A "Circle Limit III" Backbone Arc Formula

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Abstract

M.C. Escher considered *Circle Limit III* to be the most successful of his four "Circle Limit" patterns. Two artistic or mathematical questions have been raised: (1) what angle do the white backbone lines make with the bounding circle, and (2) are other such patterns of fish possible? H.S.M. Coxeter provided an exact expression to the answer the first question, and a 3-parameter family of possible fish patterns was described in Dunham's 2006 Bridges Conference paper. Dunham's 2007 Bridges Conference paper provided a sequence of calculations that determine the intersection angle for any pattern of that family. In this paper, we derive a single expression for that angle, which agrees with Coxeter's expression for the special case of *Circle Limit III*.

1. Introduction

We recall M.C. Escher's hyperbolic pattern *Circle Limit III* by showing a computer rendition of it in Figure 1. Figure 2 shows a pattern of angular fish from the family of *Circle Limit III* patterns, with four fish meeting at at both left and right fin tips. Dunham's 2006 Bridges paper introduced the concept of a 3-parameter family

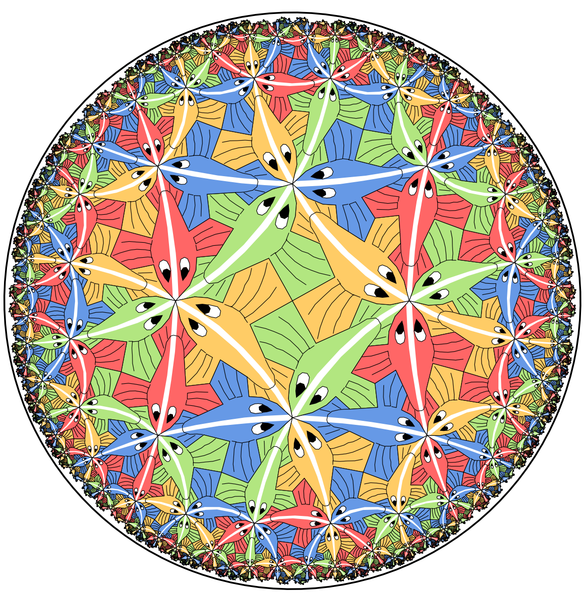

Figure 1: A rendition of Escher's *Circle Limit III*, a (4,3,3) pattern.

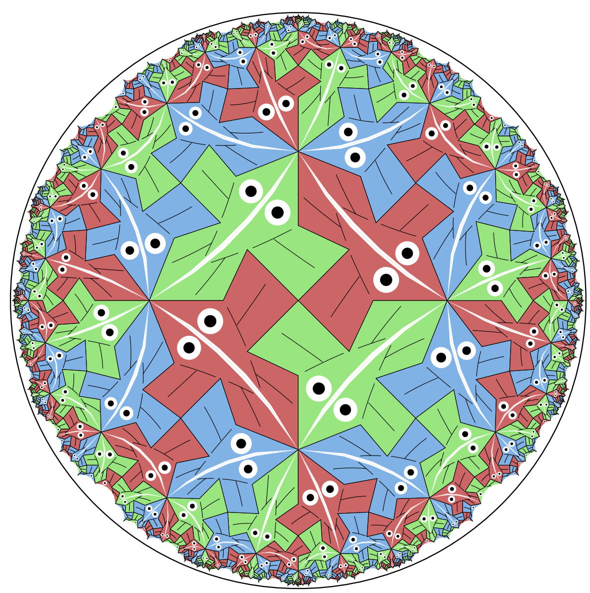

Figure 2: A (4,4,3) pattern of angular fish from the *Circle Limit III* family.

of *Circle Limit III* patterns indexed by the numbers $p, q,$ and r of fish meeting at right fin tips, left fin tips, and noses respectively [5]. Such a pattern was denoted by the triple (p, q, r). Thus *Circle Limit III* and the pattern of Figure 2 would be called $(4, 3, 3)$ and $(4, 4, 3)$ respectively. Following Escher's lead, we required r be odd so that the fish swim head-to-tail, to create the "traffic flow" Escher desired (though the results below also hold if r is even and the fish "kiss"). Also p and q should be greater than or equal to 3 in order that the fin regions define a single common point. Finally, in the style of *Circle Limit III*, we place right fin tips at the center of the bounding circle.

This paper is the culmination of a series of interactions between art and mathematics. This began in early 1958 when the mathematician H.S.M. Coxeter sent Escher a reprint of one of Coxeter's papers that showed a triangle tessellation in the Poincaré circle model of the hyperbolic plane [2]. This tessellation inspired Escher to create his "Circle Limit" patterns. In return, Escher sent Coxeter copies of those patterns. The print *Circle Limit III* later inspired Coxeter to write two papers on the geometry of the backbone lines in that pattern [3, 4]. In the issue of *The Mathematical Intelligencer* containing Coxeter's second paper, an anonymous editor wrote the following caption for the cover, which showed *Circle Limit III:*

> Coxeter's enthusiasm for the gift M.C. Escher gave him, a print of Circle Limit III, is understandable. So is his continuing curiosity. See the articles on pp. 35–46. He has not, however said of what general theory this pattern is a special case. Not as yet. [1]

We are unaware if Coxeter never described such a general theory, but that caption was the inspiration for Dunham to describe an entire family of artistic *Circle Limit III* patterns in [5]. Following Coxeter's example, Dunham presented a sequence of calculations that computed the intersection angle ω between the bounding circle and a backbone line of a general (p, q, r) pattern [6]. However, that result was less than satisfactory since it did not produce a single expression for ω, as Coxeter's papers did for *Circle Limit III*.

In this paper we find such an expression, which has two advantages: (1) it generalizes Coxeter's expression, $\cos(\omega) = (2^{\frac{1}{4}} - 2^{-\frac{1}{4}})/2$ and (2) it can be seen to be antisymmetric in p and q, a fact that is not evident in the calculations of [6]. We note that these patterns are regular when interpreted in terms of hyperbolic geometry, so all backbone lines of a pattern make the same angle with the bounding circle and there is only one angle to determine for any particular pattern.

For background we first review some hyperbolic geometry used in the calculations. Then we give a derivation of the expression for $\cos(\omega)$ using hyperbolic trigonometry, as Coxeter did in his first paper [3]. Finally, we review the results and indicate directions of further research.

2. The Poincaré Disk Model of Hyperbolic Geometry

Escher's "Circle Limit" patterns can be interpreted as repeating patterns of the hyperbolic plane. The hyperbolic plane is a surface of constant negative (Gaussian) curvature. The entire hyperbolic plane has no smooth, isometric (distance preserving) embedding in Euclidean 3-space as was proved by David Hilbert in 1901 [8]. Thus, we must rely on Euclidean *models* of hyperbolic geometry in which distance is measured differently and concepts such as hyperbolic lines have interpretations as Euclidean constructs.

We will be using the Poincaré disk model of hyperbolic geometry, as Escher did in his "Circle Limit" patterns. In the *Poincaré disk model* the points are just the (Euclidean) points within a Euclidean bounding circle, which we will take to be the unit circle in the xy-plane for computational convenience. Hyperbolic lines are represented by circular arcs orthogonal to the bounding circle (including diameters). For example, the backbone lines lie along hyperbolic lines in Figure 2. The disk model is *conformal*: the hyperbolic measure of an angle is the same as its Euclidean measure (the Euclidean measure of the angle between two circular arcs is the measure of the angle between their tangents at the point of intersection; the angle between circular arc and a line segment is formed by the line segment and the tangent to the arc). As a consequence, all fish in a "Circle Limit III" pattern have roughly the same Euclidean shape. However equal hyperbolic distances correspond to ever smaller Euclidean distances toward the edge of the disk. So all the fish are the same (hyperbolic) size in a *Circle Limit III* pattern. The Poincaré disk model appealed to Escher (and has appealed to other artists) since an infinitely repeating pattern could be shown in a bounded area and shapes remained recognizable even for small copies of the motif, due to conformality.

On first glance, it is tempting to guess that the backbone arcs of *Circle Limit III* are hyperbolic lines. Indeed, Escher seemed to think so — in a letter to Coxeter he wrote "... As all these strings of fish shoot up like rockets from infinitely far away, *perpendicularly* [emphasis ours] from the boundary, and fall back again whence they came, not one single component ever reaches the edge. ..." [3]. However, a careful measurement of the backbone arcs of the fish in *Circle Limit III* shows that they make an angle of about 80° with the bounding circle. These arcs are so-called *equidistant curves* in hyperbolic geometry: curves at a constant hyperbolic distance from the hyperbolic line with the same endpoints on the bounding circle, and Escher accurately drew them as such. For every hyperbolic line and a given distance, there are two equidistant curves, called *branches*, at that distance from the line, one each side of the line. In the Poincaré disk model, those two branches are represented by circular arcs making the same (non-right) angle with the bounding circle on either side of the corresponding hyperbolic line. Escher used only one branch for fish backbones from each pair of equidistant curves in *Circle Limit III*.

3. The Calculation of ω

For the derivation of a formula for ω, we generalize Coxeter's method that used hyperbolic trigonometry, as given in [3], but also appeal to the Euclidean representation of hyperbolic objects when needed. We start by noting that in the general (p, q, r) case, we can take the fundamental region to be a quadrilateral with vertex angles $2\pi/p, \pi/r, 2\pi/q,$ and π/r. Such quadrilateral with a pair of congruent opposite angles is sometime called a *kite*. So each (p, q, r) pattern has an associated kite tessellation, the pattern and tessellation being hyperbolic when $1/p + 1/q + 1/r < 1$. Figure 3 below shows how the *Circle Limit III* pattern is related to its kite tessellation.

For the derivation of the formula for ω we start by assuming that $p < q$ (unlike *Circle Limit III*), and show that case in Figure 4 — specifically the kite tessellation for a $(4, 8, 3)$ pattern. We will indicate below why the derivation also works for $p > q$. Of course if $p = q$, the backbone lines are hyperbolic lines and $\omega = 90^\circ$, an example of which is the $(4, 4, 3)$ pattern of Figure 2.

In Figure 4, the kite to the right of center is labeled $PRQR'$, with the p-fold point P at the origin, the q-fold point Q on the positive x-axis, and the $2r$-fold points R and R' above and below the x-axis. The x-axis is an the axis of symmetry of $PRQR'$, and of the entire kite tessellation. Another axis of reflection symmetry of the kite tessellation is the hyperbolic line m through R and perpendicular to the hyperbolic bisector of the angle $\angle PRQ$ (in the Euclidean terms of the Poincaré model, the tangent to the circular arc representing m is perpendicular to the bisector of the angle formed by RP and the tangent to the arc representing RQ). In Figure 4, RQ' forms part of m.

Next, in the Poincaré model, consider the Euclidean circular arc through R and R' that bisects (in the Euclidean sense) $\angle PRQ$ at R and $\angle PR'Q$ at R'. This arc is determined by requiring it pass through R and R', and be tangent to the Euclidean bisector of $\angle PRQ$ (and thus it also tangent to the bisector of $\angle PRQ$ by symmetry across the x-axis). This circular arc can be extended by reflection across m, since the tangent to the arc matches the tangent of its reflection at R. In fact the arc can be extended both directions to a circular arc b within the Poincaré disc. In a (p, q, r) fish pattern, the arc b is the backbone arc we wish to analyze.

We now show that the hyperbolic line ℓ associated to b lies to the left of b and to the right of P, and thus intersects the segment PR between P and R at M, as shown in Figure 4. Note that b is symmetric across the x-axis by its definition, and therefore so is ℓ. First, we show that within $PRQR'$, b lies to the right of the hyperbolic line t determined by RR' and thus to the left of t outside of $PRQR'$. Also, since t is determined by R and R', as a Euclidean circular arc, its endpoints on the bounding circle lie to the right of the vertical chord through R (and R'). Let T be the intersection of t and PQ, so by symmetry t makes a right angle with PQ. Then by one of the formulas for hyperbolic right triangles [7, page 403], applied to PRT and QRT, $\cos(\pi/p)/\sin(\angle PRT) = \cosh(RT) = \cos(\pi/q)/\sin(\angle QRT)$, so $\sin(\angle PRT) < \sin(\angle QRT)$ since $q > p$, and thus $\angle PRT < \angle QRT$, showing that b lies to the right of t inside $PRQR'$. The tangent to b at R makes an angle of $\frac{\pi}{p} + \frac{\pi}{2r}$ with the x-axis. The largest this angle can be is $\frac{\pi}{2}$ which occurs when $p = r = 3$, so that the Euclidean x-coordinate of a point on b above R (or below R') is greater than or equal to the x-coordinate of R. Thus the endpoints of b on the bounding circle lie between the endpoints of t and the endpoints of the vertical chord through R. Since the endpoints of b are to the right of the vertical chord (and b is symmetric across the x-axis), ℓ must stay to the right of the y-axis. Also, as shown above, as an equidistant curve b "bulges" to the right (since it is to the right of t inside $PRQR'$) — that is it is turning to the left as we traverse it from bottom to top. Thus, b's associated hyperbolic line (the orthogonal circular arc in the Poincaré model with the same endpoints) ℓ lies to the left of it in Figure 4, and hence ℓ intersects PR as claimed.

Finally, the points L, M, and N are determined as follows: L is the intersection of ℓ and PQ, M is the intersection of ℓ and PR, and N is the foot of the perpendicular from R to ℓ.

For the case of $p > q$, we apply transformations to reduce that case to the $p < q$ case (by relabeling). First hyperbolically slide the kite tessellation shown in Figure 4 to the left along the x-axis, putting the transformed Q at the origin. Then reflect across the y-axis to obtain the configuration of Figure 4 with P and Q interchanged (and the roles of p and q will be interchanged too). As in the previous argument, b intersects the segment between R and the the transformed Q at the origin at the point M. Thus the following derivation also applies to the case $p > q$ if we interchange P and Q, and p and q in the calculations.

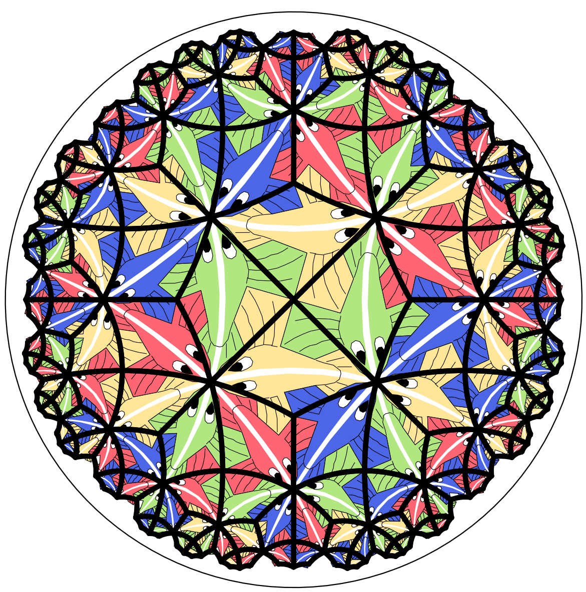

Figure 3: The *Circle Limit III* pattern with its kite tessellation superimposed.

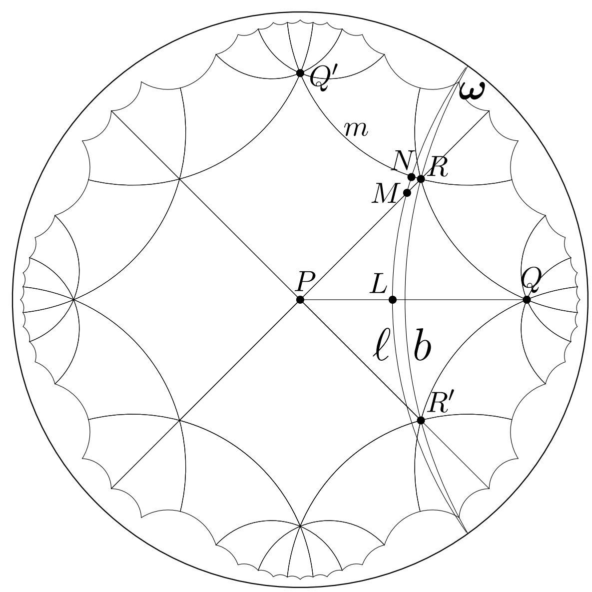

Figure 4: The diagram for the hyperbolic trigonometry calculation of ω.

Our goal is to compute the angle of intersection ω between b and the bounding circle. By a well-known formula [7, page 402], ω is given by:

$$\cos(\omega) = \tanh(RN)$$

Since RNM is a right triangle, by one of the formulas for hyperbolic right triangles [7, page 403], $\tanh(RN)$ is related to $\tanh(RM)$ by: $\tanh(RN) = \cos(\angle NRM)\tanh(RM)$. But $\angle NRM = \frac{\pi}{2} - \frac{\pi}{2r}$ since the equidistant curve bisects $\angle PRQ = \frac{\pi}{r}$ (in the Euclidean sense by construction and thus in the hyperbolic sense by conformality of the Poincaré model). Thus $\cos(\angle NRM) = \cos(\frac{\pi}{2} - \frac{\pi}{2r}) = \sin(\frac{\pi}{2r})$ and

$$\tanh(RN) = \sin(\frac{\pi}{2r})\tanh(RM) \tag{1}$$

so that our task is reduced to calculating $\tanh(RM)$.

In order to calculate $\tanh(RM)$, we note that as hyperbolic distances $RP = RM + MP$, so eliminating MP from this equation will relate RM to RP, which we can find. By the subtraction formula for $\cosh$, we have $\cosh(MP) = \cosh(RP - RM) = \cosh(RP)\cosh(RM) - \sinh(RP)\sinh(RM)$. Dividing through by $\cosh(RM)$ gives:

$$\frac{\cosh(MP)}{\cosh(RM)} = \cosh(RP) - \sinh(RP)\tanh(RM) \tag{2}$$

Also, another formula (Formula (11), page 403 of [7]) for hyperbolic right triangles applied to PML and RMN gives:

$$\cosh(MP) = \cot(\angle PML)\cot(\tfrac{\pi}{p}) \quad \text{and}$$
$$\cosh(RM) = \cot(\angle RMN)\cot(\tfrac{\pi}{2} - \tfrac{\pi}{2r})$$

Now as opposite angles, $\angle PML = \angle RMN$, so dividing the first equation by the second gives:

$$\frac{\cosh(MP)}{\cosh(RM)} = \cot(\frac{\pi}{p})\cot(\frac{\pi}{2r}) \tag{3}$$

Equating the right sides of (2) and (3) gives: $\cosh(RP) - \sinh(RP)\tanh(RM) = \cot(\frac{\pi}{p})\cot(\frac{\pi}{2r})$, which can be solved for $\tanh(RM)$ in terms of RP:

$$\tanh(RM) = \frac{\cosh(RP) - \cot(\frac{\pi}{p})\cot(\frac{\pi}{2r})}{\sinh(RP)} \tag{4}$$

Thus we have reduced the problem to finding $\cosh(RP)$ and $\sinh(RP)$. One of the hyperbolic laws of cosines [7, page 406] applied to QPR gives: $\cosh(RP) = (\cos(\frac{\pi}{p})\cos(\frac{\pi}{r}) + \cos(\frac{\pi}{q}))/\sin(\frac{\pi}{p})\sin(\frac{\pi}{r})$. We can calculate $\sinh(RP)$ from this by the formula $\sinh^2 = \cosh^2 - 1$. Substituting these values of $\cosh(RP)$ and $\sinh(RP)$ into equation (4), and inserting that result into equation (1) gives the final result:

$$\cos(\omega) = \frac{\sin(\frac{\pi}{2r})\ (\cos(\frac{\pi}{q}) - \cos(\frac{\pi}{p}))}{\sqrt{\cos(\frac{\pi}{p})^2 + \cos(\frac{\pi}{q})^2 + \cos(\frac{\pi}{r})^2 + 2\cos(\frac{\pi}{p})\cos(\frac{\pi}{q})\cos(\frac{\pi}{r}) - 1}}$$

which is antisymmetric in p and q, as we would expect. For unrestricted p and q we should replace the factor $(\cos(\frac{\pi}{q}) - \cos(\frac{\pi}{p}))$ by its absolute value.

Letting $q = r = 3$ and doing some algebraic manipulation yields the same formula given in [5] for that special case:

$$\cos(\omega) = \frac{1}{2}\sqrt{1 - \frac{3}{4\cos^2(\frac{\pi}{2p})}}$$

This expression further reduces to Coxeter's expression for the *Circle Limit III* case when $p = 4$.

Some algebraic manipulation also yields a formula for $\cot(\omega)$:

$$\cot(\omega) = \frac{\tan(\frac{\pi}{2r})(\cos(\frac{\pi}{q}) - \cos(\frac{\pi}{p}))}{\sqrt{(\cos(\frac{\pi}{p}) + \cos(\frac{\pi}{q}))^2 + 2\cos(\frac{\pi}{r}) - 2}}$$

Again, we should replace the factor $(\cos(\frac{\pi}{q}) - \cos(\frac{\pi}{p}))$ by its absolute value if $p > q$.

4. Conclusions and Future Work

For any (p, q, r) pattern, we have given a formula for the angle ω an equidistant "backbone" curve makes with the bounding circle. This formula agrees with previously obtained results by Coxeter in the *Circle Limit III*, (4,3,3) case, and by Dunham in the case $p = r = 3$.

However, there is still work to be done. In order to generate new patterns in this family of patterns, it would also be useful to be able to transform one (p, q, r) pattern to another one with different values of p, q, and r. A seemingly difficult problem is to automate the process of coloring a (p, q, r) pattern so that it has the same color along any line of fish and adheres to the map-coloring principle that adjacent fish have different colors. Currently we determine colorings "by hand", and although it may be possible to program symmetric colorings of any repeating pattern, the requirement that fish along a backbone line be the same color adds an extra degree of difficulty to coloring (p, q, r) patterns.

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