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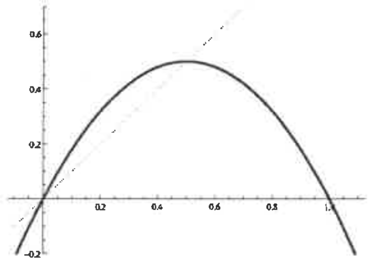
Math 5260: Dynamical Systems
Midterm 1, Fri. Oct. 18, 2013
Prof. Bruce Peckham

Directions: Do all problems. Total is 50 points.

1. Consider the dynamical system defined by $x_{n+1} = f(x_n)$ where $f(x) = 2x(1-x)$. The graph of f is below.



(a) (4pts) Determine all fixed points and their stabilities (attracting, repelling, or linearly neutral). Justify your answers with calculations.

$2x(1-x) = x \iff 2x - 2x^2 - x = 0$
$x - 2x^2 = 0$
$x(1-2x) = 0$
$x = 0, \frac{1}{2}$ are the 2 f.p.'s

$f'(x) = 2-4x \to f'(0) = 2$, $f'(\frac{1}{2}) = 0$ $\Rightarrow$ $x=0$ is repelling, $x = \frac{1}{2}$ is (super) attracting

(b) (2pts) Determine the set of initial conditions for which the corresponding orbit stays bounded. Justify briefly either on the sketch above or with an analytic argument.

$x_0 \in [0,1] \iff \{x_n\}$ is bdd

Justification: $x \in [0,1] \Rightarrow f(x) \in [0, \frac{1}{2}] \subseteq [0,1]$, so $[0,1]$ is invariant.
$x \notin [0,1] \Rightarrow f(x) < 0 \Rightarrow f^n(x) \searrow -\infty$.

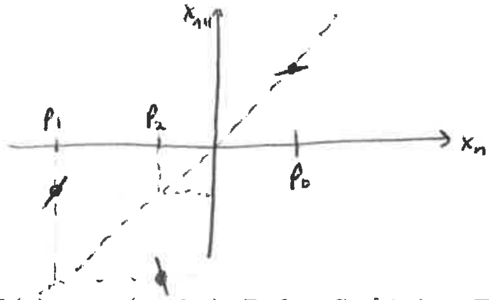
(c) (2pts) Describe the fate of all bounded orbits.

$x_0 \in (0,1) \Rightarrow x_n \to \frac{1}{2}$

$x_0 = 0 \Rightarrow x_n = 0$

$x_0 = 1 \Rightarrow x_n = 0$ for all $n \geq 1$

2. (4pts) Sketch the graph of a continuous function which has an attracting fixed point and a repelling period-2 orbit.



Any function that "connects the dots" with slopes so that: $|f'(p_0)| < 1$, $|f'(p_1) \cdot f'(p_2)| > 1$

3. (3pts) Let $D(x) = 2x \pmod 1$. Define $S : [0, 1) \to \Sigma$ (Σ is the space of sequences of 0's and 1's) by $S(x) = (s_0 s_1 s_2 \ldots)$ where

$$s_j = \begin{cases} 0 & \text{if } D^j(x) \in [0, .5) \\ 1 & \text{if } D^j(x) \in [.5, 1) \end{cases}$$

What is $S(\frac{2}{3})$? Explain briefly.

$\frac{2}{3} \in [.5, 1) \Rightarrow s_0 = 1$

$D(\frac{2}{3}) = \frac{1}{3} \in [0, .5) \Rightarrow s_1 = 0$ $\Rightarrow S(\frac{2}{3}) = (1\,0\,\overline{1\,0})$

$D(\frac{1}{3}) = \frac{2}{3} \Rightarrow s_2 = 1$

$\vdots$

4. (3 pts) If a continuous map of the unit interval has a (prime) period 30 orbit, must it have an orbit of (prime) period 28? Explain briefly?

$30 = 2 \cdot 15$ so having period-30 $\Rightarrow$ having period 28

$28 = 2^2 \cdot 7$

(lower power of 2 as a factor)

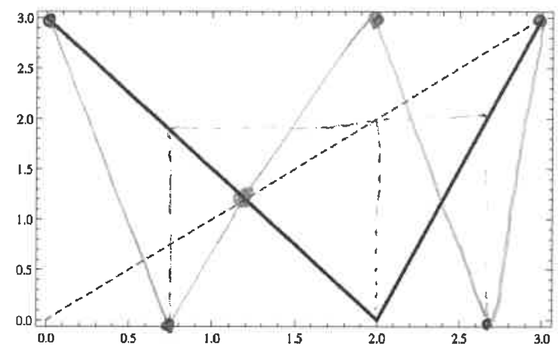
5. (3pts) How many prime period-11 *orbits* are there for the map $x^2 - 2$? Explain briefly how you determined your answer.

Per 11 pts $= 2^{11} = 2048$

$\Rightarrow$ # prime period-11 pts $= 2048 - 2$ (2 fixed pts)

$\Rightarrow \dfrac{2048 - 2}{11}$ period-11 orbits

ie, $\dfrac{2046}{11} = \underline{186}$



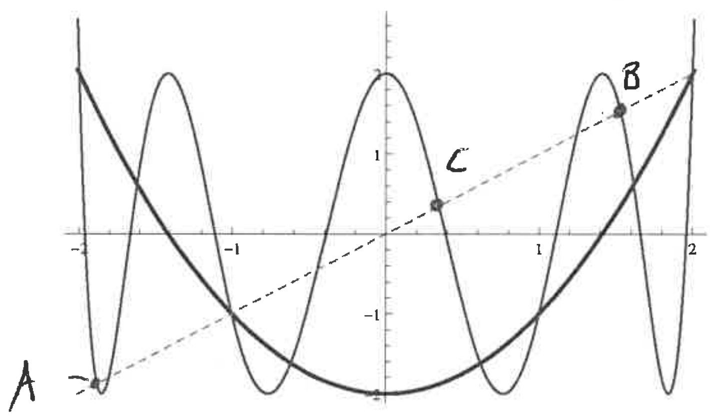
Hint: Find "pre images" of max and min (more generally, critical pts)

6. (4pts) Given the above graph of f, sketch the graph of f^2 (that is, $f \circ f$) on the same diagram.

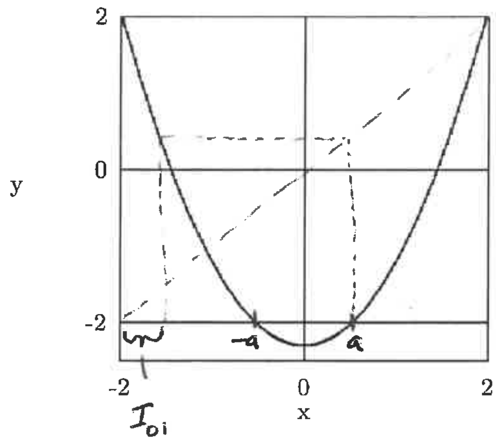
7. (3pts) List the three properties necessary for a function $f : X \to X$ to be chaotic. (No formal definitions necessary; just list the names of the three properties.)

   1. Periodic pts of f are dense in X
   2. f is transitive on X
   3. f has sensitivity to initial conditions (SIC)

8. (3pts) The graphs of Q_{-2} and Q^3_{-2} are given below for $Q_{-2}(x) = x^2 - 2$. Label on the diagonal any three points A, B, C which are on the same prime period-3 orbit and for which $A \to B \to C \to A$ under iteration of the map Q_{-2}.



(One of 6 possible answers)



9. (3pts) Let $f(x)$ be defined by the graph above. Let $I_0 = [-2, -a], I_1 = [a, 2]$, where a is the x value of the right-hand intersection of the graph with the line $y = -2$. Define $I_{ij} = \{x \in [-2, 2] | x \in I_i, f(x) \in I_j\}$. Carefully sketch and label the set I_{01}.

10. (4pts) Let $t = (000\overline{0})$ and $s = (111\overline{1})$ be points in the sequence space Σ. Find a new point $z \in \Sigma$ and an integer N which satisfy $d(z, s) < 0.1$ and $d(\sigma^N(z), t) < 0.1$.

1. Select appropriate "N": $\frac{1}{2^4} = \frac{1}{16} < .1$

$\therefore$ If we match $z_0 \cdots z_4$ then $d(z,s) \leq \sum_{i=5}^{\infty} \frac{1}{2^i} = \frac{1}{2^4} < .1$

2. Let $z = (11111 00\overline{0})$

let $N = 5$. Then $d(z,s) < .1$ as above

and $\sigma^N(z) = (0\overline{0}) \Rightarrow d(\sigma^N(z), t) = 0 < .1$

11. (3pts) List all prime period-3 *orbits* for $\sigma : \Sigma \to \Sigma$. (As in the text, Σ is the space of sequences of 0's and 1's, and σ is the shift map which drops the first term in the sequence. Indicate which the order in which points map to each other on these orbits.)

2 orbits:

$(\overline{100}) \mapsto \overline{001} \longmapsto \overline{010}$

$\overline{110} \longmapsto \overline{101} \longmapsto \overline{011}$

12. Consider the family of maps given by: $f_c(x) = x^2 + cx$.

    (a) (3pts) Find all fixed points for all maps in this family.

    (b) (3pts) Determine all intervals in the parameter c for which f_c has an attracting fixed point.

    (c) (3pts) Sketch a bifurcation diagram (in the phase $\times$ parameter space) including all fixed points with solid lines representing those that are attracting and dashed lines representing those that are repelling.

a) Fixed pts: $x^2 + cx = x$

$\Leftrightarrow x^2 + (c-1)x = 0$

$\Leftrightarrow x(x + (c-1)) = 0$

$\Rightarrow \underline{x = 0}$ or $\underline{x = 1-c}$

b) Stability $f_c'(x) = 2x + c$

$\Rightarrow f_c'(0) = c$ so 0 is attr. if $-1 < f'(0) < 1$

i.e., if $-1 < c < 1$

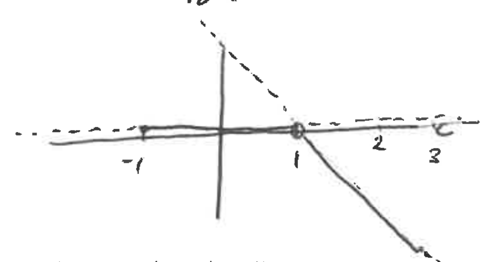
$f_c'(1-c) = 2(1-c) + c = 2 - c$ so $1-c$ is attr. if

$-1 < 2 - c < 1$

i.e., $-3 < -c < -1$

i.e., $1 < c < 3$

c)



13. Extra Credit: (3pts). Consider the space Σ consisting of sequences 0's and 1's with the "usual" metric defined by $d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$. Prove that if $s_i = t_i$ for $i = 0, 1, ..., n$, then $d(s,t) \leq \frac{1}{2^n}$.

$$d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} = \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i}$$ Since $s_i = t_i$ for $i = 0, 1, \dots, n$

$$\leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n+1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)$$

$$= \frac{1}{2^{n+1}} \cdot 2 = \frac{1}{2^n}$$