Math 3280
Differential Equations with Linear Algebra

Test 2
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Name____A.K.____________________________

Directions: Do all problems. Make no mistakes. SHOW ALL WORK. Closed book. Calculators may be used for algebraic computations, but not for solving differential equations or doing row reduction.

2.________/24

3.________/34

4.________/22

5.________/20

EC________/+6

Total________/100+6

1. Consider the differential equation $y'' - 4y' + 4y = 0$.

   (a) (4 pts) Verify that $f(x) = xe^{2x}$ is a solution to this differential equation.

$f'(x) = x\,2e^{2x} + e^{2x}$ $\Rightarrow f'' - 4f' + 4f$

$f''(x) = x\cdot 4e^{2x} + 2e^{2x} + 2e^{2x}$ $= (4xe^{2x} + 4e^{2x}) - 4(2xe^{2x} + e^{2x}) + 4xe^{2x}$

$= 4xe^{2x} + 4e^{2x}$ $= 0xe^{2x} + (4-4)e^{2x} = 0$ ✓

   (b) (4 pts) Use the information from (a) to help find the general solution to the differential equation.

Try $y = e^{rx}$: $\dots \Rightarrow r^2 - 4r + 4 = 0 \Leftrightarrow (r-2)^2 = 0 \Rightarrow e^{2x}$ is a sln.

$\therefore\ y(x) = c_1 e^{2x} + c_2 xe^{2x}$

2. (6 pts) Use the fact that e^x is one solution to help find the general solution (all solutions) to $y'' - y' - 2y = -2e^x$.

For y_c, try $y = e^{rx} \Rightarrow y' = re^{rx}, y'' = r^2e^{rx}$ $\quad \therefore\ y(x) = c_1e^{-x} + c_2e^{2x} + e^x$

$\dots \Rightarrow r^2 - r - 2 = 0 \Rightarrow (r-2)(r+1) = 0$

$\Rightarrow r = 2, -1 \quad \Rightarrow y_c(x) = c_1e^{-x} + c_2e^{2x}$

3. (6 pts) The general solution to $y'' - y = -x^2$ is $c_1e^x + c_2e^{-x} + x^2 + 2$. Find the solution to this differential equation which also satisfies the initial conditions: $y(0) = 1, y'(0) = -1$.

$y(x) = c_1e^x + c_2e^{-x} + x^2 + 2 \Rightarrow y'(x) = c_1e^x - c_2e^{-x} + 2x$

$\Rightarrow y(0) = c_1 + c_2 + 2 = 1 \qquad y'(0) = c_1 - c_2 + 0 = -1$

$\therefore\ c_1 + c_2 = -1$
$c_1 - c_2 = -1$

$\left[\begin{array}{cc:c} 1 & 1 & -1 \\ 1 & -1 & -1 \end{array}\right] \to \left[\begin{array}{cc:c} 1 & 1 & -1 \\ 0 & -2 & 0 \end{array}\right] \Rightarrow c_2 = 0, c_1 = -1$

$\Rightarrow y(x) = -e^x + x^2 + 2$

4. (4 pts) Write the vector equation $c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$ in the form $A\mathbf{x} = \mathbf{b}$. (Do not solve.)

$$\begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & 1 \\ 3 & 0 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

5. (6 pts) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{23} \end{bmatrix}$. Define a matrix B so that $BA = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+3a_{11} & a_{22}+3a_{12} & a_{23}+3a_{13} \\ a_{31} & a_{32} & a_{23} \end{bmatrix}$

Let $B = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. CK $BA = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{11}+a_{21} & 3a_{12}+a_{22} & 3a_{13}+a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ ✓

6. (8 pts) Solve the following linear system USING GAUSSIAN ELIMINATION (row reduction to echelon or reduced echelon form). Leave your answers as exact fractions - not calculator approximations.

$$\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 5 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 5 & 3 \\ 2 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 5 & 3 \\ 0 & -9 & -5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 5 & 3 \\ 0 & 1 & \frac{5}{9} \end{array}\right]$$

$$\Rightarrow x_2 = \frac{5}{9} \Rightarrow x_1 = 3 - 5x_2 = 3 - 5\left(\frac{5}{9}\right) = \frac{27-25}{9} = \frac{2}{9}$$

7. (a) (8 pts) Find all solutions to $\begin{bmatrix} 0 & 1 & 2 & 0 \\ 2 & 4 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Write your answer in vector form.

$$\to \left[\begin{array}{cccc|c} 2 & 4 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right] \Rightarrow x_4 = t,\ x_3 = s,$$

$x_2 + 2s = 0 \Rightarrow x_2 = -2s$

$x_1 + 2x_2 = 0 \Rightarrow x_1 - 4s = 0$

$\Rightarrow x_1 = 4s.$

$$\vec{x} = t\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -4 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Span →

(b) (2 pts) What is the dimension of the set of solutions to part (a)?

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8. Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(a) (6 pts) Find A^{-1} using the Gauss-Jordan (row reduction) technique.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

(b) (2 pts) Check your answer by multiplying AA^{-1}.

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ✓

9. (6 pts) Evaluate the following determinant. Show your work.

$$\begin{vmatrix} 1 & 3 & -1 & 2 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -2 & 0 \end{vmatrix} = 1\begin{vmatrix} 0 & 3 & 0 \\ 1 & -1 & -1 \\ 2 & -2 & 0 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 & 2 \\ 1 & -1 & -1 \\ 2 & -2 & 0 \end{vmatrix} + 0 - 0$$

$$= 1\cdot(-3)\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} - 1\left(2\begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix}\right)$$

$$= (-3)\cdot 2 - 1\left(2\cdot 3 + 2(-5)\right)$$

$$= -6 - (6 - 10) = \underline{\underline{-2}}$$

10. (6 pts) Give an example of two 2×2 matrices A and B for which $AB \neq BA$.

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

11. TRUE-FALSE. Justify your answer briefly. A formal proof is not required.

(a) (5pts) The set of all solutions to $y'' + 3y' - y = e^x$ is a vector subspace of the set of all functions defined on $\Re$.

False - non homogeneous. The 0-function is not a s/n.

$\therefore$ not closed under scalar mult. (by 0).

(b) (5 pts) The set of all solutions to

$$\begin{bmatrix} 2 & 1 & 4 \\ 3 & 3 & 2 \\ 6 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is a vector subspace of $\Re^3$.

True - homogeneous. (Closed under addition + Scalar mult.)

12. (4 pts) (True or False) $\begin{bmatrix}4\\5\\3\end{bmatrix}$ is in the span of $\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$. Justify using the definition of span.

Solve: $c_1\begin{bmatrix}2\\1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}4\\5\\3\end{bmatrix}$ $\left[\begin{array}{cc|c}2 & 0 & 4\\1 & 1 & 5\\0 & 1 & 3\end{array}\right] \to \begin{bmatrix}1 & 1 & 5\\0 & 1 & 3\\2 & 0 & 4\end{bmatrix} \to \left[\begin{array}{cc|c}1 & 1 & 5\\0 & 1 & 3\\0 & -2 & -6\end{array}\right]$

$\to \begin{bmatrix}1 & 1 & 5\\0 & 1 & 3\\0 & 0 & 0\end{bmatrix} \Rightarrow c_2 = 3,\quad c_1 + c_2 = 5 \Rightarrow c_1 = 2$

$\therefore$ yes

13. (8 pts) Show that the following set of two functions, $\{1, x\}$ is linearly independent. Work directly from the definition of linear independence/dependence. (For example, do not just compute a Wronskian determinant without indicating why it is being computed.)

Assume $c_1 \cdot 1 + c_2 \cdot x = 0$ } plug back in 1st eqn. $c_1 + 0 \cdot x = 0$

Diff both sides: $c_2 = 0$ } $\Rightarrow c_1 = 0$

Since c_1, c_2 both $= 0$, then $\{1, x\}$ is an indep. set!

14. (8 pts) Let $\mathcal{F}$ be the vector space of all functions with the real numbers for both domain and range. Consider the subspace $\mathcal{S}$ of $\mathcal{F}$ defined by $\mathcal{S} = \{a + bx + c(x+1) : a, b, c \in \Re\}$. Find a basis for $\mathcal{S}$. (You need not prove $\mathcal{S}$ is a subspace.) Justify briefly. (OK to use #13)

$\mathcal{B} = \{1, x\}$. $\quad a + bx + c(x+1) = (a+c) + (b+c)x$

$\therefore$ any $f \in S \Rightarrow f = c_1 \cdot 1 + c_2 \cdot x$ where $c_1 = a+c,\ c_2 = b+c$.

So $\text{span}\{1, x\} = S$. By #13, the set is lin. indep.

$\therefore \mathcal{B}$ is a basis for S.

15. Extra credit (6 pts) Consider the equation $A\vec{x} = \vec{0}$, where A is an $m \times n$ matrix. Assume that $\vec{y}$ and $\vec{z}$ are both solutions to this equation. Show that $\vec{y} + \vec{z}$ is also a solution to the same equation.

$\vec{y}, \vec{z}$ slns to $A\vec{x} = \vec{0} \Rightarrow A\vec{y} = \vec{0}$ and $A\vec{z} = \vec{0}$

$\therefore A(\vec{y} + \vec{z}) = A\vec{y} + A\vec{z} = \vec{0} + \vec{0} = \vec{0}$

i.e., $\vec{y} + \vec{z}$ is a sln to $A\vec{x} = \vec{0}$ //