Math 385 Supplement: the method of undetermined coefficients

It is relatively easy to implement the method of undetermined coefficients as presented in the textbook, but not easy to understand why it works. This handout explains the idea behind the method, and is aimed at those students who want to understand mathematics at a deeper level.

Review of homogeneous equations

The homogeneous constant coefficient linear equation $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$ has the characteristic polynomial $a_n r^n + \cdots + a_1 r + a_0 = 0$. From the roots $r_1, \ldots, r_n$ of the polynomial we can construct the solutions $y_1, \ldots, y_n$, such as $y_1 = e^{r_1 x}$. We can also rewrite the equation in a weird-looking but useful way, using the symbol $D = \frac{d}{dx}$.

Examples:

equation: $y'' - 5y' + 6y = 0$.
polynomial: $r^2 - 5r + 6 = 0$. (factored): $(r-2)(r-3) = 0$. roots: $2, 3$
weird-looking form of equation: $(D-2)(D-3)y = 0$ or $(D^2 - 5D + 6)y = 0$.
linearly independent solutions: $y_1 = e^{2x}, y_2 = e^{3x}$.
general solution: $y = c_1 e^{2x} + c_2 e^{3x}$.

equation: $y'' + 10y' + 25y = 0$.
polynomial: $r^2 + 10r + 25 = 0$. (factored): $(r+5)^2 = 0$. roots: $-5, -5$
weird-looking form of equation: $(D+5)^2 y = 0$ or $(D^2 + 10D + 25)y = 0$.
linearly independent solutions: $y_1 = e^{-5x}, y_2 = xe^{-5x}$.
general solution: $y = c_1 e^{-5x} + c_2 x e^{-5x}$.

equation: $y'' - 4y' + 8y = 0$.
polynomial: $r^2 - 4r + 8 = 0$. (factored): $(r - 2 - 2i)(r - 2 + 2i) = 0$.
roots: $2 + 2i, 2 - 2i$
w-l. f. of equation: $(D - 2 - 2i)(D - 2 + 2i)y = 0$ or $(D^2 - 4D + 8)y = 0$.
linearly independent solutions: $y_1 = e^{2x}\cos 2x, y_2 = e^{2x}\sin 2x$.
general solution: $y = e^{2x}(c_1 \cos 2x + c_2 \sin 2x)$.

equation: (already in a weird-looking form) $(D^2+1)^2(D-1)^3 y = 0$.
polynomial: $(r^2+1)^2(r-1)^3 y = 0$. roots: $i, i, -i, -i, 1, 1, 1$.
general solution: $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x + (c_5 + c_6 x + c_7 x^2)e^x$.

Annihilators

If f is a function, then the **annihilator** of f is a "differential operator"

$$\tilde{L} = a_n D^n + \cdots + a_n D + a_0$$

with the property that $\tilde{L} f = 0$. By reversing the thought process we use for homogeneous equations, we can easily find the annihilator for lots of functions:

Examples

function: $f(x) = e^x$
annihilator: $\tilde{L} = (D-1)$
check: $(D-1)f = De^x - e^x = \frac{d}{dx}e^x - e^x = 0$.

function: $f(x) = x^2 e^{-7x}$
annihilator: $\tilde{L} = (D-7)^3$

function: $f(x) = e^{-x}\cos 4x$.
corresponding "root of polynomial": $-1 \pm 4i$.
polynomial with this root: $(r+1+4i)(r+1-4i) = r^2 + 2r + 5$.
annihilator: $\tilde{L} = D^2 + 2D + 5$.

function: $f(x) = x^k e^{\alpha x}$.
corresponding "root of polynomial": α (with multiplicity $k+1$))
corresponding polynomial: $(r-\alpha)^{k+1}$.
annihilator: $\tilde{L} = (D-\alpha)^{k+1}$.

function: $f(x) = x^k e^{ax}\sin bx$.
corresponding "root of polynomial": $a \pm bi$ (with multiplicity $(k+1)$
polynomial with this root: $((r-a-bi)(r-a+bi))^{k+1} = (r^2-2ar+(a^2+b^2))^{k+1}$.
annihilator: $\tilde{L} = (D^2 - 2aD + (a^2+b^2))^{k+1}$.

function: $e^x + \cos 2x$.

To do this one, note that if $\tilde{L}_1$ annihilates f_1, and $\tilde{L}_2$ annihilates f_2, then $\tilde{L}_1\tilde{L}_2$ annihilates $f_1 + f_2$. This is true because

$$\tilde{L}_1\tilde{L}_2(f_1+f_2) = \tilde{L}_1\tilde{L}_2 f_1 + \tilde{L}_1\tilde{L}_2 f_2 = \tilde{L}_1\tilde{L}_2 f_1 + 0 = \tilde{L}_2\tilde{L}_1 f_1 = 0.$$

So the annihilator is $(D-1)(D^2+4)$.

(You can also check directly that this works.)

The Method of Undetermined Coefficients

To find a particular solution of the equation $Ly = f$, we want to guess the "correct form" of y_p and then substitute and adjust the coefficients to get a solution. How do we determine the correct form? Consider the following...

Let $\tilde{L}$ denote the annihilator of f. By definition $\tilde{L}f = 0$. If y is a solution of $Ly = f$, then

$$0 = \tilde{L}f = \tilde{L}(Ly).$$

That is, y is also a solution of the homogeneous equation $\tilde{L}Ly = 0$. We can call this the "annihilator equation" associated with the original equation $Ly = f$. We know how to find all solutions of the annihilator equation (since it is a constant coefficient homogeneous DE), and this will tell us the correct form for our guess.

So we have the following procedure for determining the particular solution y_p of $Ly = f$:

Step 1. Find the annihilator for f. Call it $\tilde{L}$.

Step 2. Let y_a denote the general solution of the annihilator equation $\tilde{L}Ly_a = 0$. Find y_a

Step 3 Find the general solution y_c of the complementary homogeneous equation $Ly_c = 0$.

Step 4 Our guess y_p will consist of the general solution y_a of the annihilator equation except with all terms from the complementary solution y_c omitted.

(Finally, of course, the general solution of $Ly = f$ is $y_c + y_p$.)

Remark

Suppose we have an equation with a complicated "right-hand side" consisting of more than one term, which we can write symbolically as

$$Ly = f_1 + f_2.$$

One way to deal with this is to consider each of the two terms on the right-hand side separately. In other words, suppose we can find functions y_1 and y_2 that solve

$$Ly_1 = f_1, \qquad\qquad Ly_2 = f_2.$$

Then let $y = y_1 + y_2$. We claim that y solves our original problem. This is easy to check, since

$$Ly = L(y_1 + y_2) = L(y_1) + L(y_2) = f_1 + f_2$$

by linearity of the equation.

For example, equation 4 below is $y^{(5)} - y''' = e^x + 2x^2 + 5$. We could break the right-hand side up into two pieces: $f_1 = e^x$, $f_2 = 2x^2 + 5$. Then by going through my procedure, we would find

$$(\text{guess for } f_1): \ y_{p,1} = Axe^x, \qquad (\text{guess for } f_2): \ y_{p,2} = Bx^3 + Cx^4 + Dx^5$$

and so

$$(\text{guess for whole problem}) \ y_p = Axe^x + Bx^3 + Cx^4 + Dx^5.$$

Examples

equation 1: $y'' + 4y' + 4y = 2e^x$, equivalently $(D+2)^2 y = 2e^x$, or $Ly = 2e^x$ for $L = (D^2 + 4D + 4)$.

Step 1: annihilator of right-hand side: $\tilde{L} = (D - 1)$

Step 2: If we apply the operator $\tilde{L} = D - 1$ to both sides of the equation $y'' + 4y' + 4y = 2e^x$, we get

$$(D-1)(D^2 + 4D + 4)y = (D-1)(2e^x) = \frac{d}{dx}(2e^x) - 1 \cdot 2e^x = 0.$$

So the annihilator equation is $(D-1)(D+2)^2 y_a = 0$.

We could have found this by just using the general expression for the annihilator equation: $\tilde{L}Ly_a = 0$.

The general solution of the annihilator equation is $y_a = c_1 e^x + (c_2 + c_3 x)e^{-2x}$.

Step 3: general solution of complementary equation is $y_c = (c_2 + c_3 x)e^{-2x}$.

Step 4: So we guess $y_p = c_1 e^x$.

If we substitute $y_p = c_1 e^x$ into the equation then we find

$$Ly_p = y_p'' + 4y_p' + 4y_p = 9c_1 e^x.$$

So to solve $Ly_p = 2e^x$ we have to take $c_1 = 2/9$. Then we get the solution $y = \frac{2}{9}e^x$.

What happens if instead we substitute $y = y_a = c_1 e^x + (c_2 + c_3 x)e^{-2x}$ into the equation? If we do this and do all the calculations, we'll find that

$$Ly = y'' + 4y' + 4y = 9c_1 e^x.$$

So to solve $Ly = 2e^x$ we have to take $c_1 = 2/9$. But there are no conditions on c_2, c_3, so they can be anything. So this gives us the general solution of the inhomogeneous equation:

$$y = \frac{2}{9}e^x + (c_2 + c_3 x)e^{-2x}.$$

So we see: if we substitute y_p into the equation and solve for the undetermined coefficients we get a particular solution. If we substitute y_a into the equation and try to solve, we get the general solution. However, in practice it is much better to substitute y_p, because the algebra is much simpler.

equation 2: $y'' + 4y' + 4y = e^{-2x}$, equivalently $(D+2)^2 y = e^{2x}$, or $Ly = e^{-2x}$ for $L = (D^2 + 4D + 4)$.

Step 1: annihilator of right-hand side: $\tilde{L} = (D+2)$

Step 2: If we apply the operator $\tilde{L} = D + 2$ to both sides of the equation $y'' + 4y' + 4y = e^{-2x}$, we get

$$(D+2)(D+2)^2 y = (D+2)(e^{-2x}) = \frac{d}{dx}(e^{-2x}) + 2 \cdot e^{-2x} = 0.$$

So the annihilator equation is $(D-2)^3 y_a = 0$.

Again we could have found this just from the formula $\tilde{L}Ly_a = 0$.

The general solution of the annihilator equation is $y_a = (c_1 + c_2 x + c_3 x^2)e^{2x}$.

Step 3: general solution of complementary equation is $y_c = (c_2 + c_3 x)e^{2x}$.

Step 4: So we guess $y_p = c_1 x^2 e^{2x}$.

As above: if we substitute y_p into the equation and solve for the undetermined coefficients we get a particular solution. If we substitute y_a into the equation and try to solve, we get the general solution. However, in practice it is much better to substitute y_p, because the algebra is much simpler.

equation 3: $y''' + 9y' = x^2 \sin 3x$, equivalently $(D+3i)(D-3i)Dy = x^2 \sin 3x$, or $Ly = x^2 \sin 3x$ for $L = (D^2+9)D$.

Step 1: annihilator of right-hand side: $\tilde{L} = (D^2+9)^3$

Step 2: annihilator equation is $\tilde{L}Ly_a = 0$, i.e. $(D^2+9)^4 Dy_a = 0$, i.e. $(D-2)^4 Dy_a = 0$. General solution is $y_a = (A + Bx + Cx^2 + Dx^3)\cos 3x + (E + Fx + Gx^2 + Hx^3)\sin x + K$

Step 3: general solution of complementary equation is $y_c = A\cos 3x + B \sin xC$.

Step 4: So we guess $y_p = (Bx + Cx^2 + Dx^3)\cos 3x + (Fx + Gx^2 + Hx^3)\sin x$.

equation 4: $y^{(5)} - y''' = e^x + 2x^2 + 5$. We can write the left-hand side as $(D^5 - D^3)y = D^3(D-1)(D+1)y$.

Step 1: annihilator of right-hand side: $\tilde{L} = (D-1)D^3$.

Step 2: annihilator equation is $\tilde{L}Ly_a = 0$, i.e. $D^6(D-1)^2(D+1)y_a = 0$. General solution is $y_a = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + (G + Hx)e^x + Ie^{-x}$.

Step 3: general solution of complementary equation is $y_c = A + Bx + Cx^2 + De^x + Ee^{-x}$

Step 4: So we guess $y_p = Ax^3 + Bx^4 + Cx^5 + Dxe^x$.

equation 5: $y'' - 6y' + 13y = xe^{3x} \sin 2x$. We can write the left-hand side as $(D-3+2i)(D-3-2i)y$ or as $(D^2 - 6D + 13)y$.

Step 1: annihilator of right-hand side: the function $xe^{3x}\sin 2x$ corresponds to the root $3 \pm 2i$, repeated twice (because of the factor of x). So $\tilde{L} = ((D-3+2i)(D-3-2i))^2 = (D^2 - 6D + 13)^2$.

Step 2: annihilator equation is $\tilde{L}Ly_a = 0$, i.e. $(D^2 - 6D + 13)^3 y_a = 0$. General solution is $y_a = (A + Bx + Cx^2)e^{3x}\cos 2x + (D + Ex + Fx^2)e^{3x}\sin 2x$.

Step 3: general solution of complementary equation is $y_c = Ae^{3x}\cos 2x + Be^{3x}\sin 2x$.

Step 4: So we guess $y_p = (Bx + Cx^2)e^{3x}\cos 2x + (Ex + Fx^2)e^{3x}\sin 2x$.

equation 6: $y^{(4)} - 2y'' + y = x^2 \cos x$. We can write the left-hand side as $(D^2 - 1)^2 y$ or as $(D-1)^2(D+1)^2 y$.

Step 1: annihilator of right-hand side: the function $x^2 \cos x$ corresponds to the root i, repeated three times (because of the factor of x^2). So $\tilde{L} = ((D-i)(D+i))^3 = (D^2+1)^3$.

Step 2: annihilator equation is $\tilde{L}Ly_a = 0$, i.e. $((D-i)(D+i))^3(D-1)^2(D+1)^2 y_a = 0$. General solution is $y_a = (A + Bx + Cx^2)\cos x + (D + Ex + Fx^2)\sin x + (G + Hx)e^x + (I + Jx)e^{-x}$.

Step 3: general solution of complementary equation is $y_c = (A + Bx)e^x + (C + Dx)e^{-x}$.

Step 4: So we guess $y_p = (A + Bx + Cx^2)\cos x + (D + Ex + Fx^2)\sin x$.