A “Circle Limit III” Backbone Arc Formula

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Abstract

M.C. Escher considered Circle Limit III to be the most successful of his four “Circle Limit” patterns. Two artistic or mathematical questions have been raised: (1) what angle do the white backbone lines make with the bounding circle, and (2) are other such patterns of fish possible? H.S.M. Coxeter provided exact expressions to the answer the first question, and a 3-parameter family of possible fish patterns was described in Dunham’s 2006 Bridges Conference paper. Dunham’s 2007 Bridges Conference paper provided a sequence of calculations that determine the intersection angle for any pattern of that family. In this paper, we derive a single expression for that angle, which agrees with Coxeter’s expression for the special case of Circle Limit III.

1. Introduction

We recall M.C. Escher’s hyperbolic pattern Circle Limit III by showing a computer rendition of it in Figure 1. Figure 2 shows a pattern of angular fish from the family of Circle Limit III patterns, with four fish meeting at both left and right fin tips. Dunham’s 2006 Bridges paper introduced the concept of a 3-parameter family of Circle Limit III patterns indexed by the numbers p, q, and r of fish meeting at right fin tips, left fin tips, and noses respectively [Dun06]. Such a pattern was denoted by the triple (p, q, r). Thus Circle Limit III and

Figure 1: A rendition of Escher’s Circle Limit III.  Figure 2: An angular fish pattern from the Circle Limit III family.
the pattern of Figure 2 would be called \((4, 3, 3)\) and \((5, 3, 3)\) respectively. We required \(r\) to be odd so that the fish swim head-to-tail, and \(p, q,\) and \(r\) should all be greater than or equal to 3. Also, in the style of \textit{Circle Limit III}, we place right fin tips at the center of the bounding circle (it was convenient in [5] to place noses at the center for those calculations). Thus we are able to distinguish the pattern \((p, q, r)\) from \((q, p, r)\).

Escher and the Canadian mathematician H.S.M. Coxeter carried on a fruitful correspondence over a number of years, as has been recounted in [2, 8]. In 1958 Escher was inspired to create his “Circle Limit” patterns by a figure showing a triangle tessellation of the hyperbolic plane in one of Coxeter’s papers, which he had sent Escher. In return, Escher gave Coxeter prints of his “Circle Limit” patterns, including \textit{Circle Limit III}. Coxeter was later inspired to write two papers on the geometry of the backbone lines in that print [2, 3]. In the issue of \textit{The Mathematical Intelligencer} containing Coxeter’s second paper, an anonymous editor wrote the following caption for the cover of that issue, which showed \textit{Circle Limit III}:

> Coxeter’s enthusiasm for the gift M.C. Escher gave him, a print of Circle Limit III, is understandable. So is his continuing curiosity. See the articles on pp. 35–46. He has not, however said of what general theory this pattern is a special case. Not as yet. [1]

We are unaware if Coxeter never described such a general theory, but that caption was the inspiration for Dunham’s quest for such a theory [4, 5].

The goal of this paper is find a single expression for the intersection angle \(\omega\) between the bounding circle and a backbone line of a general \((p, q, r)\) pattern, unlike the sequence of calculations in [5]. Since these patterns are regular when interpreted in terms of hyperbolic geometry, all backbone lines of a pattern make the same angle with the bounding circle, so there is only one angle to determine for any particular pattern.

For background we first review some hyperbolic geometry that is used in the calculations. Then we give two derivations of the expression for \(\cos \omega\). The first method is in the style of Coxeter’s first paper [2], which uses hyperbolic trigonometry. The second method is in the Euclidean calculation style of Coxeter’s second paper. The second author has also shown how Dunham’s calculations in [5] can be used to derive the expression. Finally, we review the results and indicate directions of further research.

\section*{2. Hyperbolic Geometry}

Escher’s “Circle Limit” patterns can be interpreted as repeating patterns of the hyperbolic plane. The hyperbolic plane is a surface of constant negative (Gaussian) curvature. The entire hyperbolic plane has no smooth, isometric (distance preserving) embedding in Euclidean 3-space as was proved by David Hilbert in 1901 [7]. Thus, we must rely on Euclidean models of hyperbolic geometry in which distance is measured differently and concepts such as hyperbolic lines have interpretations as Euclidean constructs.

We will mostly be using the Poincaré disk model of hyperbolic geometry, as Escher did in his “Circle Limit” patterns. But we also utilize the Weierstrass model, which was a key part in the calculations of [5]. In the \textit{Poincaré disk model} the points are just the (Euclidean) points within a Euclidean bounding circle, which we will take to be the unit circle in the \(xy\)-plane for computational convenience. Hyperbolic lines are represented by circular arcs orthogonal to the bounding circle (including diameters). For example, the backbone lines lie along hyperbolic lines in Figure 2. The disk model is \textit{conformal}: the hyperbolic measure of an angle is the same as its Euclidean measure. As a consequence, all fish in a “Circle Limit III” pattern have roughly the same Euclidean shape. However equal hyperbolic distances correspond to ever smaller Euclidean distances toward the edge of the disk. So all the fish are the same (hyperbolic) size in a \textit{Circle Limit III} pattern. The Poincaré disk model is appealed to Escher (and has appealed to other artists) since an infinitely repeating pattern could be shown in a bounded area and shapes remained recognizable even for small copies of the motif, due to conformality.
On first glance, it is tempting to guess that the backbone arcs of Circle Limit III are hyperbolic lines. Indeed, Escher seemed to think so — in a letter to Coxeter he wrote “... As all these strings of fish shoot up like rockets from infinitely far away, perpendicularly [emphasis ours] from the boundary, and fall back again whence they came, not one single component ever reaches the edge. ...” [2]. However, a careful measurement of the backbone arcs of the fish in Circle Limit III shows that they make an angle of about 80° with the bounding circle. These arcs are so-called equidistant curves in hyperbolic geometry: curves at a constant hyperbolic distance from the hyperbolic line with the same endpoints on the bounding circle, and Escher accurately drew them as such. For every hyperbolic line and a given distance, there are two equidistant curves, called branches, at that distance from the line, one each side of the line. In the Poincaré disk model, those two branches are represented by circular arcs making the same (non-right) angle with the bounding circle on either side of the corresponding hyperbolic line. Escher used only one branch for fish backbones from each pair of equidistant curves in Circle Limit III.

The points in the Weierstrass model are the points on the upper sheet of the hyperboloid of two sheets $x^2 + y^2 - z^2 = -1$. The hyperbolic distance between two points \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} and \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} is given by:

$$\cosh^{-1}(z_1 z_2 - x_1 x_2 - y_1 y_2).$$

A hyperbolic line in this model is the intersection of a Euclidean plane through the origin with this upper sheet, and thus is one branch of a hyperbola. A line can be represented by its pole, a 3-vector \begin{bmatrix} \ell_x \\ \ell_y \\ \ell_z \end{bmatrix} on the dual hyperboloid $\ell_x^2 + \ell_y^2 - \ell_z^2 = +1$, so that the line is the set of points satisfying $x \ell_x + y \ell_y - z \ell_z = 0$. Equidistant curves are represented by $x \ell_x + y \ell_y - z \ell_z = \pm d_H$, where $d_H$ is the hyperbolic distance between the equidistant curve and its line. The Weierstrass model is related to the disk model by “stereographic projection” onto the $xy$-plane toward the vertex of the lower sheet of the hyperboloid, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, which is given by the formula:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x/(1+z) \\ y/(1+z) \\ 0 \end{bmatrix}.$$ 

3. The Calculation of $\omega$ Using Hyperbolic Trigonometry

For the first derivation of a formula for $\omega$, we generalize Coxeter’s method that used hyperbolic trigonometry, as given in [2]. Figure 3 below shows how the Circle Limit III pattern is related to the regular hyperbolic tessellation \{8,3\}, the tessellation by regular (hyperbolic) octagons meeting three at a vertex. Figure 4 shows additions to the \{8,3\} tessellation, including some line segments, some labels, a backbone arc through $R$ and $R'$, and its corresponding hyperbolic line through $L$, $M$, and $N$. This figure corresponds to Figures 5 and 6 of [2]. However we want to consider the general case in which there is a $p$-fold meeting point of right fins at $P$, a $q$-fold meeting point of left fins at $Q$, and $r$-fold meeting points of noses at $R$ and $R'$. For this figure to represent the geometry, we assume $p > q$. In Figure 4, the quadrilateral $PRQR'$ is a fundamental region for the $(p,q,r)$ pattern. The backbone arc through $R$ and $R'$ is the equidistant curve whose intersection with the bounding circle makes the angle $\omega$ we wish to calculate. The hyperbolic line with the same endpoints as that curve goes through the points $L$, $M$, and $N$. The segments $LQ$ and $RN$ are perpendicular to $LN$.

By a well-known formula [6, page 402], $\omega$ is given by:

$$\cos \omega = \tanh(RN)$$

Since $RNM$ is a right triangle, by one of the formulas for hyperbolic right triangles, $\tanh(RN)$ is related to $\tanh(RM)$ by: $\tanh(RN) = \cos(\angle NRM) \tanh(RM)$ But $\angle NRM = \frac{\pi}{2} - \frac{\pi}{r}$ since the equidistant curve bisects $\angle PRQ = \frac{\pi}{r}$. Thus $\cos(\angle NRM) = \cos(\frac{\pi}{2} - \frac{\pi}{r}) = \sin(\frac{\pi}{2r})$ and

$$\tanh(RN) = \sin(\frac{\pi}{2r}) \tanh(RM) \quad (1)$$
so that our task is reduced to calculating $\tanh(RM)$.

In order to calculate $\tanh(RM)$, we note that as hyperbolic distances $RQ = RM + MQ$, so eliminating $MQ$ from this equation will relate $RM$ to $RQ$, which we can find. By the subtraction formula for $\cosh$, we have $\cosh(MQ) = \cosh(RQ - RM) = \cosh(RQ) \cosh(RM) - \sinh(RQ) \sinh(RM)$. Dividing through by $\cosh(RM)$ gives:

$$\frac{\cosh(MQ)}{\cosh(RM)} = \cosh(RQ) - \sinh(QR) \tanh(RM)$$

Also, another formula for hyperbolic right triangles applied to $QML$ and $RMN$ gives:

$$\cosh(MQ) = \cot(\angle QML) \cot\left(\frac{\pi}{q}\right) \text{ and } \cosh(RM) = \cot(\angle RMN) \cot\left(\frac{\pi}{2} - \frac{\pi}{2r}\right)$$

Now as opposite angles, $\angle QML = \angle RMN$, so dividing the first equation by the second gives another expression for $\frac{\cosh(MQ)}{\cosh(RM)}$:

$$\frac{\cosh(MQ)}{\cosh(RM)} = \cot\left(\frac{\pi}{q}\right) \cot\left(\frac{\pi}{2r}\right)$$

Equating the two expressions for $\frac{\cosh(MQ)}{\cosh(RM)}$ gives: $\cosh(RQ) - \sinh(RQ) \tanh(RM) = \cot\left(\frac{\pi}{q}\right) \cot\left(\frac{\pi}{2r}\right)$, which can be solved for $\tanh(RM)$ in terms of $RQ$:

$$\tanh(RM) = \left(\cosh(RQ) - \cot\left(\frac{\pi}{q}\right) \cot\left(\frac{\pi}{2r}\right)\right) / \sinh(RQ) \quad (2)$$

Thus we have reduced the problem to finding $\cosh(RQ)$ and $\sinh(RQ)$. A formula for general hyperbolic triangles [6, page 406] applied to $QPR$ gives: $\cosh(RQ) = \left(\cos\left(\frac{\pi}{q}\right) \cos\left(\frac{\pi}{p}\right) + \cos\left(\frac{\pi}{p}\right)\right) / \sin\left(\frac{\pi}{q}\right) \sin\left(\frac{\pi}{p}\right)$. We can calculate $\sinh(RQ)$ from this by the formula $\sinh^2 = \cosh^2 - 1$. Substituting these values of $\cosh(RQ)$ and $\sinh(RQ)$ into equation (2), and inserting that result into equation (1) gives the final result:

$$\cos(\omega) = \frac{\sin\left(\frac{\pi}{2p}\right) \left(\cos\left(\frac{\pi}{p}\right) - \cos\left(\frac{\pi}{q}\right)\right)}{\sqrt{\cos\left(\frac{\pi}{p}\right)^2 + \cos\left(\frac{\pi}{q}\right)^2 + \cos\left(\frac{\pi}{p}\right)^2 + 2 \cos\left(\frac{\pi}{p}\right) \cos\left(\frac{\pi}{q}\right) \cos\left(\frac{\pi}{p}\right) - 1}}$$
which is antisymmetric in \( p \) and \( q \), as we would expect.

Letting \( q = r = 3 \) and doing some algebraic manipulation yields the same formula given in [4] for that special case:

\[
\cos \omega = \frac{1}{2} \sqrt{1 - \frac{3}{4 \cos^2(\pi/(2p))}}
\]

Some algebraic manipulation also yields a formula for \( \tan \omega \):

\[
\tan(\omega) = \cot(\pi/2r) \sqrt{1 + \frac{4 \cos(\pi p) \cos(\pi q) + 2 \cos(\pi r) - 2}{(\cos(\pi p) - \cos(\pi q))^2}}
\]

4. The Calculation of \( \omega \) Using Euclidean Methods

The next derivation of the formula for \( \omega \) uses Euclidean methods as Coxeter did in his calculations in [3]. The calculation proceeds as follows. First find the center \((c, 0)\) and radius \( \rho \) of the circular backbone arc through \( R \) and \( R' \) in Figure 4. Then find the intersection point of the circular arc with the unit bounding circle. Finally \( \cos \omega \) is the dot product of unit radii of the circular arc and bounding circle toward the intersection point.

As can be seen in Figure 4, the tangent line to the circular arc at \( R \) makes an angle \( \theta = \pi p + \pi/2r \) with the \( x \)-axis (since the circular arc bisects \( \angle P R Q \)). Letting \((u, v)\) be the coordinates of \( R \) and using some simple trigonometry,

\[
c = u + v \tan \theta  \\
\rho = v/\cos \theta
\]

The points of intersection of the equidistant curve with the unit circle are obtained by solving:

\[
(x - c)^2 + y^2 = \rho^2 \\
x^2 + y^2 = 1
\]

The result is:

\[
x = (c^2 - \rho^2 + 1)/(2c)  \\
y = \sqrt{1 - x^2}
\]

(for the first quadrant solution).

The unit vector from the origin \( P \) toward \((x, y)\) is just \((x, y)\). The unit vector from \((c, 0)\) toward \((x, y)\) is \((-\sqrt{\rho^2 - y^2}/\rho, y/\rho)\). As mentioned above, \( \cos \omega \) is just the dot product:

\[
\cos \omega = (x, y) \cdot (-\sqrt{\rho^2 - y^2}/\rho, y/\rho) = -x \sqrt{\rho^2 - y^2} + y^2
\]

Of course \( y^2 = 1 - x^2 \), and after simplification, \( \rho^2 - y^2 = (c^2 + \rho^2 - 1)/2c \). Thus

\[
\cos \omega = \frac{1}{\rho}(-\frac{x(c^2 + \rho^2 - 1)}{2c} + 1 - x^2) = \frac{1}{2\rho}(\rho^2 + 1 - c^2)
\]

the second equality following from some simplification.

Now substitute the values of \( \rho \) and \( c \) from above, and use a common denominator of \( \cos \theta \) in \( (\rho^2 + 1 - c^2) \) to get:

\[
\cos \omega = \frac{\cos \theta}{2v} \cdot \frac{v^2 + \cos^2(\theta) - (u \cos(\theta) + v \sin(\theta))^2}{\cos^2 \theta}
\]
One factor of \( \cos(\theta) \) cancels, and expanding the squared sum, \( v^2 \) can be combined with the \( -v^2 \sin^2(\theta) \) to yield \( v^2 \cos^2(\theta) \) and so another factor of \( \cos(\theta) \) cancels giving:

\[
\cos \omega = \frac{1}{2v} ((v^2 + 1 - u^2) \cos \theta - 2uv \sin \theta)
\]

At this point, we need the coordinates \((u, v)\) of \( R \). The Weierstrass coordinates of \( R \) are given by

\[
P = \left[ \begin{array}{c}
\cos(\pi/p) \sinh(PR) \\
\sin(\pi/p) \sinh(PR) \\
\cosh(PR)
\end{array} \right]
\]

and thus projecting to the Poincaré model

\[
\begin{bmatrix}
\frac{\cos(\pi/p) \sinh(PR)}{1 + \cosh(PR)} \\
\frac{\sin(\pi/p) \sinh(PR)}{1 + \cosh(PR)} \\
0
\end{bmatrix}
\]

Of course \( \sinh(PR) = \sqrt{\cosh^2(PR) - 1} \), and by the formula used above for general hyperbolic triangles [6, page 406], \( \cosh(PR) = \frac{\cos(\pi/p) \cos(\pi/r) + \cos(\pi/q)}{\sin(\pi/p) \sin(\pi/r)} \). Now we introduce some notational shorthand to simplify the following calculations. We use:

\[
sp, cq, \text{ and } cr \text{ for } \cos(p\pi/p), \cos(p\pi/q), \text{ and } \cos(p\pi/r), \text{ and }
\]

\[
sp, sq, \text{ and } sr \text{ for } \sin(p\pi/p), \sin(p\pi/q), \text{ and } \sin(p\pi/r), \text{ and }
\]

\( C \) and \( S \) for \( \cosh(PR) \) and \( \sinh(PR) \), respectively.

With this notation, \( u = cp * S/(C + 1) \) and \( u = sp * S/(C + 1) \), so

\[
\cos \omega = \frac{C + 1}{2spS} \cdot \frac{(sp^2 S^2 + (C + 1)^2 - cp^2 S^2) \cos(\theta) - 2cspS^2 \sin(\theta)}{(C + 1)^2}
\]

\[
= \frac{1}{2spS} \cdot ((C - 1)sp^2 + (C + 1) - (C - 1)cp^2) \cos(\theta) - 2csp(C - 1) \sin(\theta)
\]

by using \( S^2 = C^2 - 1 = (C + 1)(C - 1) \) and canceling \( (C + 1)^2 \) factors. Collecting terms,

\[
\cos \omega = \frac{1}{2spS} \cdot ((C - 1)[(sp^2 - cp^2) \cos(\theta) - 2spcr \sin(\theta)]) + (C + 1) \cos(\theta))
\]

Now \( (cp^2 - sp^2) \cos(\theta) + 2spcr \sin(\theta) = \cos\left(\frac{2\pi}{p}\right) \cos(\theta) + \sin\left(\frac{2\pi}{p}\right) \sin(\theta) = \cos\left(\frac{2\pi}{p} - \left(\frac{\pi}{p} + \frac{\pi}{2}\right)\right) = \cos\left(\frac{\pi}{p} - \frac{\pi}{2r}\right), \) using double angle formulas and the cosine difference formula. Thus

\[
\cos \omega = \frac{1}{2spS} \cdot ((C + 1) \cos(\pi/p + \pi/2r) - (C - 1) \cos(\pi/p - \pi/2r))
\]

Next we expand \( S, C + 1, \) and \( C - 1 \):

\[
S = \sqrt{C^2 - 1} = \sqrt{(cpcr + cq)^2 - sp^2 cr^2 / spsr}
\]

\[
= \sqrt{cp^2 cr^2 + 2cpcqcr + cq^2 - (1 - cp^2)(1 - cr^2) / spsr}
\]

\[
= \sqrt{cp^2 + cq^2 + cr^2 + 2cpcqcr - 1 / spsr}
\]

\[
denom / spsr
\]

\[
C + 1 = (cpcr + cq + spsr) / spsr
\]

\[
C - 1 = (cpcr + cq - spsr) / spsr
\]

where \( denom = \sqrt{\cos(\frac{\pi}{p})^2 + \cos(\frac{\pi}{q})^2 + \cos(\frac{\pi}{r})^2 + 2 \cos(\frac{\pi}{p}) \cos(\frac{\pi}{q}) \cos(\frac{\pi}{r}) - 1} \) for notational convenience.

Substituting these values into the formula for \( \cos \omega \) above and canceling \( spsr \) gives:

\[
\cos \omega = \frac{1}{2sp denom} \cdot ((cpcr + cq + spsr) \cos(\pi/p + \pi/2r) - (cpcr + cq - spsr) \cos(\pi/p - \pi/2r))
\]

\[
= \frac{1}{2sp denom} \cdot ((cpcr + cq + spsr)(cpcr^2 - spsr^2) - (cpcr + cq - spsr)(cpcr^2 + spsr^2))
\]
using the cosine sum and difference formulas, and the additional shorthand $cr^2 = \cos(\pi/(2r))$ and $sr^2 = \sin(\pi/(2r))$. Grouping the $(cpcr + cq)$ terms and multiplying out the numerator $num$ in the expression for $\cos \omega$ gives:

\[
num = ((cpcr + cq) + spsr)(cpcr^2 - spsr^2) - ((cpcr + cq) - spsr)(cpcr^2 + spsr^2)
\]

\[
= 2spsr \cdot cpcr^2 - 2(cpcr + cq)spsr^2
\]

Cancelling the $2sp$ factors in the numerator and denominator of $\cos \omega$ gives:

\[
\cos \omega = \frac{1}{\text{denom}} (cp \cdot (srcr^2 - crsr^2) - cq \cdot sr^2)
\]

\[
= \frac{1}{\text{denom}} (cp \cdot sr^2 - cq \cdot sr^2)
\]

\[
= \frac{sr^2}{\text{denom}} (cp - cq)
\]

by the sine difference formula $(srcr^2 - crsr^2 = \sin(\pi/2 - \pi/2r))$, which is the same result as in Section 3 (recalling that $sr^2 = \sin(\pi/(2r))$, $cp = \cos(\pi/p)$, and $cq = \cos(\pi/q)$).

As mentioned above, this formula reduces to that of [4] for the special case $p = r = 3$. Figures 5 and 6 show examples of such patterns. It seems natural to guess that the formula for $\cos \omega$ would be easy to derive in this case with a vertical chord as an equidistant curve. In fact $\cos \omega = u$ and the expression above for $u$ does indeed reduce to that of [4], giving another derivation of that formula.

It is also worth noting that in the Circle Limit III case, $(4,3,3)$, the expressions above for $\rho$ and $c$ are exactly the same as those computed by Coxeter for $r_1$ and $d_1$ respectively in [3].

5. Conclusions and Future Work

For any $(p, q, r)$ pattern, we have given a formula for the angle $\omega$ an equidistant “backbone” curve makes with the bounding circle. This formula agrees with previously obtained results by Coxeter in the Circle Limit III, $(4,3,3)$ case, and by Dunham in the case $p = r = 3$. 

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**Figure 5:** A (3,4,3) pattern.  
**Figure 6:** A (3,5,3) pattern.
However, there is still work to be done. In order to generate new patterns in this family of patterns, it would also be useful to be able to transform one \((p, q, r)\) pattern to another one with different values of \(p, q,\) and \(r\). A seemingly difficult problem is to automate the process of coloring a \((p, q, r)\) pattern so that it has the same color along any line of fish and adheres to the map-coloring principle that adjacent fish have different colors. Currently we determine colorings “by hand”, and although it may be possible to program symmetric colorings of any repeating pattern, the requirement that fish along a backbone line be the same color adds an extra degree of difficulty to coloring \((p, q, r)\) patterns.

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References


