Escher Patterns on Triply Periodic Polyhedra

Douglas Dunham
Department of Computer Science
University of Minnesota, Duluth
Duluth, MN 55812-3036, USA
E-mail: `ddunham@d.umn.edu`
Web Site: `http://www.d.umn.edu/˜ddunham/`

Abstract

M.C. Escher drew a few of his patterns on finite polyhedra, but not on infinite polyhedra. In this paper we show some Escher-inspired patterns on triply periodic polyhedra.

1. Introduction

In this paper we show new patterns on triply periodic polyhedra that were inspired by the Dutch artist M.C. Escher. Triply periodic polyhedra have translation symmetries in three independent directions in Euclidean 3-space. Figure 1 shows a finite part of such a polyhedron decorated with butterflies. This paper

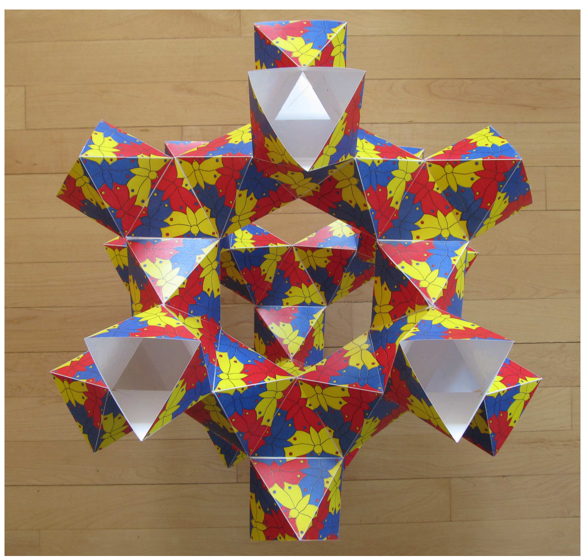

Figure 1: A piece of the $\{3, 8\}$ polyhedron decorated with butterlfies.

extends the work shown at Bridges 2012 [1], exhibiting Escher patterns on two new polyhedra.

Each of the polyhedra we discuss is composed of copies of a regular polygon, with more of them around each vertex than would be possible in the Euclidean plane. These polyhedra thus have negative curvature, and are related to regular tessellations of the hyperbolic plane. Similarly, the patterns we place on these polyhedra are related to patterns of the hyperbolic plane that are based on regular tessellations.

Escher drew a few of his patterns on closed polyhedra, which are shown in [3]. Later Doris Schattschneider and Wallace Walker placed Escher patterns on non-convex rings of polyhedra, called Kaleidocycles, which could be rotated [4]. The purpose of this paper is to extend this concept by placing Escher pattens on triply periodic polyhedra.

We begin with a review of regular hyperbolic tessellations and triply periodic polyhedra, and the relation between them, which extends to patterns on the respective surfaces. Then we show Escher patterns on two polyhedra. Finally, we indicate possibilities for other Escher patterns on triply periodic polyhedra.

2. Regular Tessellations and Triply Periodic Polyhedra

We use the Schläfli symbol $\{p, q\}$ to denote the regular tessellation formed by regular p-sided polygons or *p-gons* with q of them meeting at each vertex. If $(p-2)(q-2) > 4$, $\{p, q\}$ is a tessellation of the hyperbolic plane (otherwise it is Euclidean or spherical). Figure 2 shows the tessellation $\{4, 5\}$ in the Poincaré disk model of hyperbolic geometry. Figure 3 shows that tessellation on top of a pattern of angels and devils.

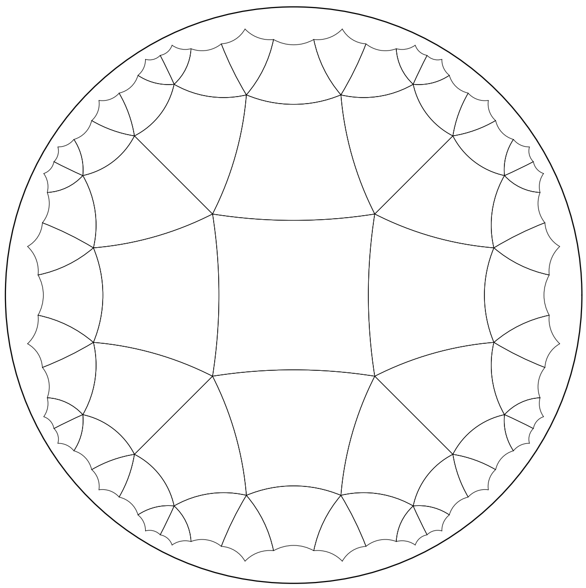

Figure 2: The $\{4,5\}$ tessellation

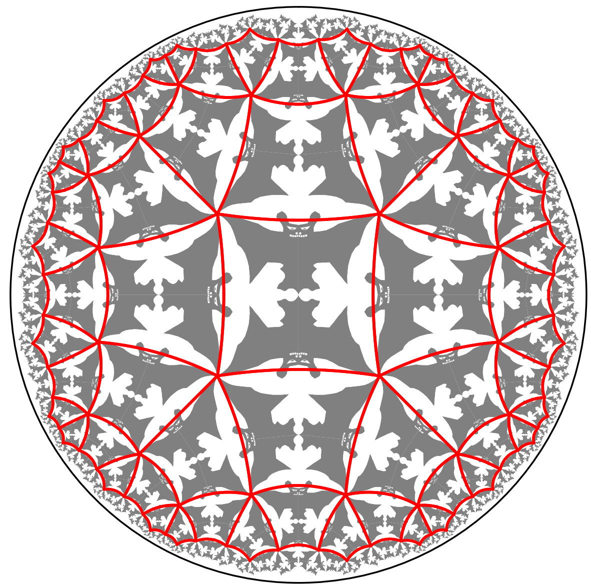

Figure 3: The $\{4, 5\}$ superimposed on a pattern of angels and devils.

We will be interested in infinite, connected *semiregular triply periodic polyhedra*. Such a polyhedron has a p-gon for each of its faces, q p-gons around each vertex, translation symmetries in three independent directions, and symmetry group that is transitive on vertices — i.e. it is *uniform*. We extend the Schläfli symbol $\{p, q\}$ to include these polyhedra (however different polyhdera can have the same $\{p, q\}$). The *infinite skew polyhedra* are the most symmetric of these polyhedra — like the Platonic solids their symmetry groups are transitive on flags (a flag is a vertex-edge-face triple where the vertex is an endpoint of the edge, which is an edge of the face). Figure 1 shows a piece of a $\{3, 8\}$ polyhedron with a butterfly pattern on it. These polyhedra are considered to be hyperbolic since the angle sum at each vertex is greater than 2π.

In some cases there is an intermediate "connecting surface" between some regular triply periodic polyhedra $\{p, q\}$ and the corresponding regular tessellations $\{p, q\}$. First, these periodic polyhedra are approximations to triply periodic minimal surfaces (TPMS). Second, each smooth surface has a *universal covering surface*: a simply connected surface with a covering map onto the original surface, which is a sphere, the Euclidean plane, or the hyperbolic plane. Since each TPMS has negative curvature (except for possible isolated points), its universal covering surface does too, and thus has the same large-scale geometry as the hyperbolic plane. In the same vein, we might call a hyperbolic pattern based on the tessellation $\{p, q\}$ the "universal covering pattern" for the related pattern on the polyhedron $\{p, q\}$. In the next two sections we show two patterned polyhedra and their corresponding "covering" patterns in the hyperbolic plane.

3. A Pattern of Angels and Devils on a $\{4,5\}$ Polyhedron

Escher only realized one pattern in each of the classical geometries, his "Heaven and Hell" patterns: his Euclidean Regular Division Drawing 45, a carved sphere with angels and devils, and his hyperbolic pattern *Circle Limit IV*, which are based on the $\{4,4\}$, $\{4,3\}$, and $\{6,4\}$ tessellations respectively. Figure 3 above shows a related angels and devils pattern based on the $\{4,5\}$ tessellation. The goal of this section is to place that pattern on a $\{4,5\}$ polyhedron also. Figure 4 shows a piece of that polyhedron [2]. It is made up of cross-shaped units in three orientations, the colors of the units indicating the orientation. Figure 5 shows one of the units, which can be thought of as a cube with four equilateral triangular prisms on it.

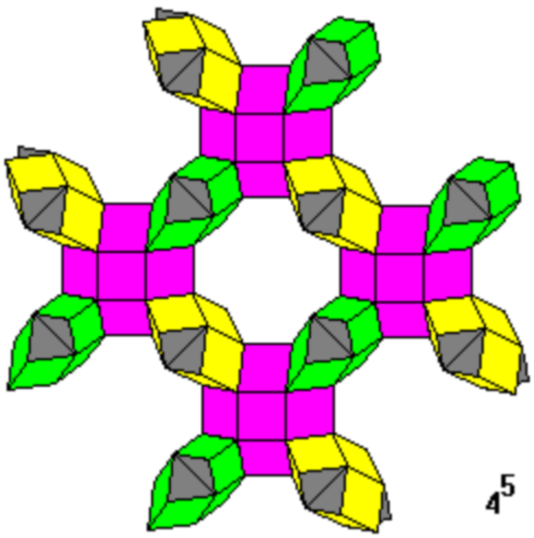

Figure 4: A piece of a $\{4,5\}$ polyhedron.

Figure 5: A "construction unit" for the $\{4,5\}$ polyhedron.

If we think of the $\{4,5\}$ polyhedron shown in Figure 4 as bounding a solid, the complementary solid may be easier to understand. That solid consists of truncated octahedral hubs with their hexagonal faces connected by regular hexagonal prisms as struts. Consequently, the $\{4,5\}$ polyhedron of Figure 4 has hexagonal holes in the four directions of the body diagonals of a cube. Figure 6 shows the $\{4,5\}$ polyhedron decorated with angels and devils, with a view down one of the hexagonal holes. Figure 7 shows another view of that patterned polyhedron, for which the angels and devils of Figure 3 form the corresponding "covering pattern".

The patterned $\{4,5\}$ polyhedron of Figures 6 and 7 thus fills a "gap" between Escher's Regular Division Drawing number 45 based on $\{4,4\}$ and the patterned $\{4,6\}$ polyhedron in Figure 3 of [1].

4. A Pattern of Butterflies on a $\{3,8\}$ Polyhedron

The inspiration for Figure 1 is Escher's Regular Division Drawing number 70, shown in Figure 8. It can be thought of as being composed of triads of butterflies, one of each color meeting at right rear wings, and is thus based on the Euclidean $\{3,6\}$ tessellation. Two colors of butterflies meet at left front wing tips and those butterflies are decorated with wing spots of the third color. Figure 9 shows a hyperbolic butterfly pattern that is based on the $\{3,8\}$ tessellation and follows Escher's coloring rules. It is this pattern that we have "wrapped around" a $\{3,8\}$ polyhedron.

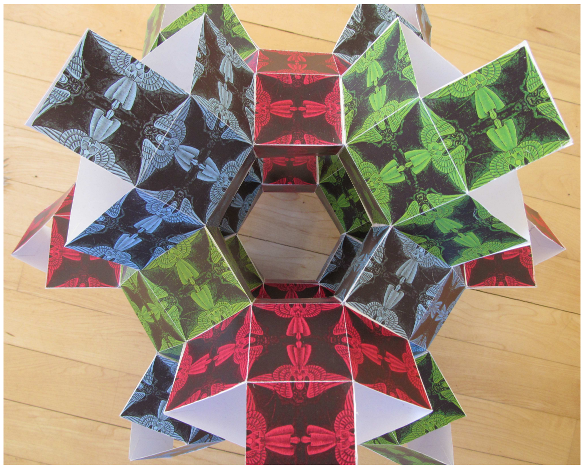

Figure 6: A view down a hexagonal hole of the $\{4,5\}$ polyhedron decorated with angels and devils.

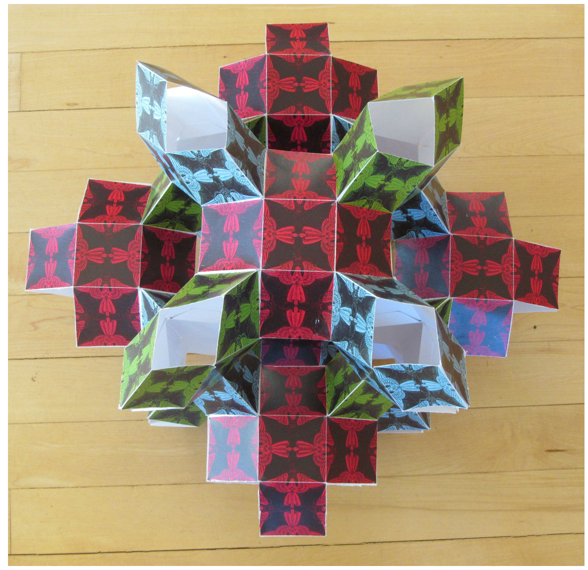

Figure 7: Another view of the $\{4,5\}$ polyhedron decorated with angels and devils.

As with the $\{4,5\}$ polyhedron of Section 3, our $\{3,8\}$ polyhedron can also be described in terms of hubs and struts, both of which are regular octahedra. A hub octahedron has strut octahedra on alternate faces, so that four hub triangles are covered by struts and four remain exposed. Each strut octahedron connects two hubs, and thus has two of its faces covered by hubs. Figure 10 shows a "construction unit" consisting of a hub and four struts. The $\{3,8\}$ polyhedron is an approximation to Schwarz's D-Surface, a TPMS. Both surfaces have the same basic shape of a fattened diamond lattice. Figure 11 shows a portion of Schwarz's D-Surface that corresponds to the "construction unit" of the polyhedron.

By careful inspection of Figures 8 and 9, one notices that the triads of butterflies meeting at right rear wings do so in two different ways. Arbitrarily, we call them "left" and "right" triads of butterflies. It turns out that all the exposed triangles of a hub of the $\{3,8\}$ polyhedron are of one kind and they alternate between adjacent hubs — that is the triads on the hub at one end of a strut are the opposite kind from those on the other end, as can be seen in Figure 12, which shows the $\{3,8\}$ polyhedron decorated with Escher butterflies. Figure 12 is a view down one of the "tunnels" of the polyhedron; Figure 1 is a view down one of the 3-fold rotation axes. Also, as mentioned above, Escher arranged that two colors of butterflies meet at left front wing tips and those butterflies are decorated with wing spots of the third color. Figure 13 shows that this coloring is maintained by showing a close-up of a vertex at which blue and red butterflies meet at wing tips and both have yellow wing spots on them.

5. Observations and Future Work

We have shown Escher patterns on two semiregular triply periodic polyhedra, which satisfied our goal of finding polyhedra that could be nicely covered with Escher-inspired patterns. There are certainly other Escher patterns that could be applied to these surfaces. Also, Escher patterns could be mapped onto other triply periodic polyhedra, either semiregular or just uniform. Several such polyhedra are known, but have not been classified (except for the most regular ones, the infinite skew polyhedra). The uniform polyhedra would be analogous to the Archimedean solids. Another project would be to paint these patterns on the intermediate TPMS's, which could be manufactured by 3D printing.

Figure 8: Escher's Regular Division Drawing 70.

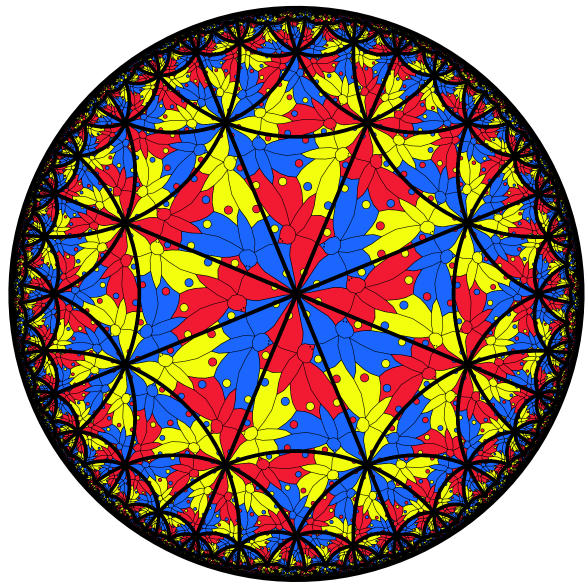

Figure 9: A pattern of Butterflies based on the $\{3,8\}$ Tessellation.

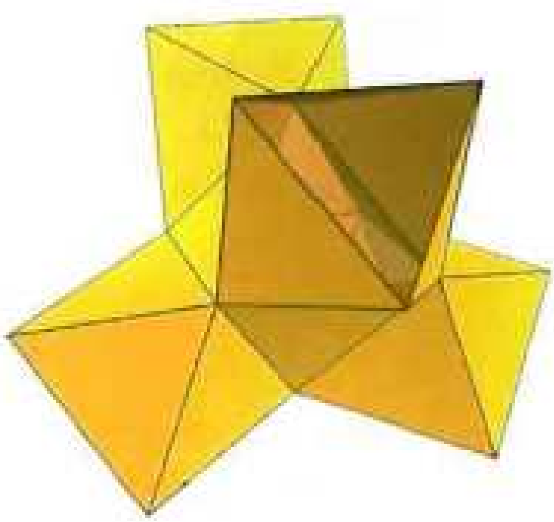

Figure 10: A "construction unit" for the $\{3,8\}$ polyhedron.

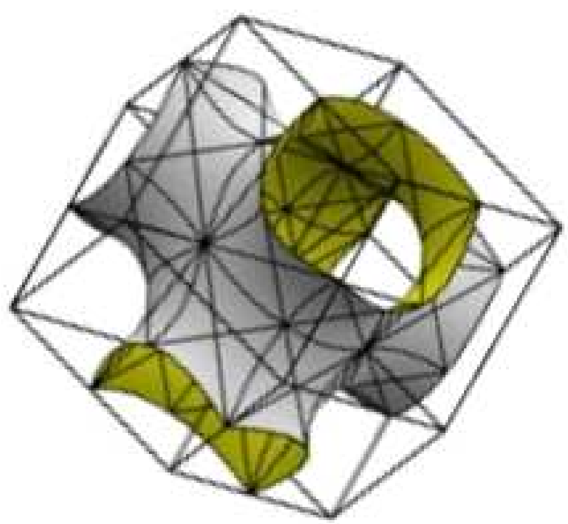

Figure 11: Part of Schwarz's D-Surface corresponding to the "construction unit".

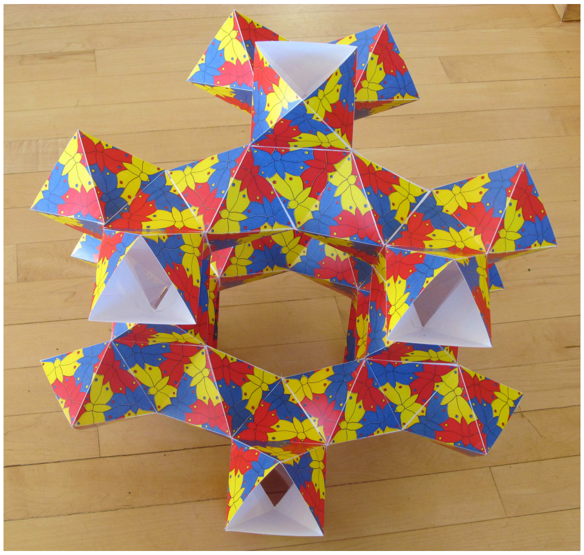

Figure 12: A view down one of the "tunnels" of the $\{3,8\}$ polyhedron.

Figure 13: A close-up showing colors of butterflies and wing spots at a vertex.

Acknowledgments

I would like to thank the reviewers for their helpful suggestions.

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