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65 Years of Art in Hyperbolic Geometry

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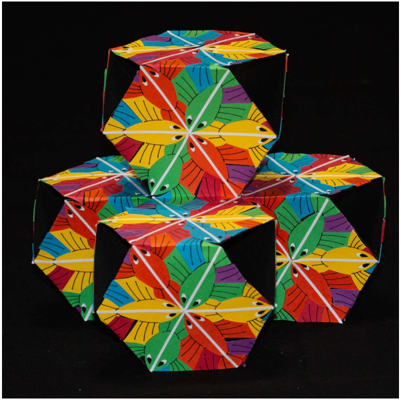


Outline

- The Beginnings: H.S.M. Coxeter and M.C. Escher
- M.C. Escher's *Circle Limit I* and *Circle Limit III*
- Regular $\{p, q \,|\, r\}$ triply periodic polyhedra
- The papercrafted part of a $\{4, 6 \,|\, 4\}$ polyhedron
- A part of the $\{6, 6 \,|\, 3\}$ polyhedron that solves all the problems
- Future work
- Contact information

Figure 7 of Coxeter's Paper

In Figure 7 we see another such group, with the important difference that now the angle-sum of each triangle is less than two right angles and the number of triangles is infinite. The group is again generated by inversions in three circles, but the figure is no longer a picture of something in space. We do not find it as easy as before to imagine that the smaller peripheral triangles are the same size as those in the middle. But in so far as we succeed in stretching our imagination to this extent, we are visualizing the non-Euclidean plane of Gauss, Bolyai and Lobatschewsky.

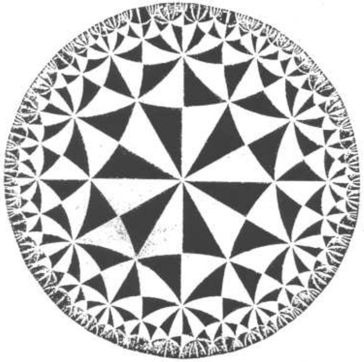
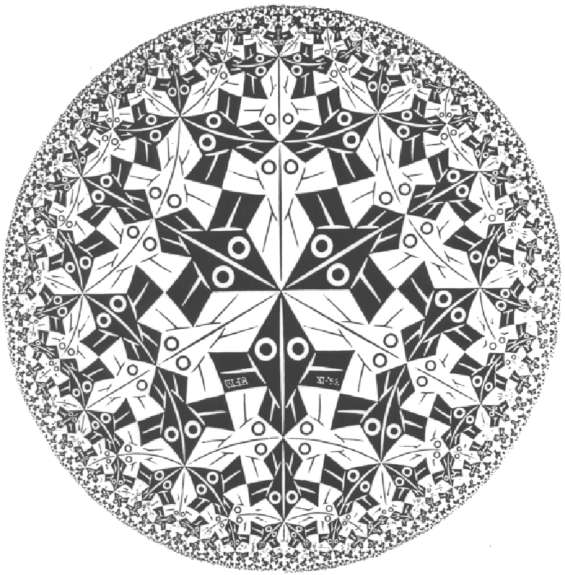


FIGURE 7

This is one way to generalize the idea of symmetry. Another is to increase he number of dimensions. Plate IV shows a wire model made by Mr. P. S. Donchian of Hartford, Conn.[3] This represents an orthogonal projection of

Escher's Woodcut Circle Limit I



Aesthetic Problems with Circle Limit I — per Escher

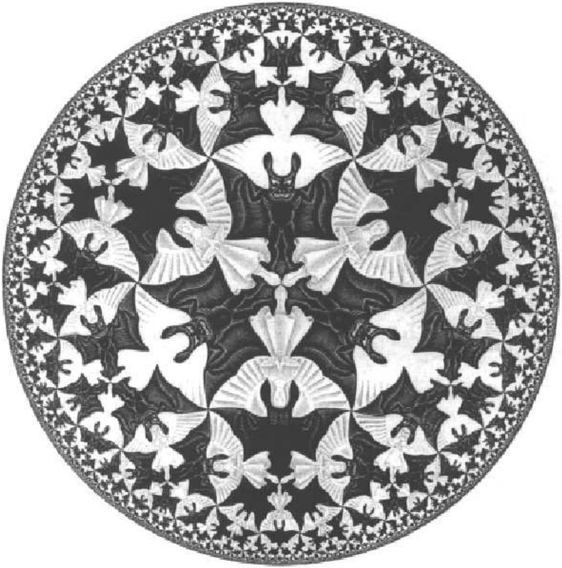
1. The fish were not consistently colored along backbone lines — they alternated from black to white and back every two fish lengths.
2. The fish also changed direction every two fish lengths — thus there was no “traffic flow” (Escher’s words) in a single direction along the backbone lines.
3. The fish are very angular and not “fish-like”

Escher's Woodcut Circle Limit III

— solved the problems

Escher's Woodcut Circle Limit II

Escher's Woodcut Circle Limit IV



Computer Rendition of Escher's *Circle Limit* Patterns

From 1978 to 1980 Dunham, John Lindgren, and David Witte designed and wrote a program that could reproduce each of Escher's four *Circle Limit* patterns.

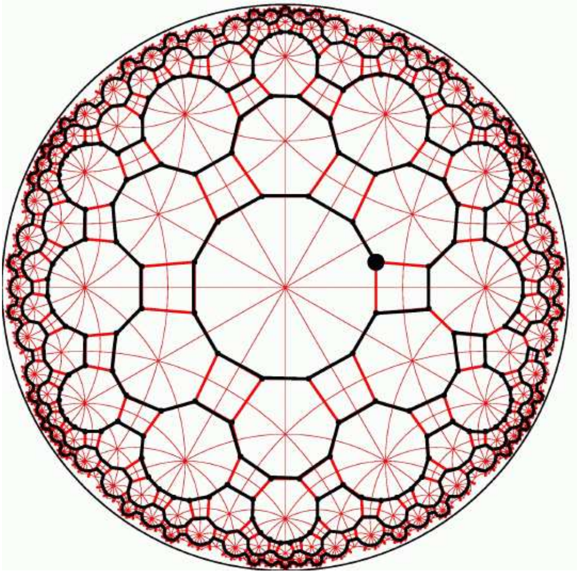
The symmetry groups of *Circle Limit I* and *Circle Limit IV* are subgroups of the group [6,4] (in Coxeter notation), which is generated by reflections across the sides of a 30-45-90 degree hyperbolic triangle.

Similarly, the symmetry groups of *Circle Limit I* and *Circle Limit IV* are subgroups of the group [8,3], which is generated by reflections across the sides of a 22.5-60-90 degree hyperbolic triangle.

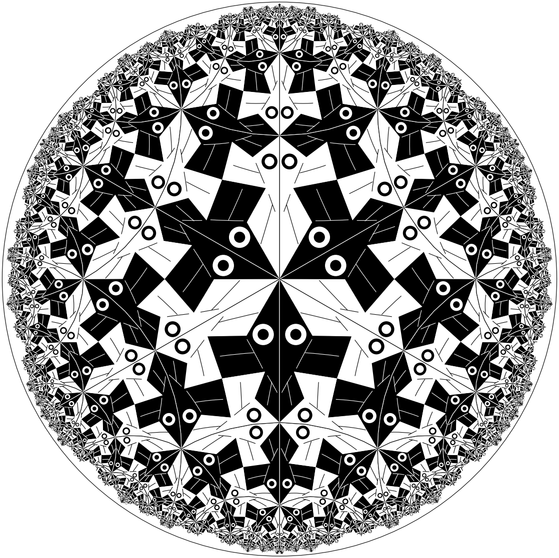
The replication algorithm worked by iterating along a Hamiltonian path in the symmetry group of the pattern.

The algorithm also worked more generally for patterns whose symmetry group is a subgroup of [p,q], where $(p-2)(q-2) > 4$ (for the pattern to be hyperbolic.

A Hamiltonian Path in the group [6,4]

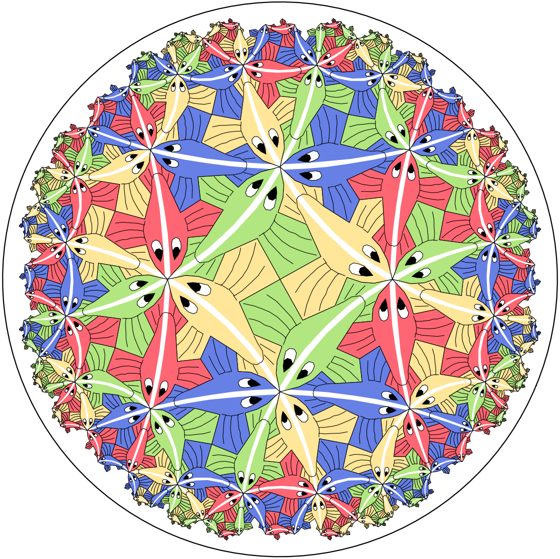


A Rendition of Circle Limit I

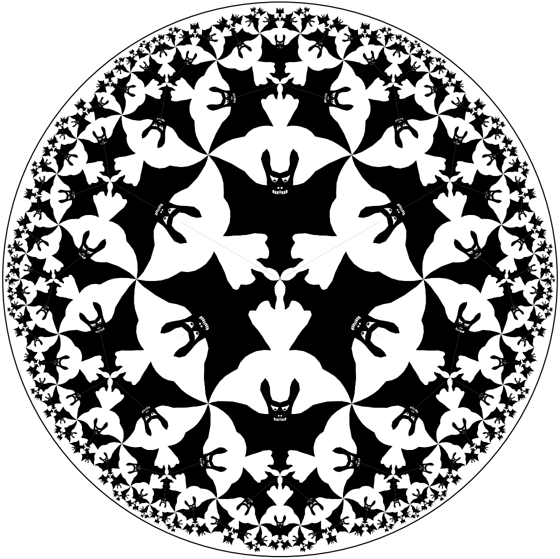


A Rendition of Circle Limit II

A Rendition of Circle Limit III



A Rendition of Circle Limit IV



A Fish Pattern Based on a subgroup of [10,3]

An embroidery designed by me and implemented by Lisa Shier in 2018.

Regular Triply Repeating Polyhedra

In 1926 H.S.M. Coxeter defined *regular skew polyhedra* (apeirohedra) to be infinite polyhedra repeating in three independent directions in Euclidean 3-space, with the symmetry group of isometries being transitive on flags.

Coxeter denoted them by the extended Schläfli symbol $\{p, q \mid r\}$ which denotes the polyhedron composed of p-gons meeting q at each vertex, with regular r-sided polygonal holes.

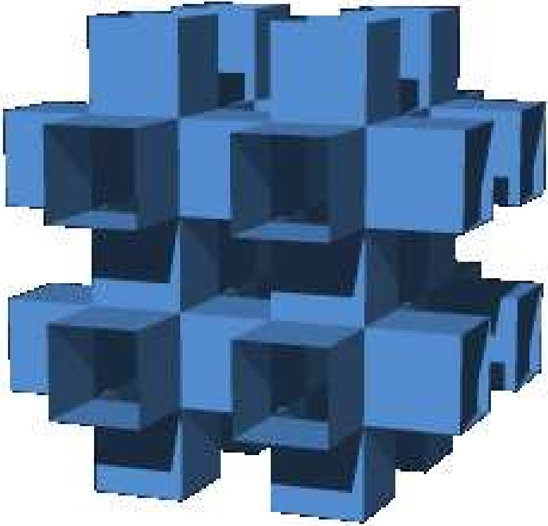
Coxeter and John Flinders Petrie proved that there are exactly three of them: $\{4, 6 \mid 4\}$, $\{6, 4 \mid 4\}$, and $\{6, 6 \mid 3\}$.

Since the sum of the vertex angles is greater than 2π, they are considered to be the hyperbolic analogs of the Platonic solids and the regular Euclidean tessellations $\{3, 6\}$, $\{4, 4\}$, and $\{6, 3\}$

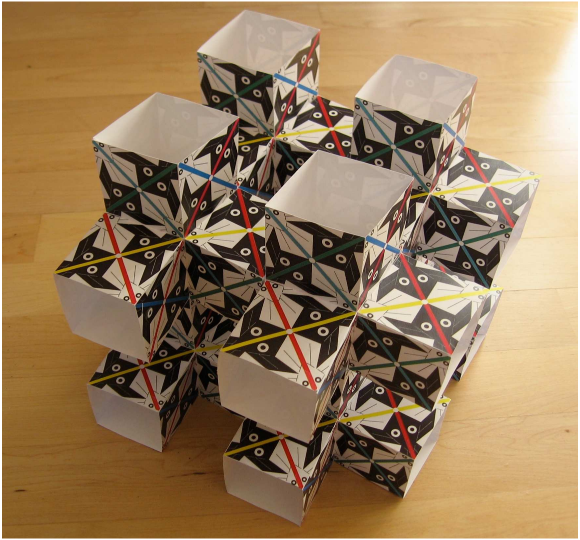
In 2012 Dunham was the first person to decorate those solids with Escher-inspired patterns.

The simplest regular skew polyhedron:

Also called the *Mucube* (for Multi-cube). It consists of invisible “hub” cubes connected by “strut” cubes, hollow cubical cylinders with their open ends connecting neighboring hubs.



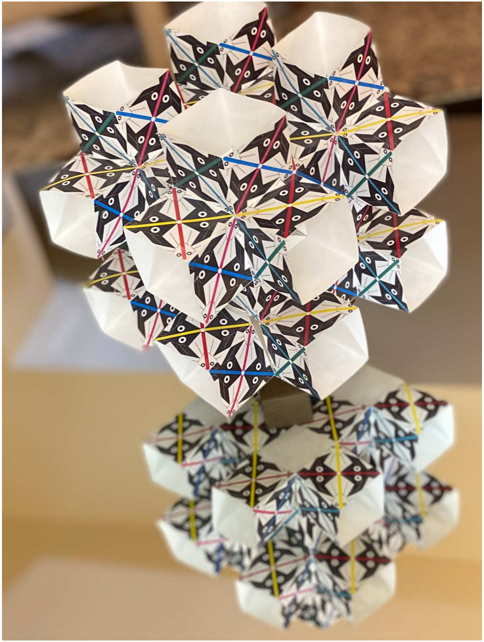
A 2012 patterned $\{4,6\,|\,4\}$ polyhedron with fish



Problems with that fish polyhedron

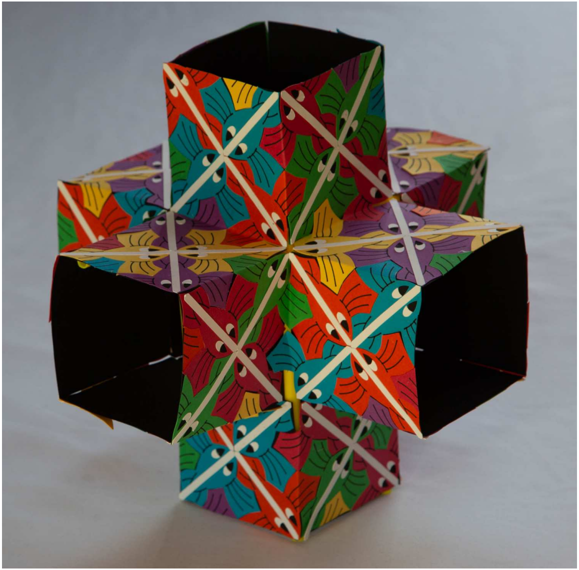
1. The same three problems Escher saw in *Circle Limit I*.
2. A fourth problem: the backbone lines of a particular color are not parallel — which can be seen in a mirror.

The fish polyhedron on a mirror



A papercrafted fish pattern on $\{4, 6 \,|\, 4\}$ by Lisa Shier (2020)

Fixes the first and third problems.



The papercrafted $\{4, 6 \,|\, 4\}$ polyhedron on a mirror

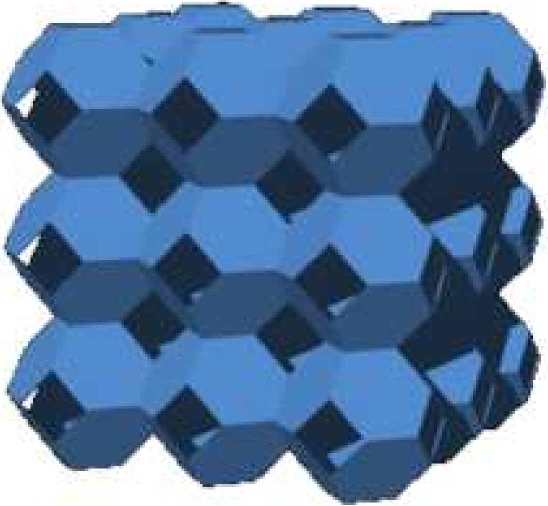
Fixes the fourth problem too, but not the second one.

Colors of fish on the $\{4, 6 \,|\, 4\}$ polyhedron

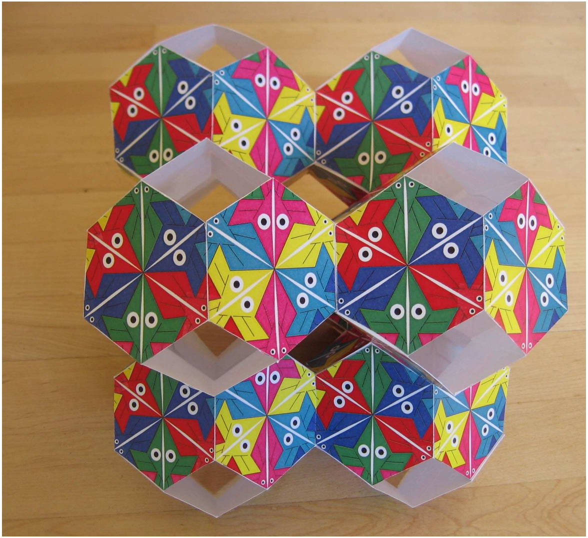
1. There are six families of fish backbone lines that are parallel to the face diagonals of a cube.
2. All the fish in one family are the same color.

The dual of the Mucube is the $\{6, 4 \,|\, 4\}$ polyhedron

Also called the *Muoctahedron* (for Multi-octahedron). It consists of truncated octahedra in a cubic lattice arrangement, connected on their invisible square faces (which are also the square holes between the truncated octahedra).

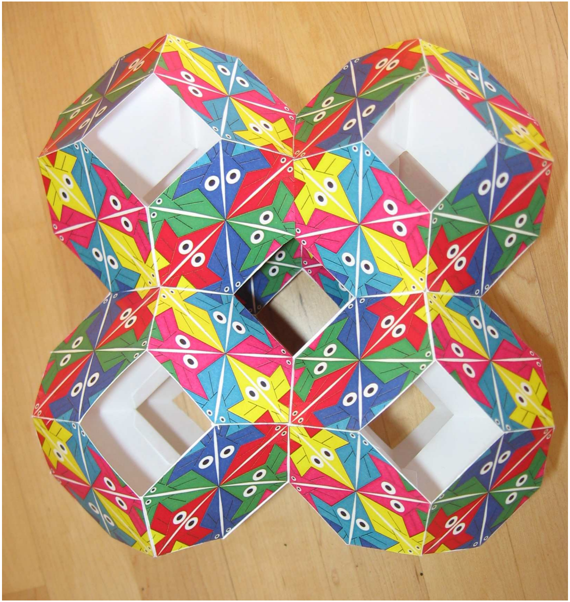


An angular fish pattern on the $\{6, 4 \mid 4\}$ polyhedron



A top view of the fish pattern on the $\{6, 4 \,|\, 4\}$ polyhedron

It solves Escher's first problem, but still has problems two and three.



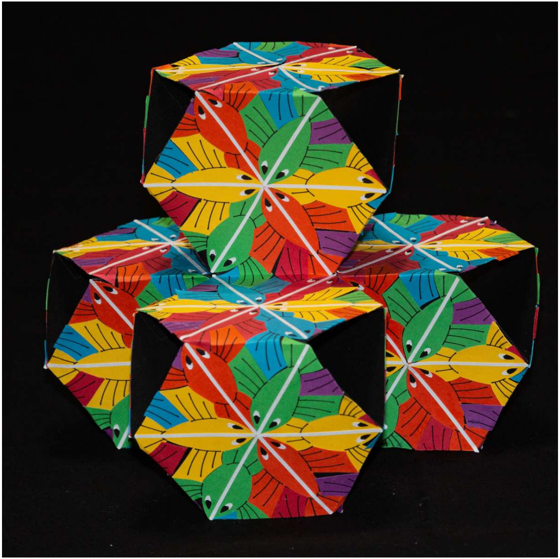
The $\{6,6\,|\,3\}$ polyhedron is self-dual

Also called the *Mutetrahedron* (for Multi-tetrahedron). It consists of truncated tetrahedra in a diamond lattice arrangement, connected by their missing triangular faces to faces of invisible regular tetrahedra between them.

The hand-designed $\{6, 6 \,|\, 3\}$ patterned polyhedron

Which fixed the second, "traffic flow", problem.

The papercrafted $\{6, 6 \,|\, 3\}$ polyhedron



Colors of fish on the $\{6, 6 \,|\, 3\}$ polyhedron

1. There are six families of fish backbone lines that go through the centers of the hexagon faces of the $\{6, 6 \,|\, 3\}$ polyhedron.
2. And as with the patterned $\{4, 6 \,|\, 4\}$ polyhedron, the fish in one family are the same color.
3. Each of the families is parallel to one of the sides of a tetrahedron — which can be one of the truncated tetrahedra, since all the (patterned) truncated tetrahedra in the $\{6, 6 \,|\, 3\}$ polyhedron are translates of one another.
4. In the $\{6, 6 \,|\, 3\}$ polyhedron, each family half the lines of fish go one direction, and the other half go the opposite direction — so that fish of one color on one truncated tetrahedron go in opposite directions on adjacent faces (unlike the fish lines on the $\{4, 6 \,|\, 4\}$ polyhedron).

Comparison of fish patterns on the $\{4,6\,|\,4\}$ and $\{6,6\,|\,3\}$ polyhedra

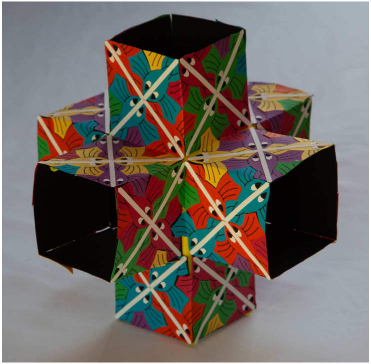


Figure: Fish pattern on $\{4,6\,|\,4\}$

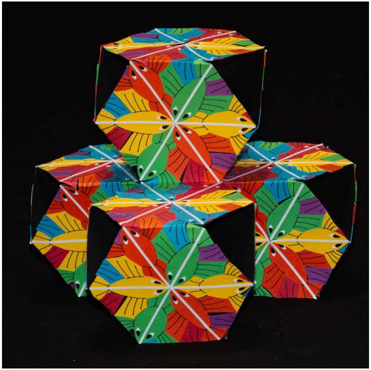
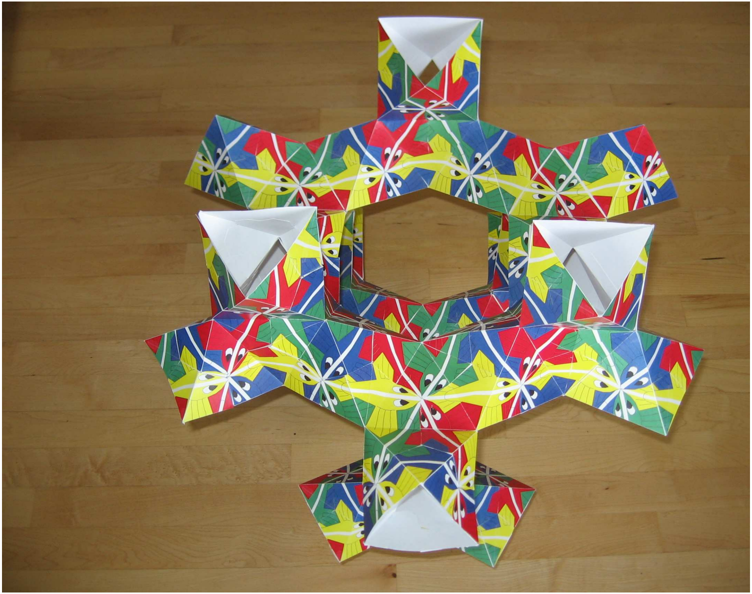


Figure: Fish pattern on $\{6,6\,|\,3\}$

Future Work

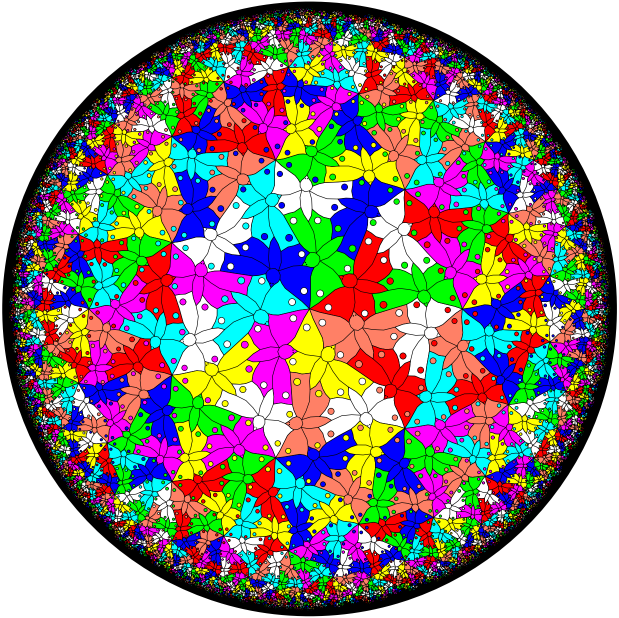
- One problem with our fish pattern on the $\{6, 6 \,|\, 3\}$ polyhedron is that there are two kinds of fish — those with fins sweeping forward and those with fins sweeping back.
- We believe that there is no natural fish pattern on any $\{p, q \,|\, r\}$ polyhedron with only one kind of fish.
- So a possible solution would be to put a fish pattern on a more general triply repeating polyhedron.
- One possibility is to use a $\{3, 8\}$ polyhedron.
- We have previously done this, but with the fish swimming through the centers of the triangles as shown below. But a more satisfying solution might have the fish swimming along triangle edges.

Fish on a $\{3, 8\}$ polyhedron.



A Butterfly Pattern Based on the Group [7,3]

Lisa Shier is working on machine-embroidering this butterfly pattern.



Acknowledgements and Contact

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