expand along 2^{nd} row:

$$\begin{vmatrix}1 & 3 & -1 & 2\\1 & 0 & 3 & 0\\2 & 1 & -1 & -1\\0 & 2 & -2 & 0\end{vmatrix} = (-1)\,1\begin{vmatrix}3 & -1 & 2\\1 & -1 & -1\\2 & -2 & 0\end{vmatrix} + 0\,|\cdots| - 3\begin{vmatrix}1 & 3 & 2\\2 & 1 & -1\\0 & 2 & 0\end{vmatrix} + 0\,|\cdots|$$

$$= \underbrace{(-1)\big((0+2-4) - (-4+6+0)\big)} + 0 + \underbrace{(-3)\left(0\cdot|\cdot| - 2\begin{vmatrix}1 & 2\\2 & -1\end{vmatrix} + 0|\cdots|\right)} + 0$$

$$= \cancel{0 + 4 - 6} = \underline{+2} - 4 + 6 \qquad \overline{-30}$$

$$= 4 \qquad - 30 = \underline{-26}$$

10. (6+1 pts) Give an example of a 2×2 matrix A and a vector $\vec{b}$ for which $A\vec{b} = \vec{0}$, but the entries of A are not all zero, and the entries of $\vec{b}$ are not all zero. Bonus point if no entry of A is zero and no entry of $\vec{b}$ is zero.

$$A\vec{b} = \begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

11. Let $W = \left\{\begin{bmatrix}n\\0\end{bmatrix} \in \mathbb{R}^2 : n \in \mathbb{Z}\right\}$. Recall that $\mathbb{Z}$ is the set of all integers, or whole numbers: $\{.., -2, -1, 0, 1, 2, 3, ...\}$.

(a) (4pts) Is W closed under vector addition? Explain briefly.

Yes. $\begin{bmatrix}n\\0\end{bmatrix} + \begin{bmatrix}m\\0\end{bmatrix} = \begin{bmatrix}n+m\\0\end{bmatrix}$. $\quad n, m \in \mathbb{Z} \Rightarrow n+m \in \mathbb{Z}$

(b) (4 pts) Is W closed under scalar multiblication? Explain briefly.

No. $\frac{1}{2}\cdot\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} \notin W$, but $\begin{bmatrix}1\\0\end{bmatrix} \in W$.

(c) (2pts) Is W a vector subspace of $\mathbb{R}^2$? Justfy briefly.

No. W Not closed under ~~[illegible]~~ scalar multiplication.

12. (6 pts) (True or False) $\begin{bmatrix}4\\6\\3\end{bmatrix}$ is in the span of $\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$. Justify using the definition of span.

Solve $c_1\begin{bmatrix}2\\1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}4\\6\\3\end{bmatrix}$

i.e., $\begin{bmatrix}2 & 0\\1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}4\\6\\3\end{bmatrix}$

Row reduction:

$$\left[\begin{array}{cc:c}2 & 0 & 4\\1 & 1 & 6\\0 & 1 & 3\end{array}\right] \xrightarrow{\text{swap}} \left[\begin{array}{cc:c}1 & 1 & 6\\2 & 0 & 4\\0 & 1 & 3\end{array}\right] \xrightarrow{R2-2R1} \left[\begin{array}{cc:c}1 & 1 & 6\\0 & -2 & -8\\0 & 1 & 3\end{array}\right]$$

$$\xrightarrow{R2+2R3} \left[\begin{array}{cc:c}1 & 1 & 6\\0 & 0 & -2\\0 & 1 & 3\end{array}\right] \xrightarrow{\text{swap}} \left[\begin{array}{ccc}1 & 1 & 6\\0 & 1 & 3\\0 & 0 & -2\end{array}\right]$$

Last row $\Rightarrow$ $0c_1 + 0c_2 = -2$

$\therefore$ No solution

13. (10 pts) Let $\mathcal{P}_1 = \{a + bx : a, b \in \mathbb{R}\}$. It turns out that $\mathcal{P}_1$ is a subspace of the set of all functions (with domain all real numbers and range in the real numbers). Show that the set $\{1, x+1\}$ is a basis for $\mathcal{P}_1$. Work directly from the definitions of linear independence and span.

1.) Lin. indep.: Solve $c_1 \cdot 1 + c_2 \cdot (x+1) = 0$

$\Rightarrow$ by differentiation $0c_1 + c_2 \cdot 1 = 0$

2nd eq. $\Rightarrow$ $c_2 = 0$, 1st eq becomes $1 \cdot c_1 + 0 \cdot (x+1) = 0$

i.e., $c_1 = 0$.

$\therefore$ $c_1 = 0$ and $c_2 = 0$ $\Rightarrow$ $\{1, x+1\}$ is a lin indep. set.

2.) Span: Solve $c_1 \cdot 1 + c_2 \cdot (x+1) = a + bx$ ($a+bx$ is an arbitrary function in P_1)

$\Rightarrow (c_1 + c_2) + c_2 x = a + bx$

$\Rightarrow c_1 + c_2 = a$, $c_2 = b$

$\therefore c_1 + b = a \Rightarrow c_1 = a - b$

i.e., $\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}a-b\\b\end{bmatrix}$.

$\therefore$ Any $a+bx$ is in the span of $\{1, x+1\}$.

i.e., Span$\{1, x+1\} = P_1$.

1) and 2) $\Rightarrow$ $\{1, x+1\}$ is a basis for P_1.

14. Extra credit (6 pts) Consider the vector equation $A\vec{x} = \vec{0}$, where A is an $m \times n$ matrix. Let W be the subset of all solutions to this vector equation. Assume that $\vec{y}$ is a function in W. Show that $c\vec{y}$ is also in W, where c is any real number.

i.e., vector

$\vec{y} \in W \Rightarrow A\vec{y} = \vec{0}$.

$\therefore A(c\vec{y}) = c(A\vec{y}) = c \cdot \vec{0} = \vec{0}$

$\therefore c\vec{y} \in W$.