Patterned Triply Periodic Polyhedra

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Abstract

This paper discusses repeating patterns on infinite skew polyhedra, which are triply periodic polyhedra. We exhibit patterns on each of the three regular skew polyhedra. These patterns are each related to corresponding repeating patterns in the hyperbolic plane. This correspondence will be explained in the paper.

1. Introduction

A number of people, including M.C. Escher, created convex polyhedra with patterns on them. Later, in 1977 Doris Schattschneider and Wallace Walker designed non-convex rings of polyhedra, called Kaleidocycles, that could be rotated, which are described in [Sch05]. The goal of this paper is to start an investigation of repeating patterns on infinite skew polyhedra — i.e. triply periodic polyhedra. Figure 1 shows a finite piece of such a pattern.

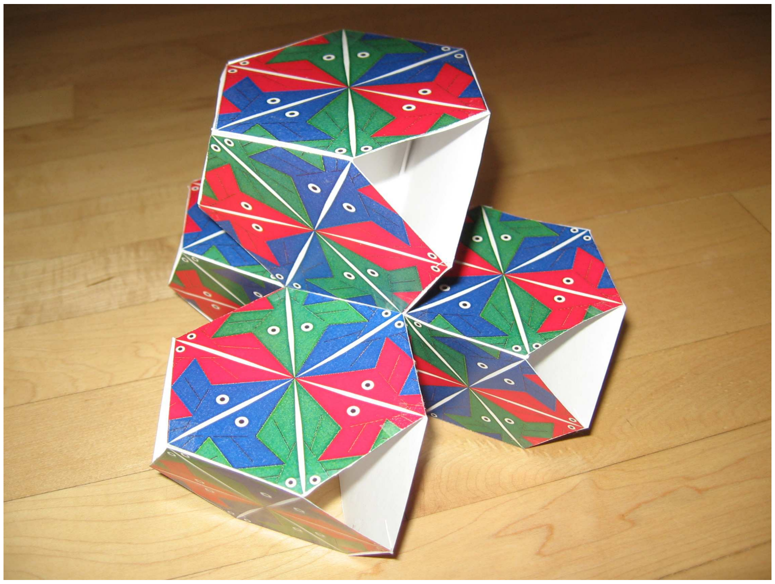

Figure 1: A pattern of fish on the tessellation $\{6, 6|3\}$.

We begin with a discussion of infinite skew polyhedra and show how they are related to tessellations of the hyperbolic plane. This relationship can also be applied to repeating patterns on those respective surfaces. Then we present patterns on each of the three regular triply periodic polyhedra. Finally, we indicate possible directions of further investigation.

2. Patterns, Hyperbolic Geometry, and Infinite Skew Polyhedra.

A *repeating pattern* is a pattern made up of congruent copies of a basic subpattern or *motif*. There can be repeating patterns on the Euclidean plane, hyperbolic plane, sphere, and polyhedra. For hyperbolic geometry, we use the *Poincaré disk* model whose points are represented by Euclidean points within a bounding circle. Hyperbolic lines are represented by (Euclidean) circular arcs orthogonal to the bounding circle (including diameters). This model distorts distances in such a way that equal hyperbolic distances correspond to ever-smaller Euclidean distances as figures approach the edge of the disk.

A *regular tessellation* is a special kind of repeating pattern on the Euclidean plane, the sphere, or the hyperbolic plane. It is formed by regular p-sided polygons or p-gons with q of them meeting at each vertex, and is denoted by the Schläfli symbol $\{p, q\}$. If $(p-2)(q-2) > 4$, the tessellation is hyperbolic, otherwise it is Euclidean or spherical. Figure 2 shows the regular hyperbolic tessellation $\{4, 6\}$, and

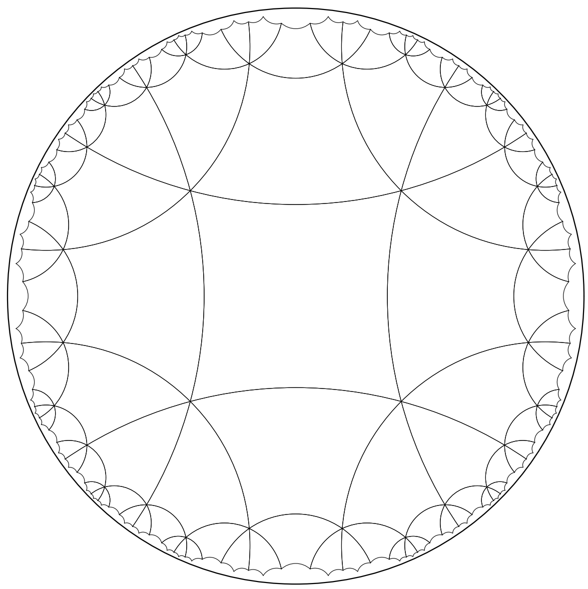

Figure 2: The $\{$4,6$\}$ tessellation.

An *infinite skew polyhedron* (in Euclidean 3-space) has regular polygon faces, a non-planar vertex figure, and repeats infinitely in three independent directions [Wiki1]. Such polyhedra have been called *hyperbolic tessellations* since they have negative angle defects at their vertices, but we don't use this designation since it conflicts with our definition above. (They have also been named *polyhedral sponges* since they can be seen to divide space into polyhedral cells.)

Regular skew polyhedra are special cases of infinite skew polyhedra whose symmetry groups are flag-transitive. There are three of them, as discovered by John Petrie in 1926 [Wiki1]. H.S.M. Coxeter used the modified Schläfli symbol $\{p, q|n\}$ to denote them, indicating that there are q p-gons around each vertex and n-gonal holes [Cox73, Cox99]. Figure 1 above shows a fish pattern on $\{6, 6|3\}$. The other possibilities are $\{4, 6|4\}$ and $\{6, 4|4\}$, which we show below.

A smooth surface has a *universal covering surface*: a simply connected surface with a covering map onto the original surface. If the original surface is negatively curved, universal covering surface is the hyperbolic plane. We can extend this idea to regular skew polyhedra: the hyperbolic tessellation $\{p, q\}$ is the universal covering polyhedron for $\{p, q|n\}$. Since regular skew polyhedra have negative angle defect, their universal covering polyhedra must be hyperbolic. We also extend the covering idea to repeating patterns on infinite skew polyhedron.

Infinite skew polyhedra are also related to triply periodic minimal surfaces (TPMS), since some TRMS surfaces are the (unique) minimal surfaces formed from the wire-frames (collection of edges) of infinite skew polyhedra. Alan Schoen has done extensive investigations into TPMS [Schoen].

In the next three sections we show examples of patterns on the regular skew polyhedra and their associated hyperbolic patterns.

3. A Pattern on the $\{4, 6|4\}$ Polyhedron

The $\{4, 6|4\}$ polyhedron is the easiest to understand. It is based on the tessellation of 3-space by cubes. One way to visualize it is to index the cubes by integers in each of the three directions and include only those with one or three even indices as a solid figure (the complement is congruent to it). The $\{4, 6|4\}$ polyhedron is the boundary of that solid figure. Escher's "Heaven and Hell" pattern was the only one that he realized in each of the classical geometries: Euclidean, spherical, and hyperbolic. So it seems appropriate to also place such an "angels and devils" pattern on a regular skew polyhedron, the $\{4, 6|4\}$ polyhedron as shown in Figure 3.

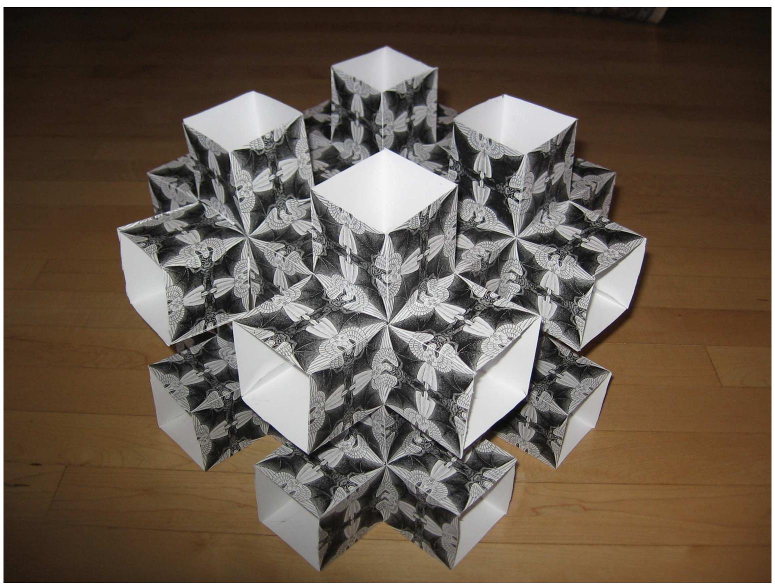

Figure 3: A pattern of angels and devils on the $\{4, 6|4\}$ polyhedron.

Figure 4 shows the corresponding universal covering pattern based on the $\{4, 6\}$ tessellation, which is shown in red. One can see the six hyperbolic "squares" around each vertex.

4. A Pattern on the $\{6, 4|4\}$ Polyhedron

The $\{4, 6|4\}$ polyhedron is the dual of the $\{4, 6|4\}$ polyhedron. The $\{4, 6|4\}$ polyhedron is based on the Bitruncated cubic space-filling tessellation by truncated octahedra [Wiki3]. If we index rectangular lattice positions in 3-space as in the previous section, we can place one set of truncated octahedra at positions of all even indices, and a complementary set a positions of all odd indices such that all octahedra are congruent and fill space. The boundary between these two sets is the$\{4, 6|4\}$ polyhedron. Figure 5 shows another pattern of angels and devils on that polyhedron, with axes of bilateral symmetry of the angels and devils shown in red, blue, and green.

Figure 6 shows the corresponding universal covering pattern based on the $\{6, 4\}$ tessellation. In Figure 6 we have emphasized the bilateral symmetry of the figures with three families of lines colored red, green, and

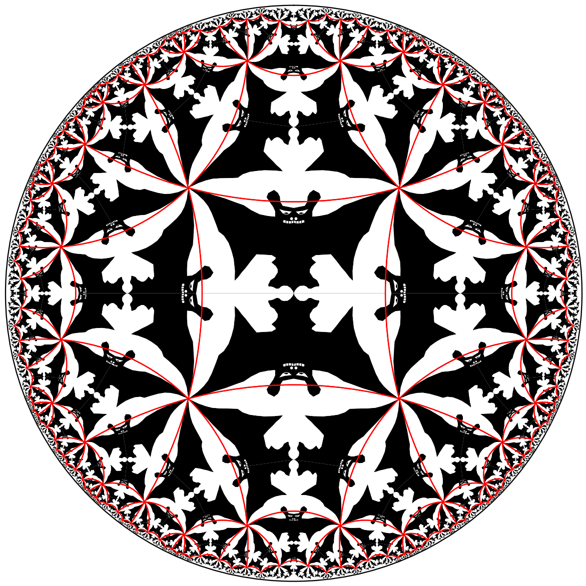

Figure 4: A pattern of angels and devils showing the underlying $\{4, 6\}$ tessellation.

Figure 5: A pattern of angels and devils on the $\{6, 4|4\}$ polyhedron.

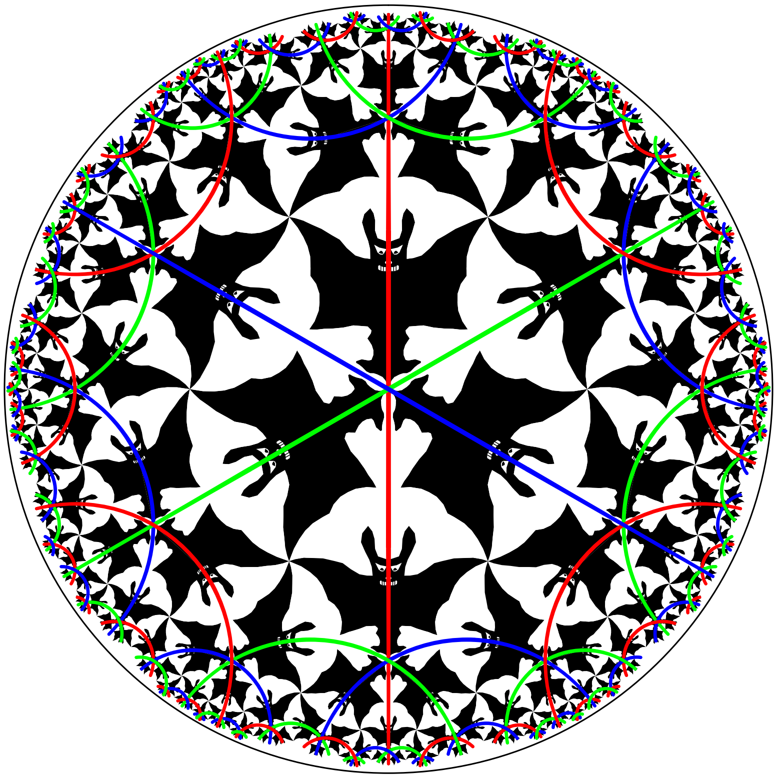

Figure 6: A pattern of angels and devils based on the $\{6, 4\}$ tessellation.

blue, such that no two lines of a family intersect. These lines correspond zigzagging polylines in Figure 5, with the red lines going roughly left-to-right, the blue lines going front-to-back, and the green lines oriented approximately vertically. We could have similarly emphasized the bilateral symmetry in Figure 3, in which the axes of bilateral symmetry would be square "loops" around the cubic arms and holes.

5. A Pattern on the $\{6, 6|3\}$ Polyhedron

The $\{6, 6|3\}$ polyhedron may be the trickiest to understand. It is formed from truncated tetrahedra with the triangular faces removed. Such triangular faces from four truncated tetrahedra are then placed in a tetrahedral arrangement (around a small invisible tetrahedron) [Wiki2]. A side view is shown in Figure 1. Figure 7 shows a "top" view looking down at one of the vertices (where six hexagons meet). We placed a pattern of angular fish on this polyhedron. Figure 8 shows the corresponding universal covering pattern based on the $\{6, 6\}$ tessellation.

All the fish along a backbone line in Figure 8 are the same color and swim the same direction. No two backbone lines of the same color intersect. In fact the pattern has (perfect) 3-color symmetry. The same comments also apply to the pattern of Figures 1 and 7. In the upward facing planes in Figure 1, the red fish swim lower right to upper left, the blue fish swim lower left to upper right, and the green fish swim toward the viewer. In fact the backbone lines on the $\{6, 6|3\}$ polyhedron are embedded Euclidean lines.

6. Observations and Future Work

We have shown patterns on each of the regular skew polyhedra, but certainly many more patterns could be drawn on them. It would also be possible to draw patterns on other infinite but non-regular skew polyhedra. In creating such patterns, it is desirable to take advantage of the combinatorics and any underlying geometry

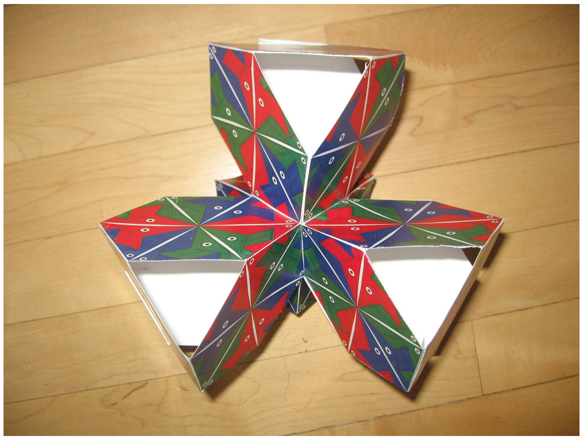

Figure 7: A top view of a pattern of fish on the $\{6,6|3\}$ polyhedron.

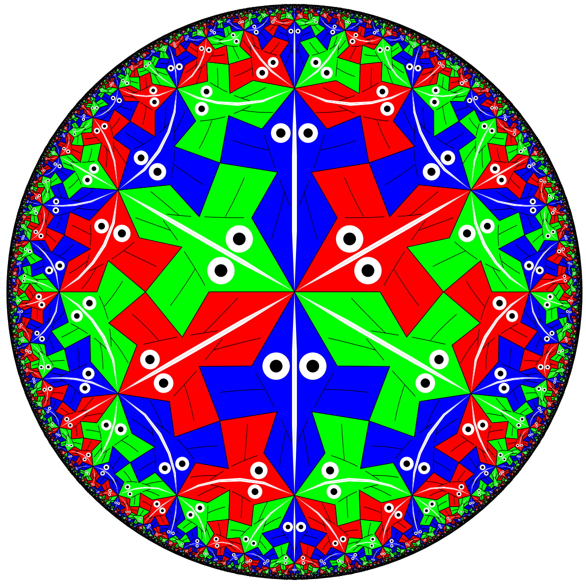

Figure 8: A pattern of fish based on the $\{6,6\}$ tessellation.

of the skew polyhedra. This was perhaps best done above by the pattern on the $\{6,6|3\}$ polyhedron. It would also be nice to similarly take advantage of the embedded lines in the $\{6,4|4\}$ polyhedron, which could be done if we used the hexagon pattern of fish that we used for the $\{6,6|3\}$ polyhedron. However, the fish would then alternate directions along a backbone line. In summary, there are many more patterns on skew polyhedra to investigate.

Acknowledgments

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[Wiki12 Wikipedia entry for "Regular skew polyhedron"
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[Wiki3] Wikipedia entry for "Truncated Octahedron"
`http://en.wikipedia.org/wiki/Truncated_octahedron`