# Hyperbolic Truchet Tilings

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**Abstract**

About 300 years ago Sébastien Truchet systematically studied patterns that could be formed from square tiles that were divided by a diagonal into a white triangle and a black triangle. Other pattern creators have been inspired by him to make Truchet-like tilings composed of circular arcs and other motifs. These tilings are all based on Euclidean tessellations, usually the tiling by squares. In this paper we extend the concept of a Truchet tiling to the hyperbolic plane and show some sample patterns.

## 1. Introduction

The French Dominican Father Sébastien Truchet systematically studied Euclidean patterns that could be formed by using square tiles that are divided into two isosceles right triangles, one white and one black. The goal of this paper is to extend Truchet's ideas to the hyperbolic plane. Figure 1 is a hyperbolic Truchet pattern of such "squares".

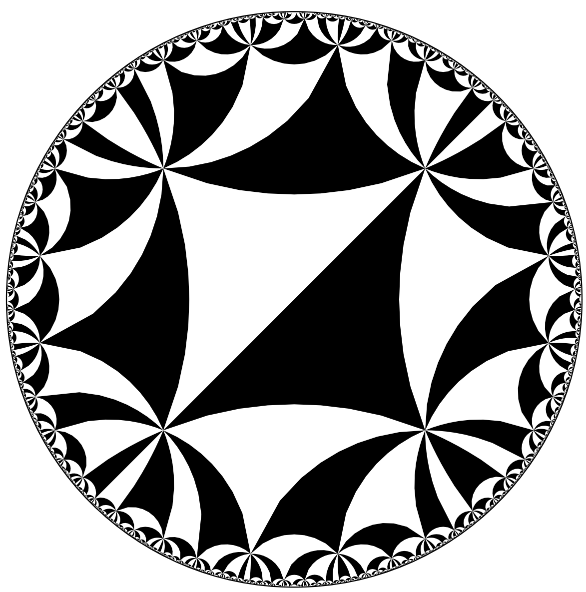


**Figure 1:** A hyperbolic Truchet tiling based on the $\{4,8\}$ grid.

We begin with a brief biography of Truchet. Then we review hyperbolic geometry and regular tessellations, upon which our tilings are based. Next we discuss hyperbolic patterns based on "square" grids, which are most directly related to Truchet's tilings. Then we show how to put divided squares together to form more general patterns, as Truchet did. Other pattern creators have designed Truchet-like tilings using arcs and other figures, which we also extend to the hyperbolic plane. Finally, we indicate possible directions of further research.

## 2. A Short Biography of Truchet

Sébastien Truchet was born in Lyon, France in 1657, becoming a Dominican Father as an adult. In addition to Truchet tilings, he is probably best known for his work in typography and the "Roman Du Roi" typeface that is an ancestor of "Times New Roman". However, Truchet also designed many French canals and invented sundials, weapons, and special implements for transporting trees in the Versailles gardens (from Wikipedia [9]). He published his seminal work on tilings "Memoir sur les Combinaisons" in the *Memoires de l'Académie Royale des Sciences* in 1704 [8]. This treatise has been translated into English (by Pauline Bouchet), with some history and comments on Truchet's theory (by Cyril Smith) in a *Leonardo* paper which reproduces Truchet's figures [7]. Truchet died February 5, 1729.

## 3. Hyperbolic Geometry and Regular Tessellations

Truchet used the familiar Euclidean square tessellation for his tiling patterns. Others who have created Truchet-like tilings have also used the other two regular Euclidean tessellations, by equilateral triangles and by regular hexagons, as a basis for their tilings. We show how to extend Truchet tilings to the hyperbolic plane, which has an infinite number of regular tessellations.

Since there is no smooth embedding of the hyperbolic plane into Euclidean 3-space, we must rely on models of hyperbolic geometry. Specifically, we use the *Poincaré disk* model. The hyperbolic points in this model are represented by Euclidean points within a bounding circle. Hyperbolic lines are represented by (Euclidean) circular arcs orthogonal to the bounding circle (including diameters). The hyperbolic measure of an angle is the same as its Euclidean measure in the disk model (i.e the model is *conformal*), but equal hyperbolic distances correspond to ever-smaller Euclidean distances as figures approach the edge of the disk, as is shown in Figure 1.

There is a *regular tessellation*, $\{p, q\}$, of the hyperbolic plane by regular $p$-sided polygons, which we call *p-gons*, with $q$ of them meeting at each vertex, provided $(p-2)(q-2) > 4$. If $(p-2)(q-2) = 4$, one obtains one of the three Euclidean tessellations, the square grid $\{4, 4\}$, the hexagon grid $\{6, 3\}$, and the equilateral triangle grid $\{3, 6\}$. Figure 2 shows the regular hyperbolic tessellation $\{4, 6\}$, and Figure 3 shows the regular tessellation $\{4, 8\}$ superimposed on the Figure 1 pattern.

## 4. Hyperbolic Truchet Patterns Based on a Divided Square

Perhaps the simplest Euclidean Truchet tiling is the one created by translations of the *basic square* — a square divided into two isosceles right triangles by a diagonal, one triangle being white and the other black, as shown in Figure 4 on the left. We obtain another Truchet tiling by rotating the basic squares about its vertices, so that the $45°$ vertices all meet at alternate vertices of the $\{4, 4\}$ grid, as shown on the right of Figure 4. These are patterns A and D of Truchet's *Memoire* [7] and the only ones adhering to the map-coloring principle: no triangles of the same color share and edge.

In the hyperbolic plane, if one translates a decorated 4-gon of a $\{4, q\}$ to the next $4$-gon to the right, then upward, then to the left, etc., in a counter-clockwise manner about a $q$-vertex, the decorated $4$-gon will return to its orginal position after $q$ steps. However, the decoration will be rotated by an angle of $q\pi/2$. This phenomenon is called *holonomy*. Thus to obtain a consistent tiling (i.e. with a large symmetry group), using a decorated $4$-gon, $q\pi/2$ must be a multiple of $2\pi$, so $q$ must be divisible by 4. Figure 1, with $q = 8$, is the "smallest" hyperbolic Truchet tiling related to Figure 4a. Figure 5 shows the next example with $q = 12$.

If we apply the rotation construction in the hyperbolic case, the base angles of the black and white isosceles triangles all meet at some of the vertices of $\{4, q\}$ and the vertex angles of the isosceles triangles

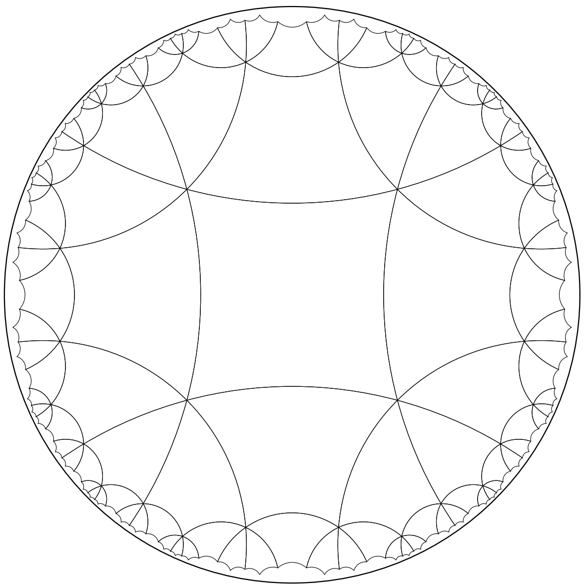


**Figure 2:** The $\{4,6\}$ tessellation

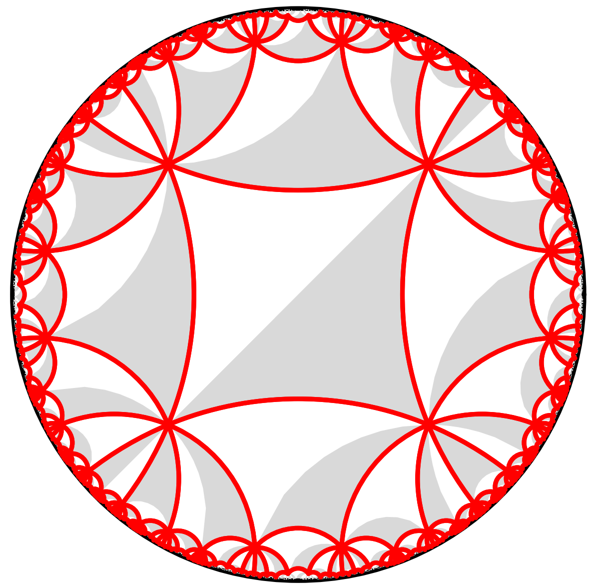
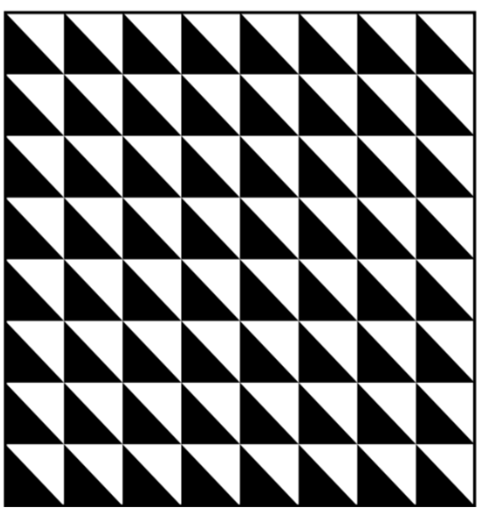


**Figure 3:** The $\{4, 8\}$ superimposed on the Figure 1 pattern.



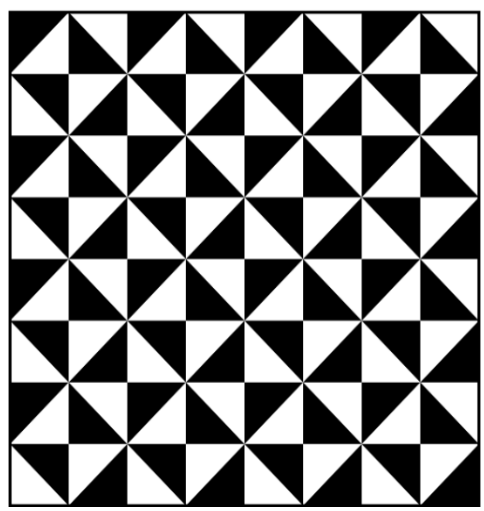


**Figure 4:** (a) A "translation" Truchet tiling, (b) A "rotation" Truchet tiling.

meet at the other vertices of $\{4, q\}$. In this case $q$ must be even to satisfy the map-coloring principle. Figure 6 shows the pattern when $q = 6$. Figures 7 and 8, respectively, illustrate the differences between the

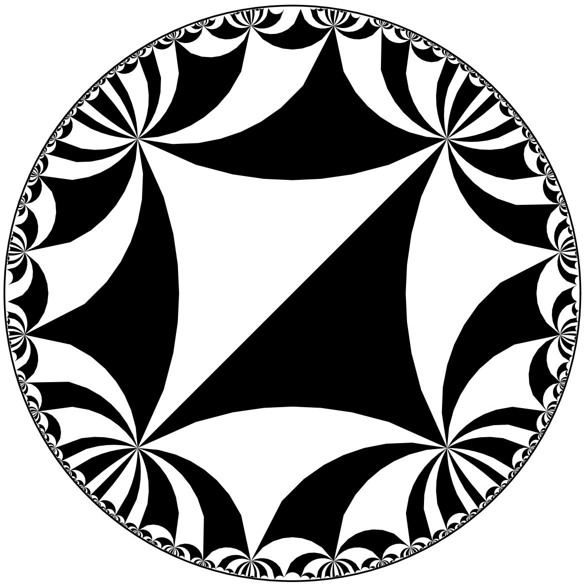


**Figure 5:** A "translation" Truchet pattern based on the $\{4, 12\}$ tessellation.

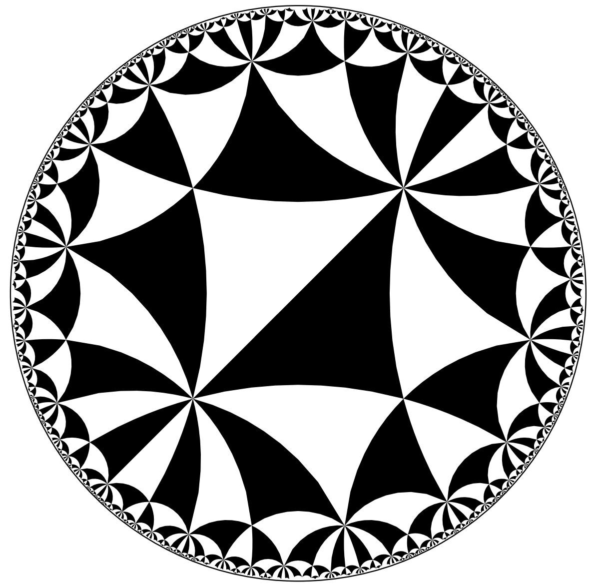


**Figure 6:** A "rotation" Truchet pattern based on the $\{4, 6\}$ tessellation.

hyperbolic "translation" and "rotation" patterns by placing small circles at the vertex angles of the black and white isosceles triangles (both based on the $\{4, 8\}$ tessellation).

In his classification of patterns, Truchet did not restrict himself to the map-coloring principle, allowing triangles of the same color to share an edge. Figure 9 shows such a pattern, which mixes "translation" and "rotation" edge matchings. It is pattern F in Truchet's Plate 1 of his Memoire [7]. Figure 10 shows a hyperbolic version of this pattern based on the $\{4, 6\}$ tessellation, which shows large, alternately colored hexagons (since $q = 6$) instead of the squares of pattern F.

## 5. Hyperbolic Truchet Tilings with Multiple Triangles per $p$-gon

In his analysis, Truchet considered all rectangles composed of two basic squares (each divided into a black and white triangle). The first square could be given one of four orientations, as could the second square, and the second square could be placed adjacent to each of the four edges of the first square. Thus there are 64 different rectangles. But many pairs of rectangles are equivalent by rotation, yielding 10 inequivalent rectangles, as shown in Truchet's Table 1 [7]. There are six inequivalent rectangles if reflections are allowed, but Truchet did not consider them. Truchet showed 24 patterns, six on each of Plates 1, 2, 3, and 4 of his *Memoire*, that could be built from his rectangles. The patterns were labeled with the letters A through Z and &, omitting J, K, and W.

While it is natural to consider Euclidean tilings by rectangles, "rectangles" (quadrilaterals with congruent opposite sides) do not tile the hyperbolic plane so easily. So, instead, we consider $p$-gons (of a $\{p, q\}$) divided into black and white $\frac{\pi}{p}$ - $\frac{\pi}{q}$ - $\frac{\pi}{2}$ *basic triangles* by radii and apothems. Figure 11 shows such a pattern based on the $\{4, 6\}$ tessellation with alternating black and white triangles — probably a better hyperbolic analog to Truchet's pattern A of Plate 1 than Figures 1 and 5 above. Figure 12 shows a $\{4, 6\}$ pattern with pairs of black and white triangles adjacent across apothems, which is analogous to Truchet's pattern E of Plate 1.

Figure 13 is also based on $\{4, 6\}$ and uses the same triangles within the $4$-gon as Figure 12, but extended differently across the $4$-gon edges. Like Figure 10, it is analogous to Truchet's Pattern F of Plate 1. Figure

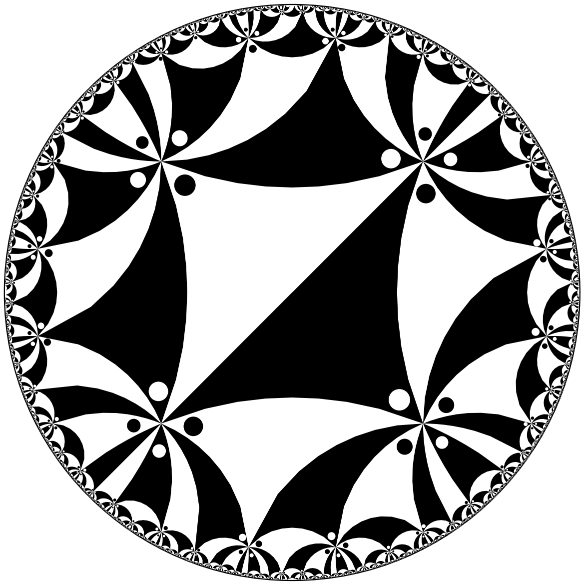


**Figure 7:** The $\{4, 8\}$ "translation" pattern with marked vertex angles of the isosceles triangles.

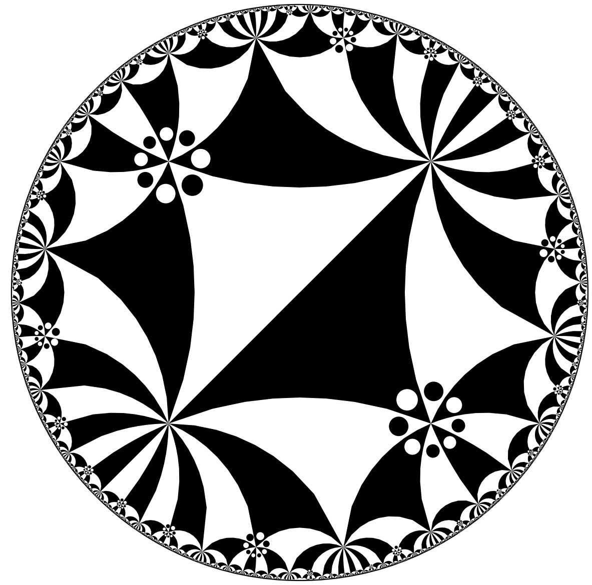


**Figure 8:** The $\{4, 8\}$ "rotation" pattern with marked vertex angles of the isosceles triangles.

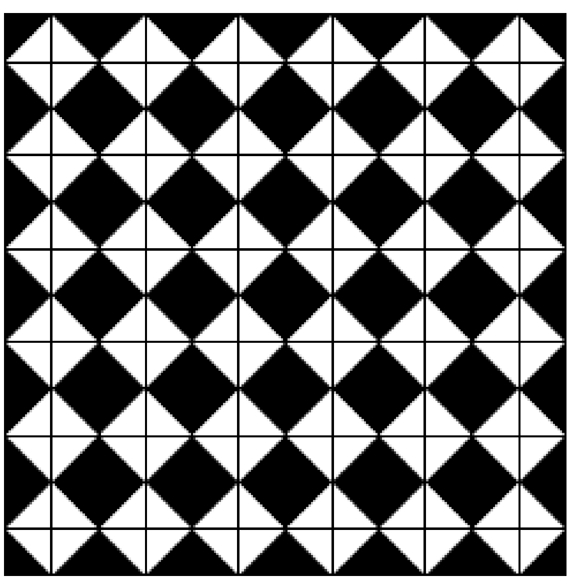


**Figure 9:** Truchet's pattern F, which does not adhere to the map-coloring principle.

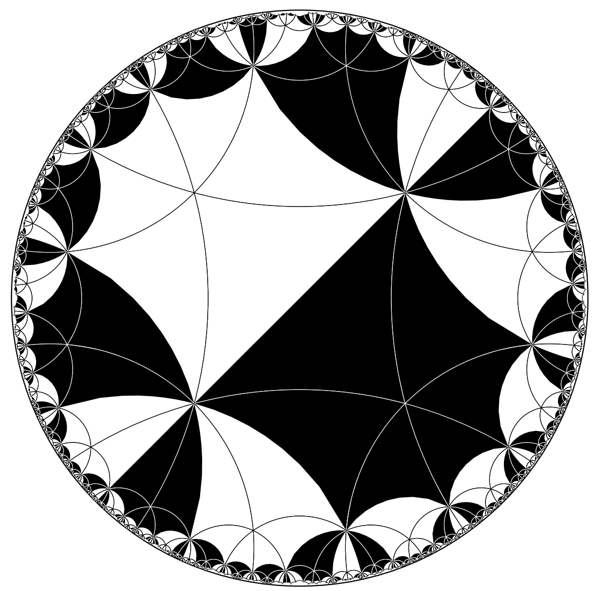


**Figure 10:** A hyperbolic Truchet pattern corresponding to Truchet's pattern F.

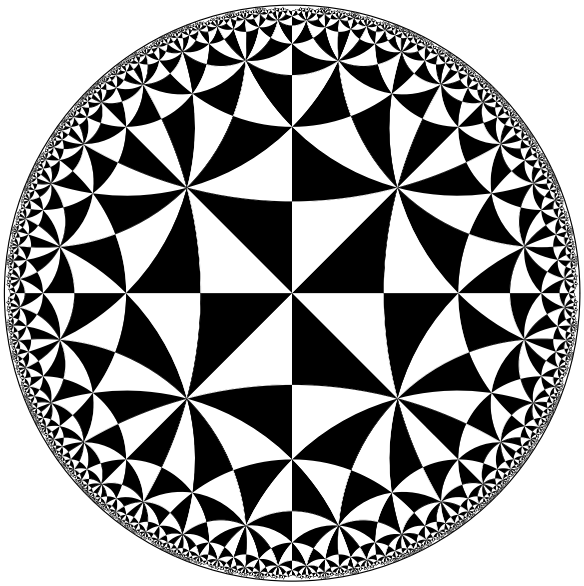


**Figure 11:** A pattern generated by alternate black and white triangles in a $4$-gon.

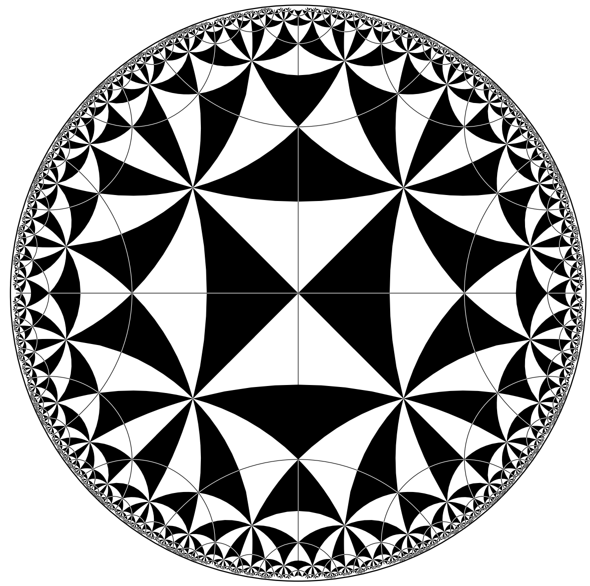


**Figure 12:** A pattern generated by paired black and white triangles in a $4$-gon.

14 is more complicated, having two adjacent and two nonadjacent basic black triangles within a $4$-gon, and the same for basic white triangles. It is based on $\{4, 8\}$ and is analogous to Truchet's pattern N on Plate 2.

We end this section by showing patterns based on $p$-gons with $p \neq 4$. Figure 15 shows a tiling generated by alternating pairs of black and white basic triangles within a $6$-gon. So far $p$ has been even, but Figure 16 shows a tiling generated by a symmetric arrangement of basic triangles within a $5$-gon. These two patterns are not related to any patterns in Truchet's *Memoire*.

## 6. Patterns with Other Motifs

Inspired by Truchet, other pattern designers have used motifs other than triangular arrangements within a square. One popular motif consists of two quarter arcs of circles with each arc connecting the midpoints of adjacent edges of the square as first described by Smith [7]. Patterns based on this motif can have the motifs randomly placed or arranged to spell words [4]. Figure 17 shows a hyperbolic pattern based on that motif (superimposed on the underlying $\{4, 6\}$ tessellation). In the early 1700's Pierre Simon Fournier continued Truchet's work on patterns by designing more complicated motifs which could be repeatedly applied to create Truchet-like patterns [2]. One such motif looks like a wasp and is placed in the corners of squares by pattern makers. Figure 18 shows such a motif in the corners of the $4$-gons of a $\{4, 5\}$.

Other possible Truchet-like tilings include random tilings that contain just one diagonal from each square, which can produce mazes. Other pattern makers have colored Truchet arc patterns [1], and have defined 3-dimensional Truchet tilings [3] and [6]. Truchet patterns have also been placed on the equilateral triangle and regular hexagon tessellations ($\{3, 6\}$ and $\{6, 3\}$) [1]. Browne [1] and Reimann [5] have also created Truchet patterns on Archimedean tessellations.

## 7. Observations and Future Work

We have created some Truchet patterns in the hyperbolic plane based on the regular $\{p, q\}$ tessellations. This leads to a combinatorial question: how many ways can black and white basic triangles be placed in a $p$-gon? A related question is: how many arc patterns are there on a $p$-gon. The answer to this question for $q = 2m$ seems to be the same as the number of ways $m$ non-intersecting chords can be arranged in a circle.

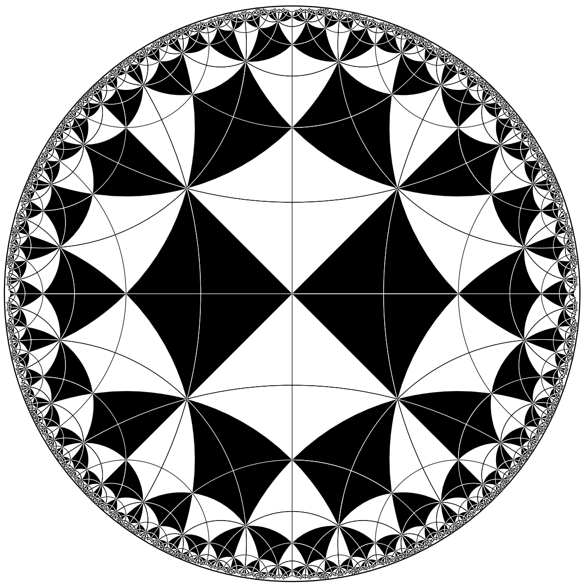


**Figure 13:** Another pattern generated by paired black and white triangles in a $4$-gon.

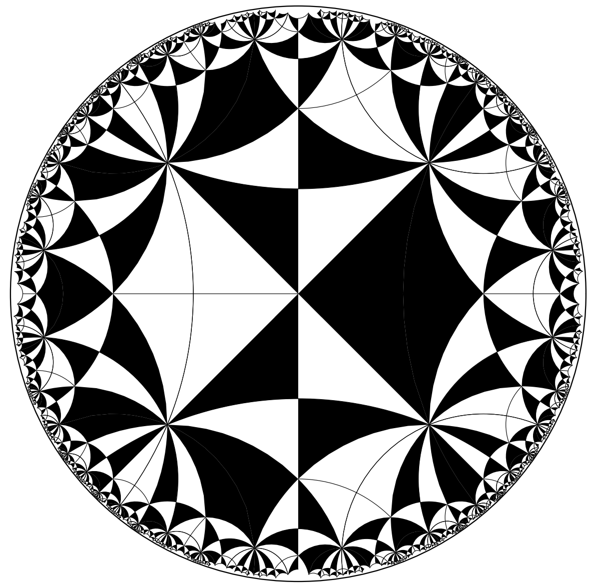


**Figure 14:** A more complicated pattern based on a $\{4, 8\}$ grid.

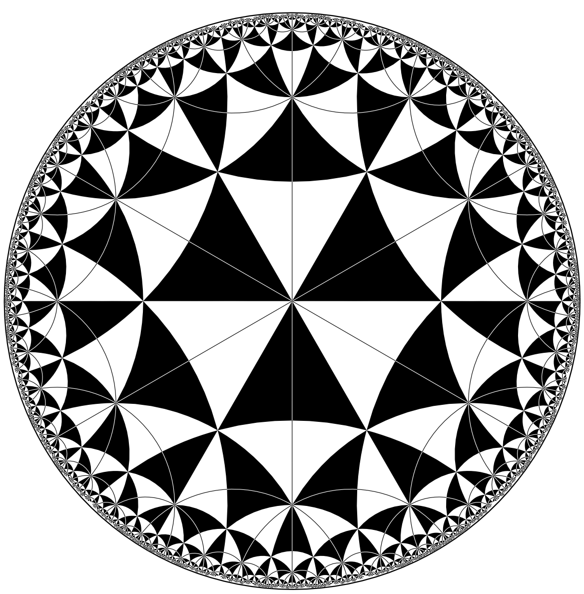


**Figure 15:** A simple $\{6, 4\}$ pattern.

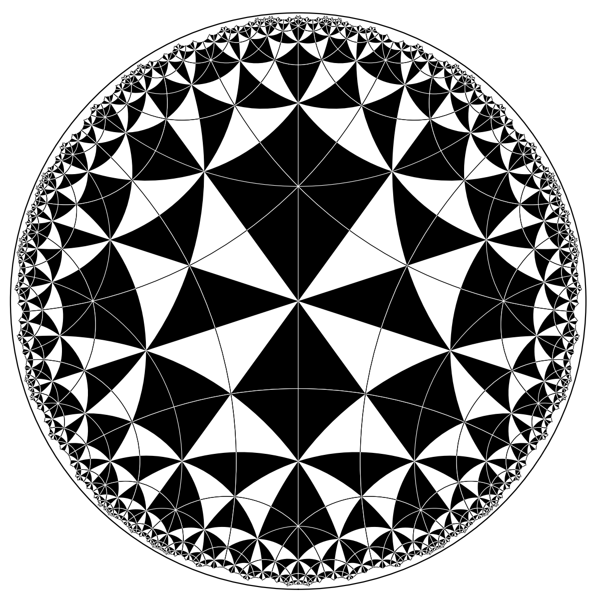


**Figure 16:** A new $\{5, 4\}$ Truchet-like tiling.

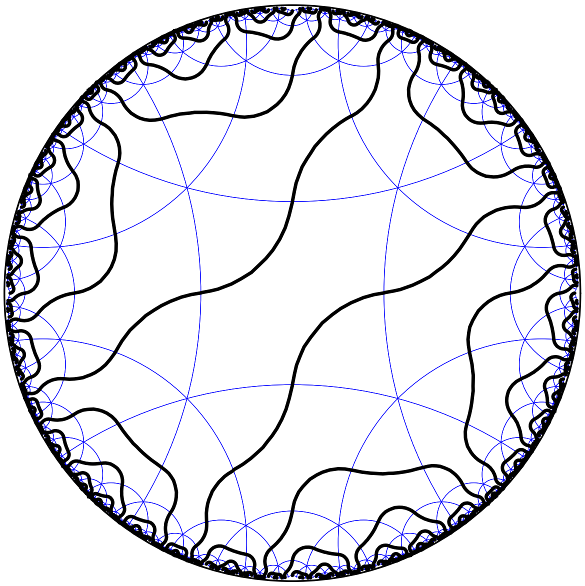


**Figure 17:** A hyperbolic Truchet arc pattern on a $\{4, 6\}$ grid.

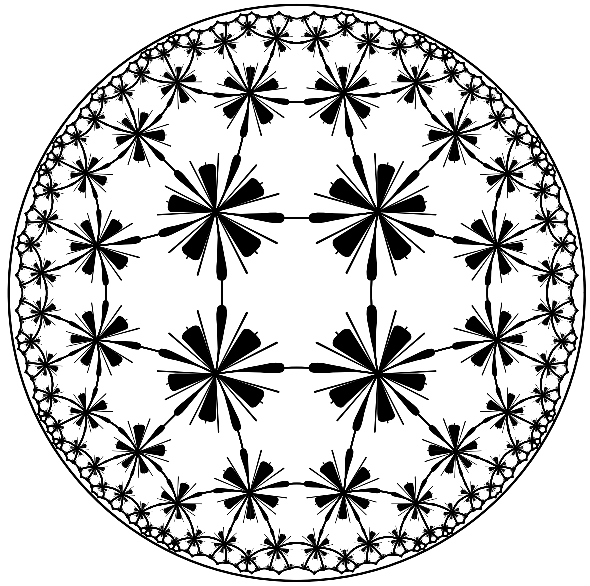


**Figure 18:** A hyperbolic Fournier pattern of "wasps" on a $\{4, 5\}$ grid.

Since some Truchet patterns have black-white color symmetry, it would seem natural to investigate the coloring of Truchet tilings with more than two colors. Another direction of future research would be to create Truchet patterns on hyperbolic Archimedean tessellations.

## Acknowledgments

I would like to thank David Reimann for his inspiration and useful discussions. I would also like to thank Lisa Fitzpatrick, director, and the staff of the Visualization and Digital Imaging Lab at the University of Minnesota Duluth.

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