The Symmetry of “Circle Limit IV” and Related Patterns

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Abstract

M.C. Escher’s print *Circle Limit IV* is the last of his four “Circle Limit” patterns. There are two questions one can ask about the symmetry of *Circle Limit IV.* First, what is the correct orientation in which to display the print? Second, what is the symmetry group of the pattern? We answer those questions and show some new patterns related to *Circle Limit IV.*

1. Introduction

M.C. Escher’s print *Circle Limit IV* [1] was the last of his four “Circle Limit” patterns. Figure 1 is an early computer generated pattern inspired by that print. It has more symmetry than Escher’s pattern, which is

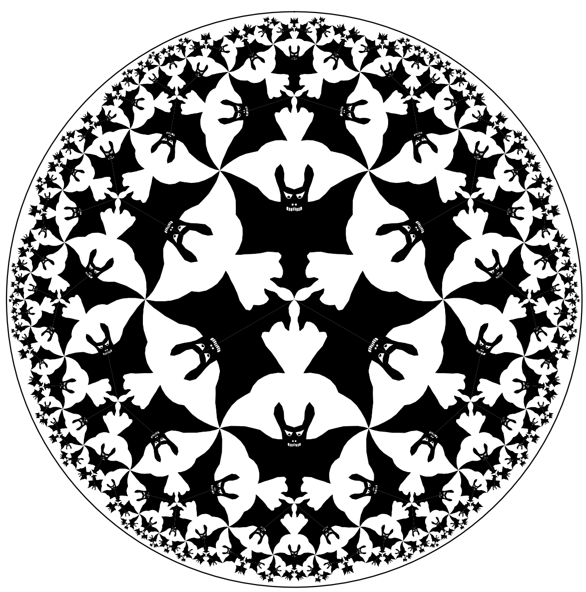

Figure 1: A pattern based on M.C. Escher’s print *Circle Limit IV.*

shown in Figure 2 below with Escher’s initials circled. The angels and devils motif was also the only one that Escher realized in each of the classical geometries: Euclidean, spherical, and hyperbolic.

The *Circle Limit IV* pattern seems to have confused some people in that it has appeared rotated from its correct orientation. In the next section we will explain how to determine the correct orientation. In the following section, we review the Poincaré disk model of hyperbolic geometry and use it to analyze the

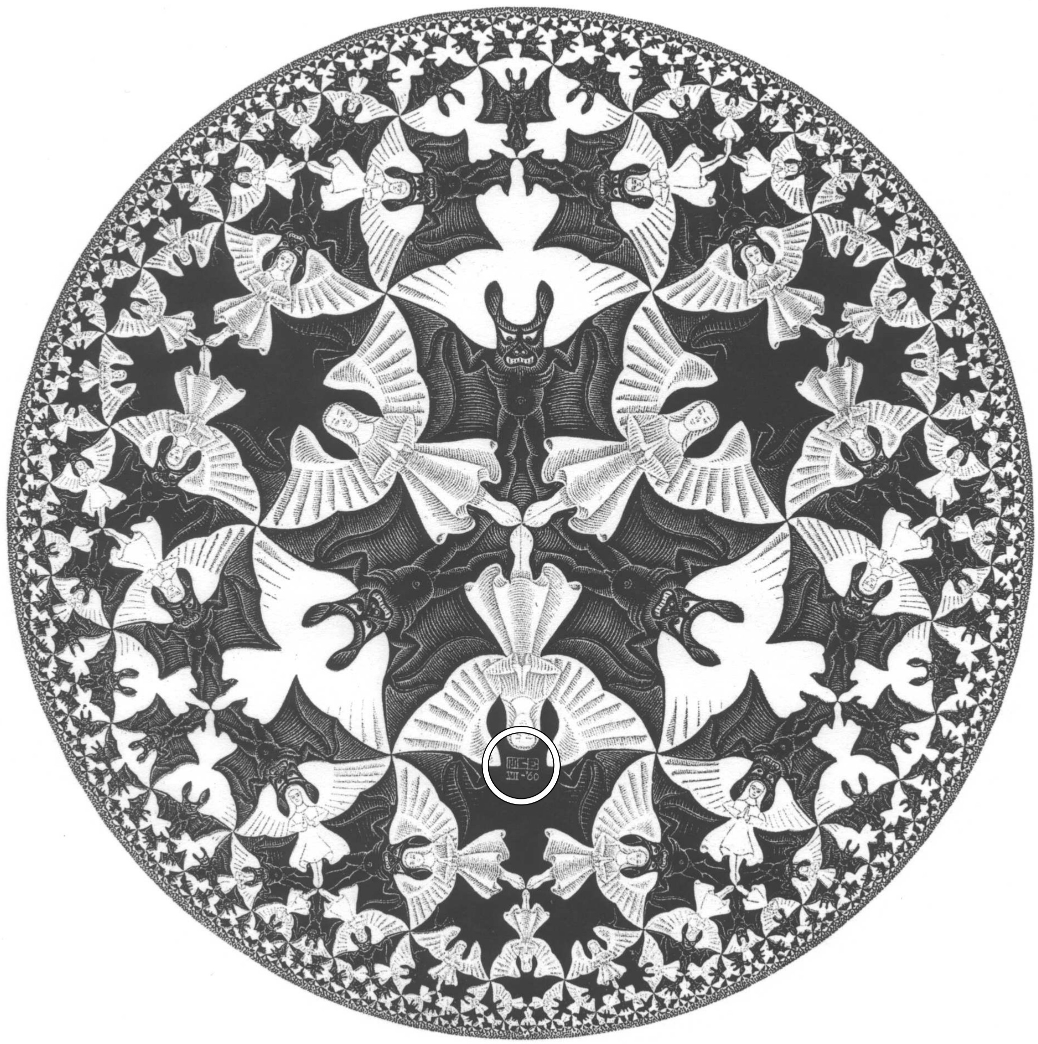

Figure 2: M.C. Escher's print *Circle Limit IV* with his initials circled.

symmetry group of *Circle Limit IV,* which can also be a source of confusion — for many years I thought the symmetry group was different from what it actually is. We also show new patterns related to *Circle Limit IV,* including one with my imagined symmetry group.

2. The Correct Orientation of *Circle Limit IV*

Recently, four books have been published that show *Circle Limit IV* with incorrect orientations, on pages 224 and 166 of [1] and [8] respectively, and on the covers of [6] and [7]. Interestingly, the previous edition of [6] also used *Circle Limit IV* on its cover — correctly oriented [5].

However, the mis-orientation of *Circle Limit IV* is not just a recent phenomenon. In one of the definitive books on Escher, *M.C. Escher, His Life and Complete Graphic Work* [9], a large version of the print is correctly oriented on page 98, yet a small view of it is incorrectly oriented on page 322.

How is the correct orientation determined? In fairness to the books mentioned, it is a bit subtle. It is determined by Escher's initials (`MCE` in block letters) and the date, which he placed on many of his prints. The problem is made more difficult in some reproductions by very faint intials. Figure 3 shows those initials and date as they appear on *Circle Limit IV.* Escher dated his prints by month and year. At first he used

Figure 3: Escher's initials and date on *Circle Limit IV.*

Arabic numerals for the months, but switched to Roman numerals in 1937. Thus, *Circle Limit IV* is dated `VII-'60`, indicating that it was completed in July, 1960. Escher's block letter initials look almost the same upside down. However, the requirement that the date be upright establishes the correct orientation of the print. There are three each of fully developed angels and devils in the center. Outward from the center and adjacent to each angel, is a devil shown mostly in black background. The initials and date appear in the head of the bottom "background" devil.

Here is a question for the reader. To print a copy of *Circle Limit IV,* did Escher use two black woodblocks — one with the initials (for the bottom) and one without initials (for the top left and for the top right)? If not, how did he get by with just one woodblock? Here is the answer.[2]

3. The Symmetry Group of *Circle Limit IV*

Though Escher considered his "Circle Limit" patterns to be just patterns with a circular limit, they can also be viewed as repeating patterns in the Poincaré disk model of hyperbolic geometry. The points of that model are just the (Euclidean) points within a Euclidean bounding circle. Hyperbolic lines are represented by circular arcs orthogonal to the bounding circle (including diameters). Figure 4 shows some hyperbolic lines over an angels and devils pattern. The measure of an angle between hyperbolic lines is the same as the Euclidean measure of the angle between their arcs; however, equal hyperbolic distances correspond to ever smaller Euclidean distances toward the edge of the disk. Thus all angels are the same hyperbolic size, as are all devils. A hyperbolic reflection across a hyperbolic line is represented by inversion in the circular arc representing that line (reflection across a diameter is just a Euclidean reflection). As in Euclidean geometry, successive hyperbolic reflections across two intersecting lines results in a hyperbolic rotation about the intersection point by twice the angle between the lines.

Using this hyperbolic interpretation, we can analyze the symmetries of the *Circle Limit IV* pattern. If we only look at the outlines of the angels and devils, the reflection axes of the pattern coincide with

[2]There was just one woodblock — with initials. After making three impressions with that woodblock, Escher took a small plain block of wood slightly larger than the initials, inked it, and blacked out two of the initials/dates.

bilateral symmetry axes of the angels and devils. These axes divide the hyperbolic plane into hyperbolically congruent "squares", that is equilateral, equi-angular quadrilaterals, as in Figure 4. The outlines also have 4-fold rotation centers at the centers of the squares where wing tips of the angels and devils meet. Thus, the outline pattern has symmetry group $4 * 3$ in orbifold notation, or $[4^+, 6]$ in H.S.M. Coxeter's notation [2]. Escher's first angels and devils pattern, the Euclidean periodic Drawing 45 (page 150 of [10]) created in 1941, has symmetry group $4 * 2$ (or $[4^+, 4]$ in Coxeter's notation).

However, in *Circle Limit IV* some of the angels and devils are have been filled in with interior detail, others are just background with no interior detail, and yet others are partially filled in. If this were done in such a way that maintained the reflection symmetries across the edges of the "squares" mentioned above, yet destroyed any rotational symmetry about their centers, the symmetry group would be $*3333$. Such a pattern is shown in Figure 5 in which different colors destroy the rotational symmetry. In fact I erroneously

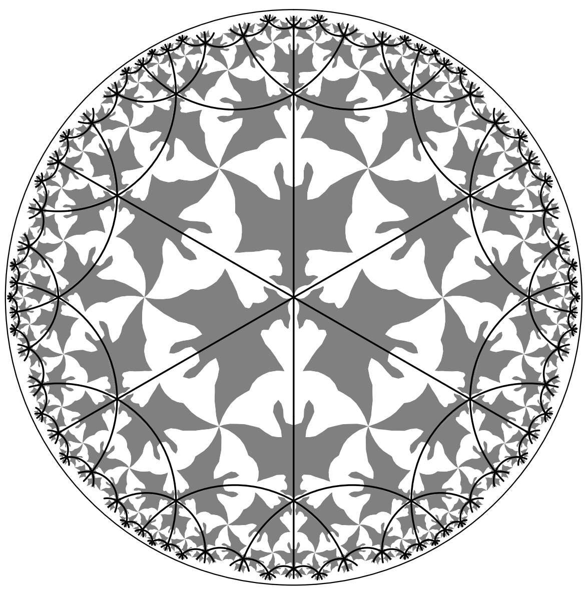

Figure 4: The hyperbolic plane divided into "squares" by the bilateral symmetry axes of the angels and devils.

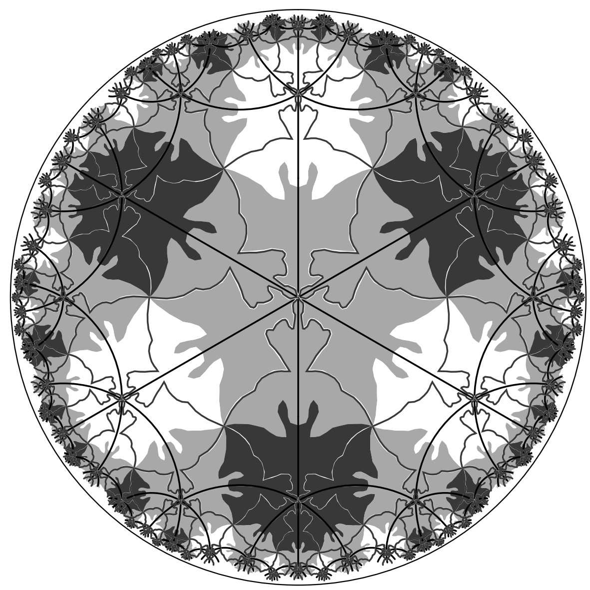

Figure 5: A pattern based on *Circle Limit IV* with symmetry group $*3333$

thought that the symmetry group of *Circle Limit IV* was $*3333$ until very recently.

What is the actual symmetry group of *Circle Limit IV*? First, we ignore Escher's initials and date, otherwise the only symmetry would be the identity. Then there are three reflection axes through the center of the circle. It turns out that those reflections and the rotations they generate (by ± 120 degrees and the identity) are the only symmetries. Careful inspection reveals that the devils are fully developed and the angels are just background on the upper end of the vertical reflection axis and the reverse is true at the lower end of that axis. The same switching of roles occurs on the other two reflection axes. Thus the symmetry group is just D_3, the Euclidean dihedral group about the center of the circle.

This was Escher's intention as he explained on page 10 of [3]:

> Here too, we have the components diminishing in size as they move outwards. The six largest (three white angels and three black devils) are arranged about the centre and radiate from it. The disc is divided into six sections in which, turn and turn about, the angels on a black background and then the devils on a white one gain the upper hand. In this way, heaven and hell change place six times. In the intermediate, "earthly" stages, they are equivalent.

Escher had previously used this idea in 1942 when he carved a maple ball with angels and devils on its surface. Here is his description of it in a letter to the collector C.V.S. Roosevelt ([10] page 245):

> It has two poles and an equator. One pole represents "heaven," with only white angels on a black background, which I carved much deeper than the angel figures. The other pole shows "hell," with only black devils on a deeply carved white background. At the equator both angels and devils are visible and equivalent, carved at the same sphere-level.

So, due to the relative indentations of the angels and devils, the symmetry group of the carved ball is just D_2, generated by orthogonal reflections through the poles. There is a picture showing this relative indentation on page 244 of [10]. If all the angels and devils were on the same sphere-level on the ball, the symmetry group would be 3*2 in orbifold notation (or $[3^+, 4]$ in Coxeter's notation). Such balls (without the relative indentation of Escher's original) are available from the M.C. Escher web site [4].

So Escher's first angels and devils pattern was the Euclidean Notebook Drawing 45 done in 1941, and was followed by the carved sphere in 1942. Almost 20 years passed before he completed the classical geometry series with his hyperbolic *Circle Limit IV.*

4. Other Angels and Devils Patterns.

We have already seen in Figures 4 and 5 two hyperbolic patterns with more symmetries than *Circle Limit IV.* If we left out the interior details of all angels and devils as in Figure 4, or filled in those details for all the figures as in Figure 6 below, there would be 4-fold rotations about the meeting points of wing tips in addition to the reflections in the sides of the "squares" of Figure 4. As mentioned above, the symmetry group would be $4 * 3$ or $[4^+, 6]$ in Coxeter's notation.

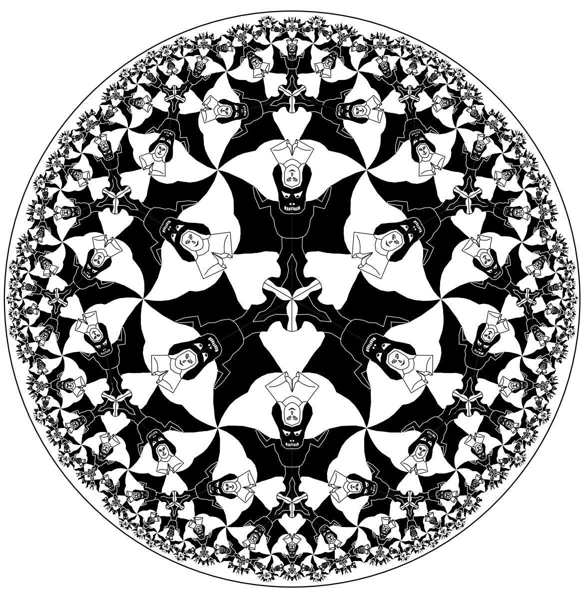

Figure 6: An angels and devils pattern with the interior details filled in for all the figures.

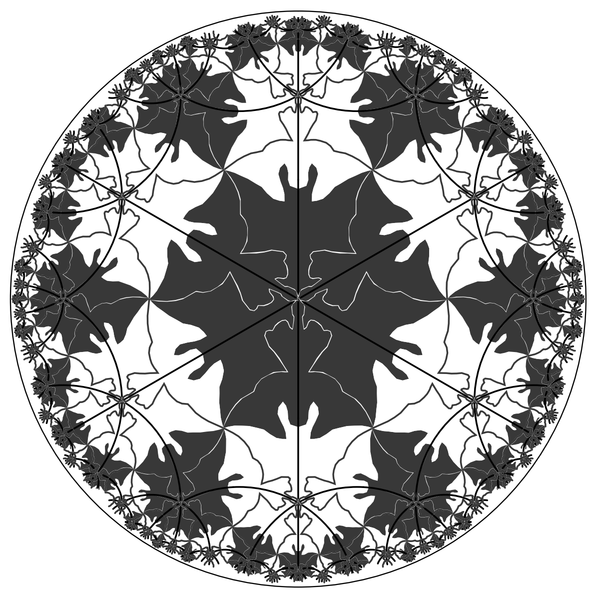

Figure 7: A pattern based on *Circle Limit IV* with symmetry group $2 * 33$

In Figure 5, following Escher's lead, we can interpret the black and white areas as places where the devils and angels dominate respectively. The gray areas represent "earthly" places where the angels and devils hold equal sway. As mentioned above, by taking into account the different colors, the symmetry group does not have rotational symmetries about the centers of the "squares" and is therefore $*3333$.

Figure 7 shows a pattern with 2-fold rotational symmetry about the centers of the "squares" of Figure 3, and thus has intermediate symmetry between that of Figures 4 and 5. The symmetry group of this pattern is $2 * 33$ or $cmm_{3,3}$ using a notation suggested by Coxeter. Again, following Escher's lead, we may interpret the white areas as places where the angels dominate, and the dark areas as places where the devils dominate. In this case there are no intermediate "earthly" regions.

5. Conclusions and Future Work

We have explained how to determine the correct orientation for *Circle Limit IV* and analyzed its symmetries. We have also shown angels and devils patterns with other symmetries.

For future work it would be useful to extend the current version of our hyperbolic pattern program so that it can draw patterns with symmetries like *Circle Limit IV* — it can currently only draw patterns that repeat the same way in all directions. We are just starting the process of converting this C program, which is not very portable, to Java so that others may use it.

Also, it would be interesting to use 3D printing technology to create a "heaven and hell" disk pattern with areas where angels are raised and devils are recessed, areas where the opposite is true, and intermediate areas where neither is dominant, in the manner of Escher's carved maple ball. It is likely that the techniques developed by Yen and Séquin for spherical geometry [11] would be useful for this task.

Acknowledgments

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