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Patterns on Triply Periodic Polyhedra

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Outline

- Triply periodic polyhedra
- Review of hyperbolic geometry and regular tessellations
- Relation between periodic polyhedra and regular tessellations
- A fish pattern on the $\{4,6 \mid 4\}$ polyhedron
- A fish pattern on the $\{6,4 \mid 4\}$ polyhedron
- A fish pattern on the $\{6,6 \mid 3\}$ polyhedron
- A pattern of butterflies on a $\{3,8\}$ polyhedron
- A pattern of butterflies on another $\{3,8\}$ polyhedron
- A pattern of fish on a $\{3,8\}$ polyhedron
- Future research

Triply Periodic Polyhedra

- A *triply periodic polyhedron* is a (non-closed) polyhedron that repeats in three different directions in Euclidean 3-space.
- We consider *semi-regular* polyhedra composed of copies of a regular p-sided polygon, or p-gon, and which are *uniform* — there is a symmetry of the polyhedron that takes any vertex to any other vertex. If q is the number of p-gons at a vertex, we can denote the polyhedron by the Schläfli symbol $\{p, q\}$.
- We start by discussing more regular triply periodic polyhedra that are "flag-transitive" — there is a symmetry of the polyhedron that takes any vertex, edge containing that vertex, and face containing that edge to any other such (vertex, edge, face) combination. These are natural analogs of the Platonic solids and were called *regular skew polyhedra* by H.S.M. Coxeter. In 1926, with John Petrie, Coxeter proved there are exactly three such polyhedra, which he denoted $\{4, 6 \mid 4\}$, $\{6, 4 \mid 4\}$, and $\{6, 6 \mid 3\}$, where $\{p, q \mid r\}$ is the extended Schläfli symbol that denotes a polyhedron made up of p-gons meeting q at a vertex, and with regular r-sided holes.

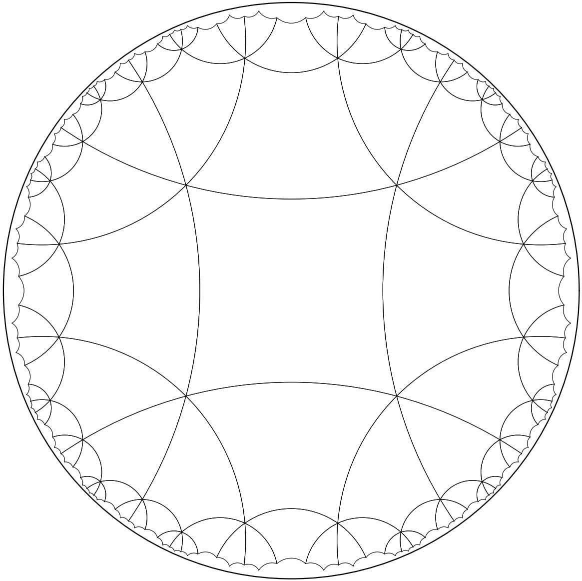
Hyperbolic Geometry and Regular Tessellations

- In 1901, David Hilbert proved that, unlike the sphere, there was no isometric (distance-preserving) embedding of the hyperbolic plane into ordinary Euclidean 3-space.
- Thus we must use *models* of hyperbolic geometry in which Euclidean objects have hyperbolic meaning, and which must distort distance.
- One such model is the *Poincaré disk model*. The hyperbolic points in this model are represented by interior point of a Euclidean circle — the *bounding circle*. The hyperbolic lines are represented by (internal) circular arcs that are perpendicular to the bounding circle (with diameters as special cases).
- This model is appealing to artists since (1) angles have their Euclidean measure (i.e. it is conformal), so that motifs of a repeating pattern retain their approximate shape as they get smaller toward the edge of the bounding circle, and (2) it can display an entire pattern in a finite area.

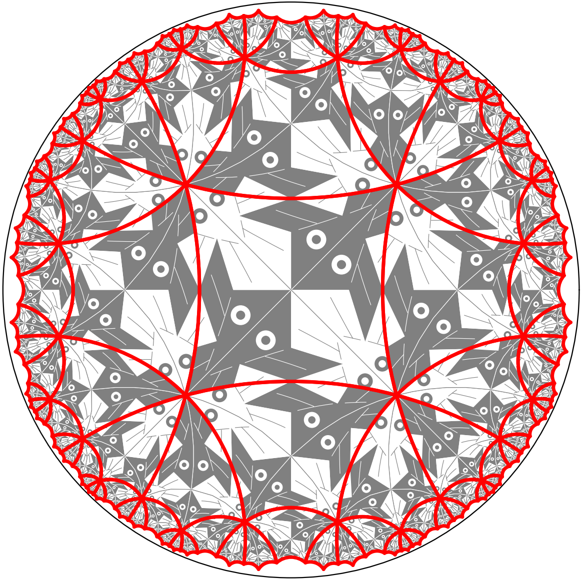
Repeating Patterns and Regular Tessellations

- A *repeating pattern* in any of the 3 "classical geometries" (Euclidean, spherical, and hyperbolic geometry) is composed of congruent copies of a basic subpattern or *motif*.
- The *regular tessellation*, denoted by the Schläfli symbol $\{p, q\}$, is an important kind of repeating pattern composed of regular p-sided polygons meeting q at a vertex.
- If $(p-2)(q-2) < 4$, $\{p, q\}$ is a spherical tessellation (assuming $p > 2$ and $q > 2$ to avoid special cases).
- If $(p-2)(q-2) = 4$, $\{p, q\}$ is a Euclidean tessellation.
- If $(p-2)(q-2) > 4$, $\{p, q\}$ is a hyperbolic tessellation. The next slide shows the $\{6, 4\}$ tessellation.
- Escher based his 4 "Circle Limit" patterns, and many of his spherical and Euclidean patterns on regular tessellations.

The Regular Tessellation $\{4, 6\}$



The tessellation $\{4,6\}$ superimposed on the pattern of angular fish of the title slide pattern



Relation between periodic polyhedra and regular tessellations — a 2-Step Process

- (1) Some triply periodic polyhedra approximate triply periodic minimal surfaces (TPMS's).

  As a bonus, some triply periodic polyhedra contain embedded Euclidean lines which are also lines embedded in the corresponding TPMS. The backbones of our fish lie along those lines and form skew rhombi for regular skew polyhedra. If these skew rhombi are spanned by "soap films", one obtains the corresponding TPMS.
- (2) As a minimal surface, a TPMS has negative curvature (except for isolated points of zero curvature), and so its universal covering surface also has negative curvature and thus has the same large-scale geometry as the hyperbolic plane.

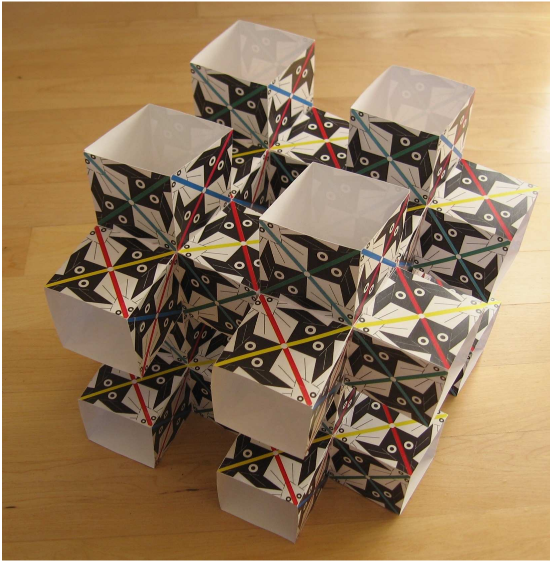
  So the polygons of the triply periodic polyhedron can be transferred to the polygons of a corresponding regular tessellation of the hyperbolic plane, and similarly a pattern on such a polyhedron can be "lifted" to a *universal covering pattern* in the hyperbolic plane.

A Fish Pattern on the $\{4, 6 \mid 4\}$ Polyhedron

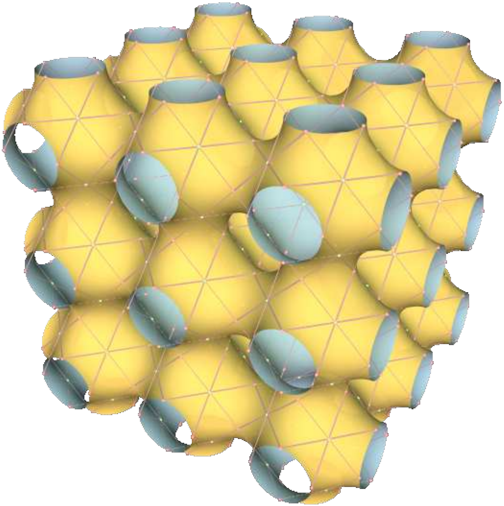
The $\{4, 6 \mid 4\}$ polyhedron is easiest to understand. It consists of invisble "hub" cubes connected by "strut" cubes on all 6 faces of the hubs. We show the 2-step relation between the patterned $\{4, 6 \mid 4\}$ polyhedron and its "universal covering pattern" as follows:

- The pattern of the Title Slide, which we have seen.
- Schwarz's P-surface, the TPMS that is approximated by the $\{4, 6 \mid 4\}$ polyhedron, showing the embedded lines that correspond to the backbone lines of the fish.
- A close-up of Schwarz's P-surface showing the skew rhombi.
- A close-up of one of the vertices of the Title Slide polyhedron.
- The hyperbolic "universal covering pattern" of the Title Slide polyhedron.

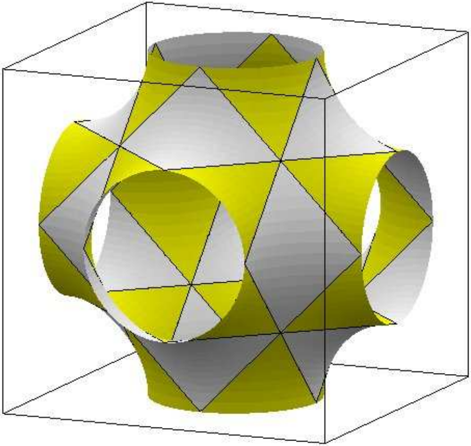
The triply periodic polyhedron of the Title Slide — showing colored embedded lines and skew rhombi



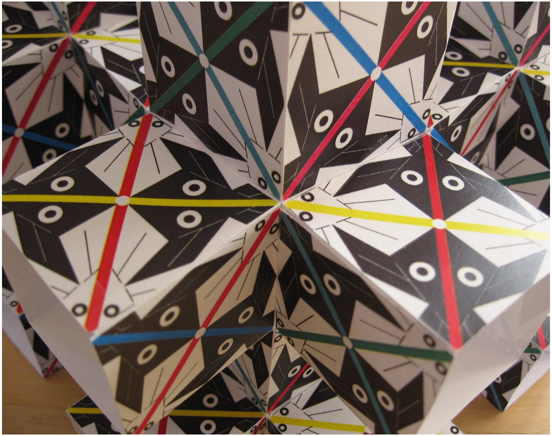
Schwarz's P-surface — approximated by the previous triply periodic polyhedron, and showing corresponding embedded lines



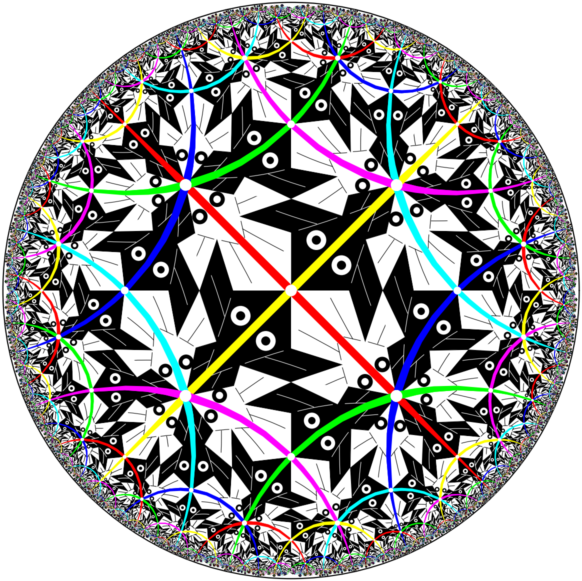
A close-up of Schwarz's P-surface showing corresponding embedded lines and "skew rhombi"



A close-up of a vertex of the Title Slide polyhedron



The pattern of the Title Slide “lifted” to its hyperbolic “universal covering pattern” — showing the embedded lines as hyperbolic lines, which bound the “skew rhombi”.

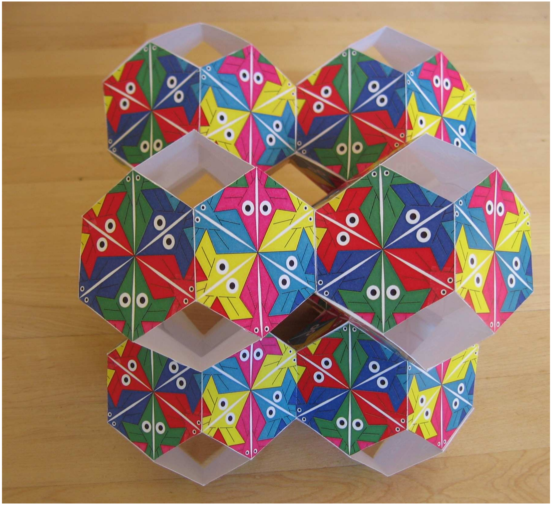


A Fish Pattern on the $\{6, 4 \mid 4\}$ Polyhedron

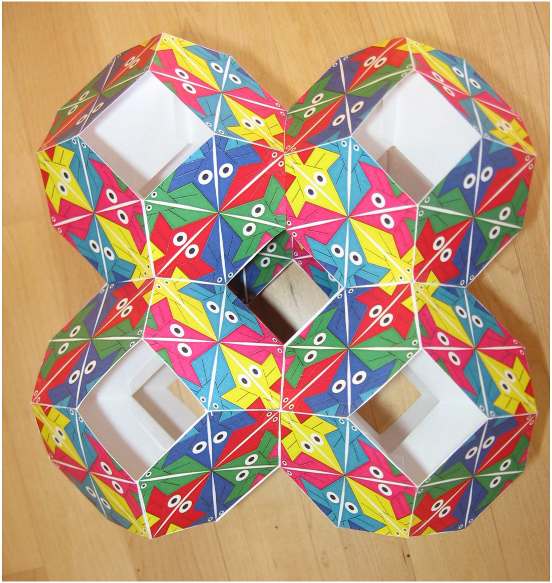
The $\{6, 4 \mid 4\}$ polyhedron is dual to the $\{4, 6 \mid 4\}$ polyhedron which we just saw. In fact the backbone lines of the fish can be taken to be the same lines in 3-space for both polyhedra. Thus they both approximate the same TPMS, Schwarz's P-surface. The $\{4, 6 \mid 4\}$ polyhedron consists of truncated octahedra in a cubic lattice arrangement and connected on their (invisible) square faces. For this polyhedron we show:

- The pattern of fish on the $\{6, 4 \mid 4\}$ polyhedron.
- A top view of the patterned polyhedron that shows how fish of a single color line up along backbone lines.
- The hyperbolic "universal covering pattern" of the patterned polyhedron.

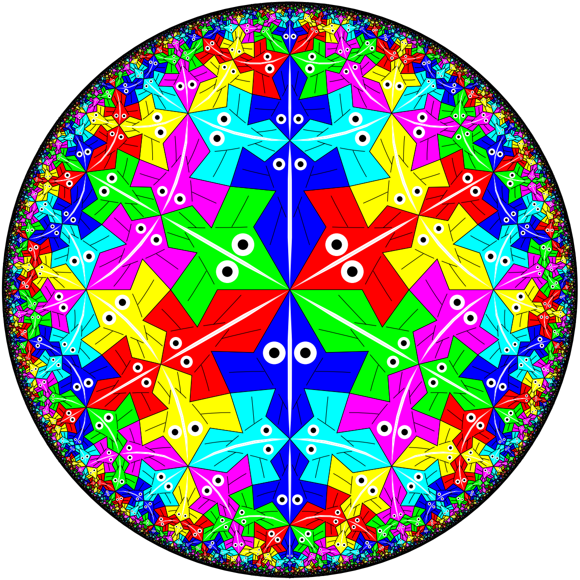
The Pattern of Fish on the $\{6, 4 \mid 4\}$ Polyhedron



A top view of the fish on the $\{6, 4 \mid 4\}$ polyhedron — showing fish along embedded lines



The hyperbolic universal covering pattern of fish — a version of Escher's Circle Limit I pattern with 6-color symmetry

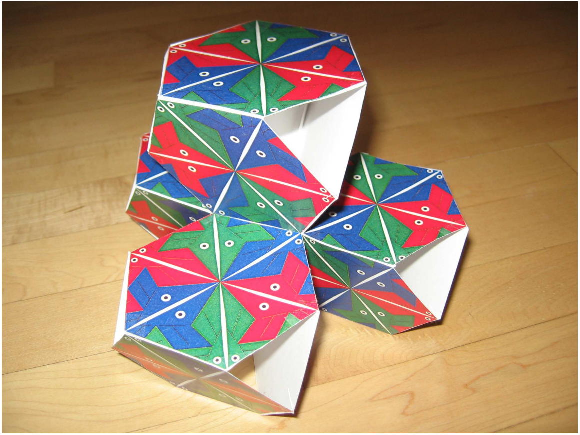


A Pattern of Fish on the $\{6, 6 \mid 3\}$ Polyhedron

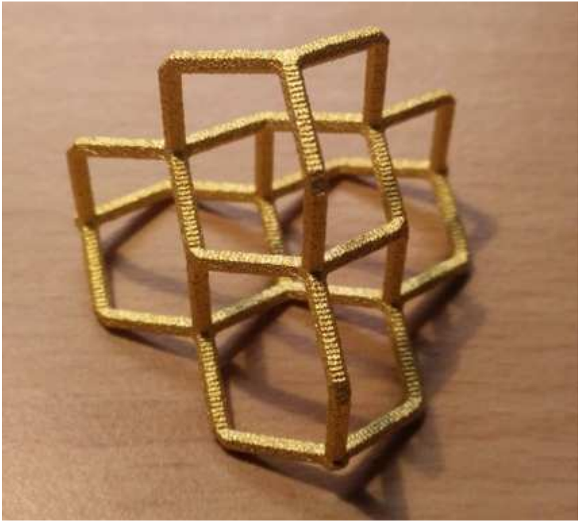
The $\{6, 6 \mid 3\}$ polyhedron is self-dual. It consists of truncated tetrahedra, four of which share (invisible) equilateral triangular faces with an invisible small regular tetrahedron. The embedded backbone lines of the fish also form skew rhombi (but different than for the $\{4, 6 \mid 4\}$ and $\{6, 4 \mid 4\}$ polyhedra). If we span these skew rhombi with "soap films", we obtain the corresponding TPMS, Schwarz's D-surface which has the topology of a thickened diamond lattice. For this polyhedron we show:

- The pattern of fish on the $\{6, 6 \mid 3\}$ polyhedron.
- The diamond lattice.
- A "construction unit" of Schwarz's D-surface within a rhombic dodecahedron. Since rhombic dodecahedra tile space, this gives the entire D-surface.
- A top view of the patterned polyhedron that shows a vertex.
- The hyperbolic "universal covering pattern" of the patterned polyhedron.

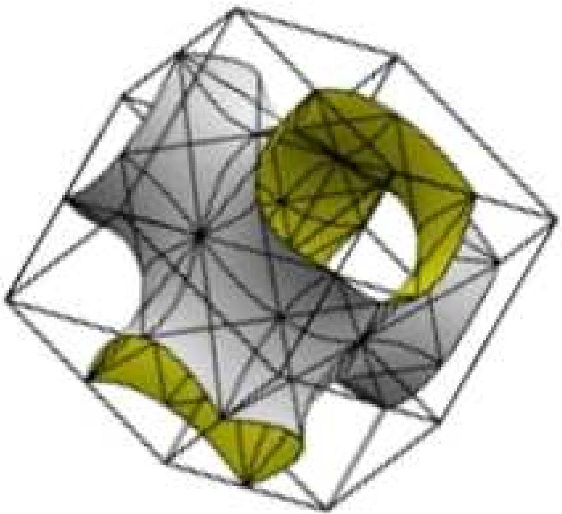
The Pattern of Fish on the $\{6, 6 \mid 3\}$ Polyhedron — showing an invisible tetrahedral hub with 4 truncated tetrahedral "struts"



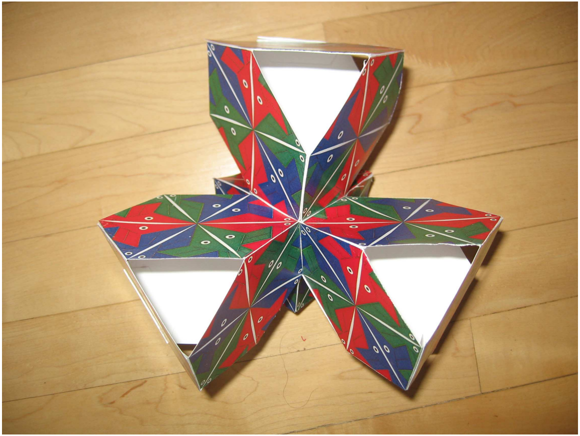
The diamond lattice.



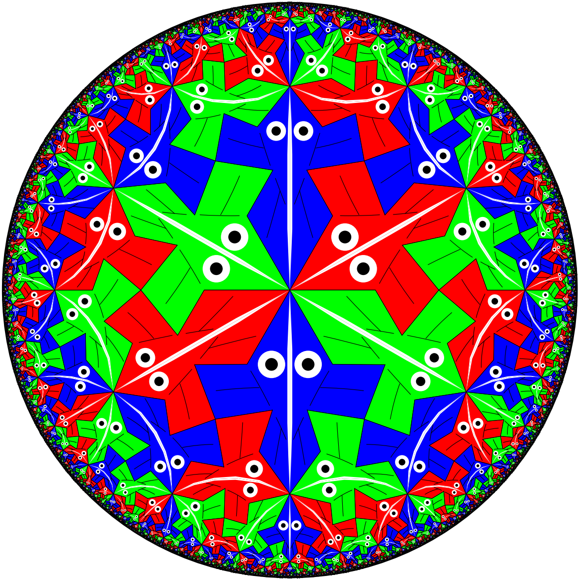
A piece of Schwarz's D-surface showing embedded lines



A top view of the fish on the $\{6, 6 \mid 3\}$ polyhedron — showing a vertex



The corresponding universal covering pattern of fish — based on the $\{6, 6\}$ tessellation

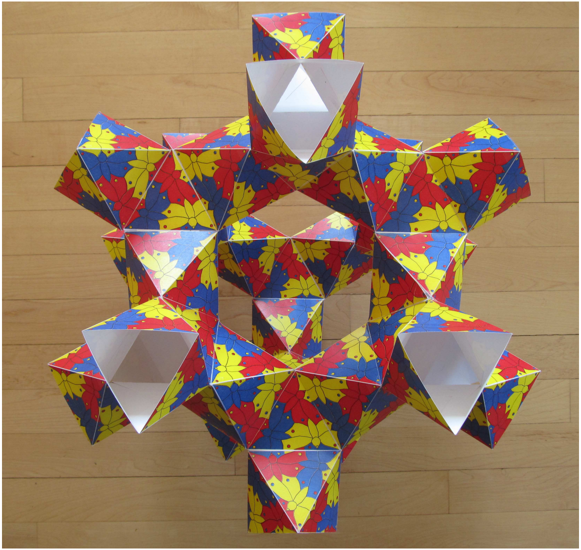


A Pattern of Butterflies on a $\{3, 8\}$ Polyhedron

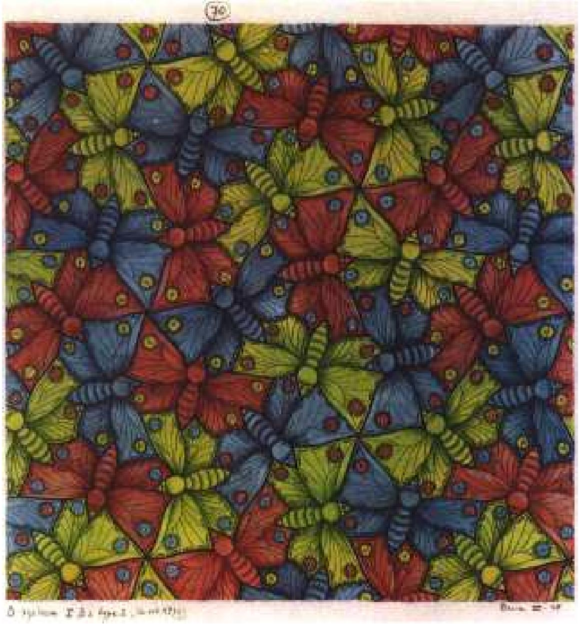
We show a $\{3, 8\}$ polyhedron decorated with a butterfly pattern that was inspired by Escher's Regular Division Drawing # 70. This polyhedron is also related to Schwarz's D-surface. We show:

- Butterflies on the $\{3, 8\}$ polyhedron.
- Escher's Regular Division Drawing # 70.
- A hyperbolic pattern of butterflies based on the $\{3, 8\}$ tessellation — the "universal covering pattern" of the patterned polyhedron.
- A construction unit of the $\{3, 8\}$ polyhedron consisting of a regular octahedral "hub" and four octahedral "struts" placed on alternate faces of the hubs.
- Part of Schwarz's D-surface corresponding to the construction unit.
- Another view of the patterned $\{3, 8\}$ polyhedron down one of its "tunnels".

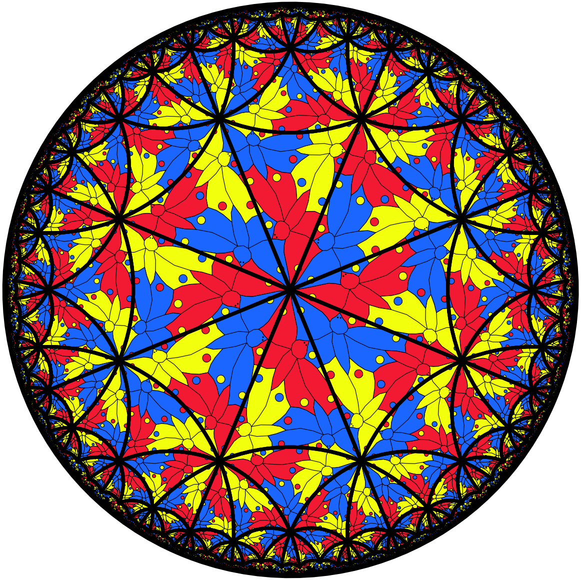
Butterflies on the $\{3, 8\}$ polyhedron.



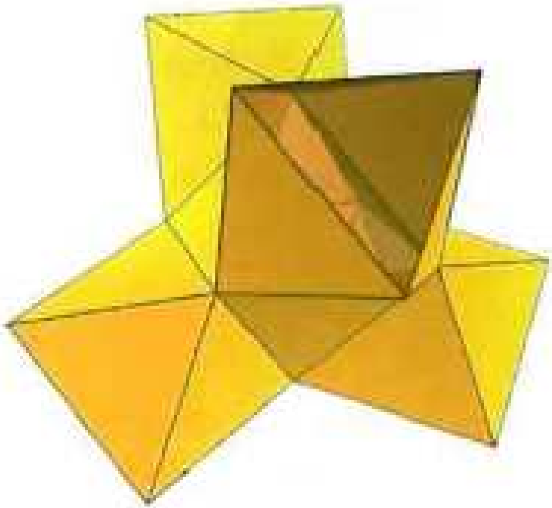
Escher's Regular Division Drawing # 70 based on the $\{3, 6\}$ tessellation.



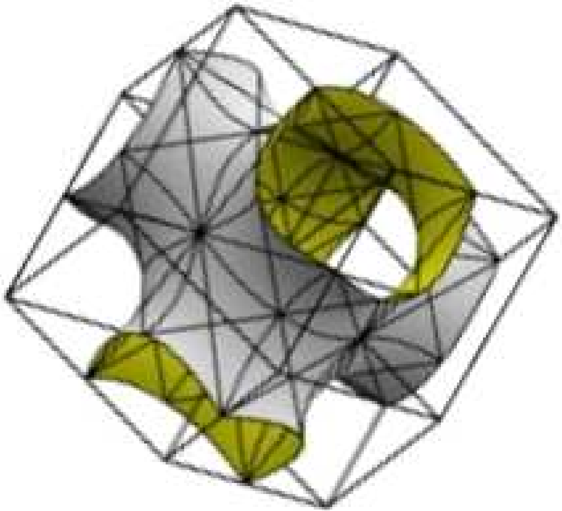
A pattern of butterflies based on the $\{3, 8\}$ tessellation — the "universal covering pattern" for the polyhedron.



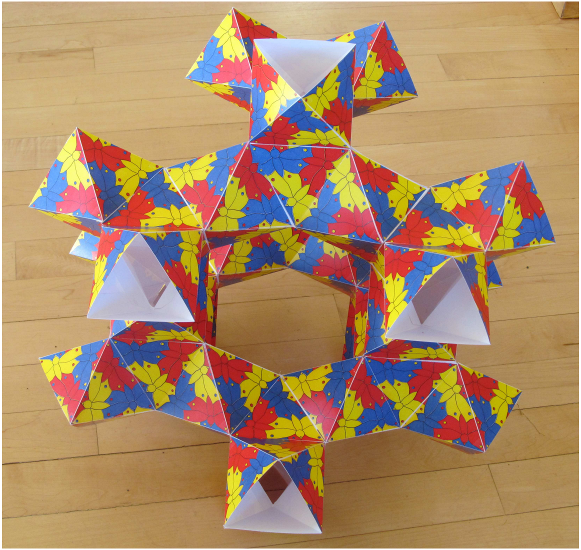
A “construction unit” of the triply periodic polyhedron



A corresponding piece of Schwarz's D-surface



A view down one of the "tunnels" of the $\{3, 8\}$ polyhedron.

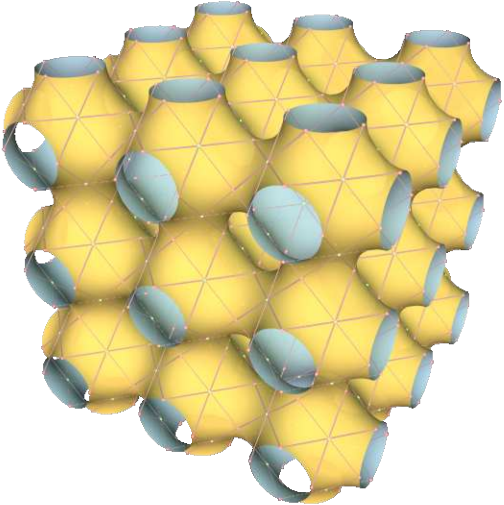


Butterflies on Another $\{3, 8\}$ Polyhedron

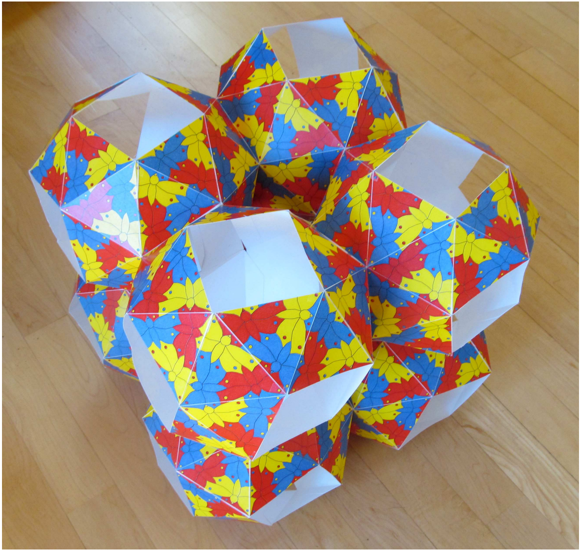
We show the pattern of butterflies on a different the triply periodic $\{3, 8\}$ polyhedron. This butterfly pattern was also inspired by Escher's Regular Division Drawing # 70. Thus the hyperbolic "covering pattern" is the same as for the previous polyhedron. This polyhedron has the same topology as Schwarz's P-surface, a TPMS with the topology of a thickened version of the 3-D coordinate lattice. We show:

- Schwarz's P-surface again — to compare with the polyhedron.
- The $\{3, 8\}$ polyhedron, which is made up of snub cubes arranged in a cubic lattice, attached by their (missing) square faces, and alternating between left-handed and right-handed versions.
- A close-up of the patterned polyhedron.

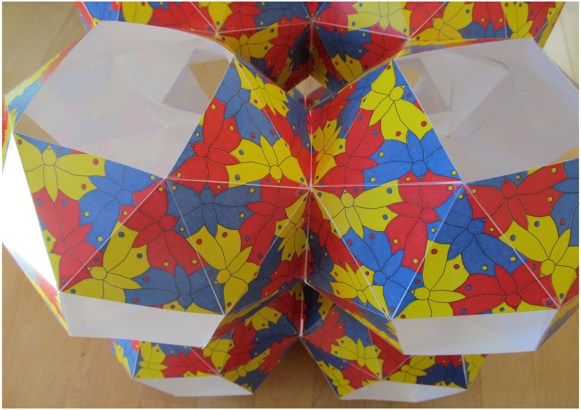
Schwarz's P-surface



Another Patterned $\{3, 8\}$ Polyhedron



A Close-up of the $\{3, 8\}$ Polyhedron

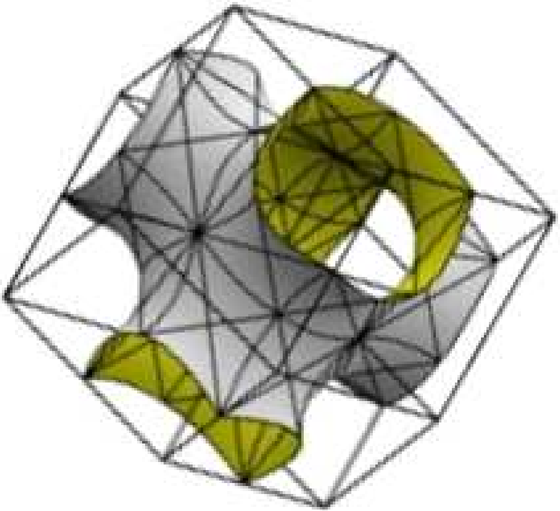


A Fish Pattern on a $\{3,8\}$ Polyhedron

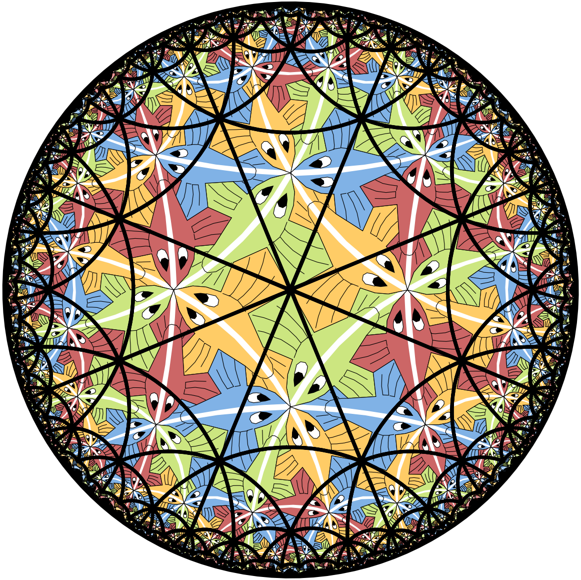
We show a fish pattern on the first $\{3,8\}$ polyhedron shown above. The fish pattern was inspired by Escher's hyperbolic print *Circle Limit III*, which is based on the regular $\{3,8\}$ tessellation. This polyhedron, like the $\{6,6 \mid 3\}$, polyhedron is also an approximation to Schwarz's D-surface. The red, green, and yellow fish swim along the embedded lines of the D-surface (the blue fish swim in loops around the "waists"). We show:

- A piece of Schwarz's D-surface within a rhombic dodecahedron showing embedded lines.
- Escher's *Circle Limit III* pattern with the $\{3,8\}$ equilateral triangle tessellation superimposed — the universal covering pattern.
- The patterned polyhedron.
- A top view of the polyhedron.

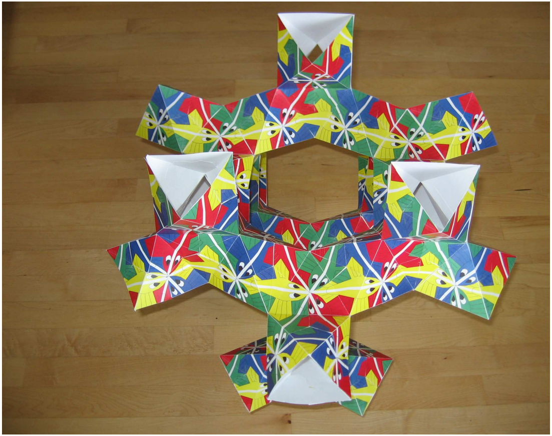
A piece of Schwarz's D-surface showing embedded lines



Escher's Circle Limit III pattern with the equilateral triangle tessellation superimposed — the universal covering pattern



The patterned polyhedron



A top view of the patterned polyhedron

Future Work

- Put other patterns on the regular skew polyhedra, exploiting their embedded lines.
- Place patterns on semi-regular triply periodic polyhedra.
- Put patterns on other triply periodic polyhedra — especially those that more closely approximate triply periodic minimal surfaces.
- Draw patterns on TPMS's — the gyroid, for example.

Thank You!

To all the AMS-EMS-SPM Meeting organizers.

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