Repeating Fractal Patterns with 4-Fold Symmetry

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Abstract

Previously we described an algorithm that can fill a region with an infinite sequence of randomly placed and progressively smaller shapes, producing a fractal pattern. If the algorithm is appropriately modified and the region is a fundamental region for one of the 17 "wallpaper" groups, one can obtain a fractal pattern with that symmetry group. This produces artistic patterns which have a pleasing combination of global symmetry and local randomness. In this paper we focus on such patterns with 4-fold symmetry, and we show several sample patterns.

Introduction

In the past we have created pleasing patterns with an algorithm [2, 5, 7] that can fill a planar region with a series of progressively smaller randomly-placed motifs. Also we have extended that algorithm to create "wallpaper" patterns with reflection symmetry [3]. In this paper we extend the algorithm to create wallpaper patterns with 4-fold symmetry, i.e. with 4-fold rotation centers. Thus the groups we are interested in are $p4, p4mm,$ and $p4mg$. Figure 1 shows a random pattern of circles with symmetry group $p4$. In order to

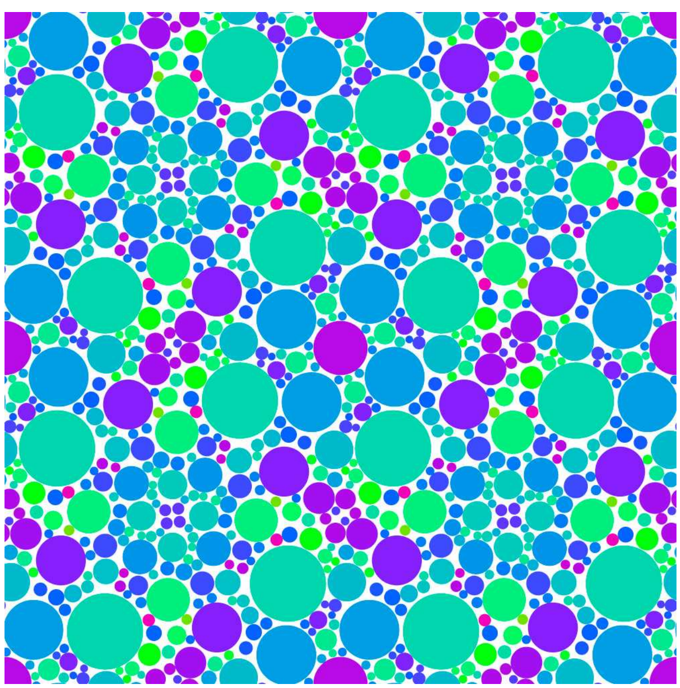

Figure 1: *A locally random circle fractal with global $p4$ symmetry.*

create our wallpaper patterns, we fill a fundamental region for one of the 2-dimensional crystallographic groups with randomly placed, progressively smaller copies of a motif, possibly with different colors (or orientations), such as the circles of Figure 1. This randomness generates a fractal pattern. Then copies of the filled fundamental region can be used to tile the plane, yielding a locally fractal, but globally symmetric pattern.

In the next section we recall how the basic algorithm works. Then we recall a few facts about wallpaper groups, especially those with 4-fold symmetry, describing the modifications needed to create the new patterns discussed here. Next, we discuss and exhibit sample patterns for the wallpaper groups $pp4, p4mm$, and $p4mg$. Finally, we draw conclusions and summarize the results.

The Algorithm

The idea of the algorithm is to place progressively smaller motifs m_i within a region R so that a motif does not overlap any previously placed motif. Random placements are tried until a non-overlapping one is found. As noted in [2, 4, 6], for many choices of R and motifs m_i of area A_i, the following algorithm proceeds without halting:

For each $i = 0, 1, 2, \dots$

  Repeat:

    Randomly choose a point within R to place the i-th motif m_i.

  Until (m_i doesn't intersect any of $m_0, m_1, ..., m_{i-1}$)

  Add m_i to the list of successful *placements*

Until some stopping condition is met, such as a maximum value of i or a minimum value of A_i.

It has been found experimentally by the second author that this non-halting phenomenon is achieved by a wide range of choices of shapes of R and the motifs if the motifs obeyed an inverse power law *area rule*: if A is the area of R, then for $i = 0, 1, 2, \dots$ the area of A_i of m_i can be taken to be:

$$A_i = \frac{A}{\zeta(c, N)(N + i)^c} \tag{1}$$

where $c > 1$ and $N > 1$ are parameters, and $\zeta(c, N)$ is the Hurwitz zeta function: $\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(q+k)^s}$. Thus $\lim_{n\to\infty} \sum_{i=0}^{n} A_i = A$, that is, the process is space-filling if the algorithm continues indefinitely. In Figure 1 $c = 1.44$ and $N = 2.5$. In the limit, the fractal dimension D of the placed motifs can be computed to be $D = 2/c$ [7]. Examples of the algorithm written in C code can be found at Shier's web site [8].

It is conjectured by the authors that the algorithm does not halt for non-pathological shapes of R and m_i, and "reasonable" choices of c and N (depending on the shapes of R and m_i). In fact this has been proved for $1 < c < 1.0965...$ and $N \geq 1$ by Christopher Ennis when R is a circle and the motifs are also circles [4].

Circles make good candidates for both the enclosing region R and the motif since, by their symmetry, they play a significant role in both mathematics and decorative art. In Figure 1, mathematics provides the arrangement of the circular motifs while art provides the colors.

Wallpaper Groups

It has been known for over a century that there 17 different kinds of patterns that repeat in two independent directions in the Euclidean plane. Such patterns are called *wallpaper patterns* and their symmetry groups are called plane crystallographic groups or *wallpaper groups*. In 1952 the International Union of Crystallography (IUC) established a notation for these groups, and a shorthand notation soon followed. In 1978 Schattschneider wrote a paper clarifying the notation and giving an algorithm for identifying the symmetry group of a wallpaper pattern [6]. Later, Conway popularized the more general orbifold notation [1]. Each wallpaper pattern has a *fundamental region* containing the basic subpattern which can be used to generate the entire pattern by applying transformations from the symmetry group. It is often convenient to first combine (by rotation or reflection) copies of the fundamental region to form *translation units* that can generate the entire pattern using translations alone. We create wallpaper patterns by filling a fundamental region R with motifs as above, and then extend the pattern using transformations of the wallpaper group.

In our previous paper [2], we showed examples of patterns that had $p1$ (or $\mathbf{o}$ in orbifold notation) symmetry, the simplest kind of wallpaper symmetry, with only translations in two independent directions. Figure 2 shows such a pattern. In Figure 2 peppers on the left edge "wrap around" and are continued on the right edge; similarly peppers on the top edge "wrap around" to the bottom. In [3] we have also shown patterns

Figure 2: *A pattern of peppers with* $p1$ *symmetry.*

with "reflection" symmetry groups whose fundamental regions (which becomes our region R) are bounded by mirror lines. There are four such groups: $p2mm, p3m1, p4mm,$ and $p6mm$, or $*2222, *333, *442,$ and $*632$ in orbifold notation, respectively. In fact reflections across those mirrors generate the groups.

As mentioned above, this paper focuses on the wallpaper groups $p4, p4mm,$ and $p4mg$ (442, $*442$, and 4*2 in orbifold notation). The group $p4$ has two 4-fold rotation points at the 45-degree vertices of a 45-45-90 triangle (two such adjacent triangles comprise a fundamental region) and a 2-fold rotation point at the 90-degree angle. The group $p4mm$ is generated by reflections across the sides of 45-45-90 triangle, and $p4mg$ is generated by a 4-fold rotation about the right angle of a 45-45-90 triangle and a reflection across the opposite side.

An issue that arose for reflection groups, but not $p1$, is what to do if a trial placement of the motif crosses a mirror boundary of the fundamental region. One solution is to let that happen, as shown in Figure 3.

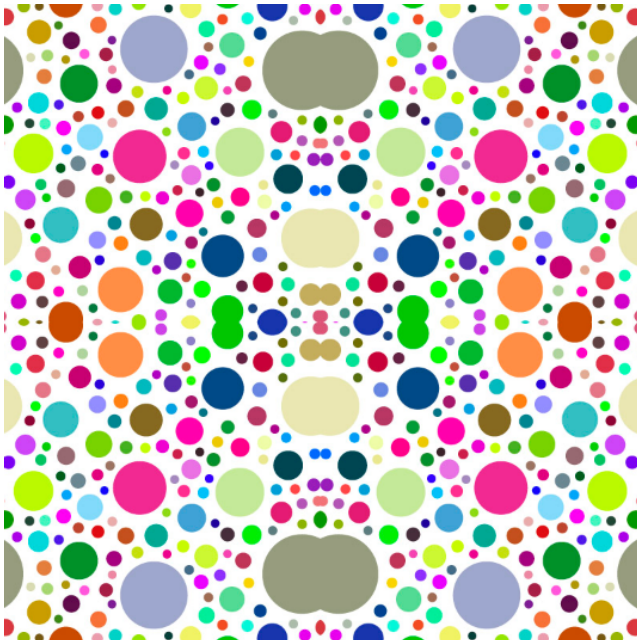

Figure 3: *A random circle pattern with $p2mm$ symmetry, and partial circles on mirror boundaries.*

However, as can be seen in Figure 3, this leads to the unsatisfactory situation in which combined objects are not the shape of the original motif. One solution is to avoid the mirror boundaries, either by rejecting that trial placement or by moving that placement away from the mirror. Figure 4 shows an example. The mirror boundaries are quite evident using this method.

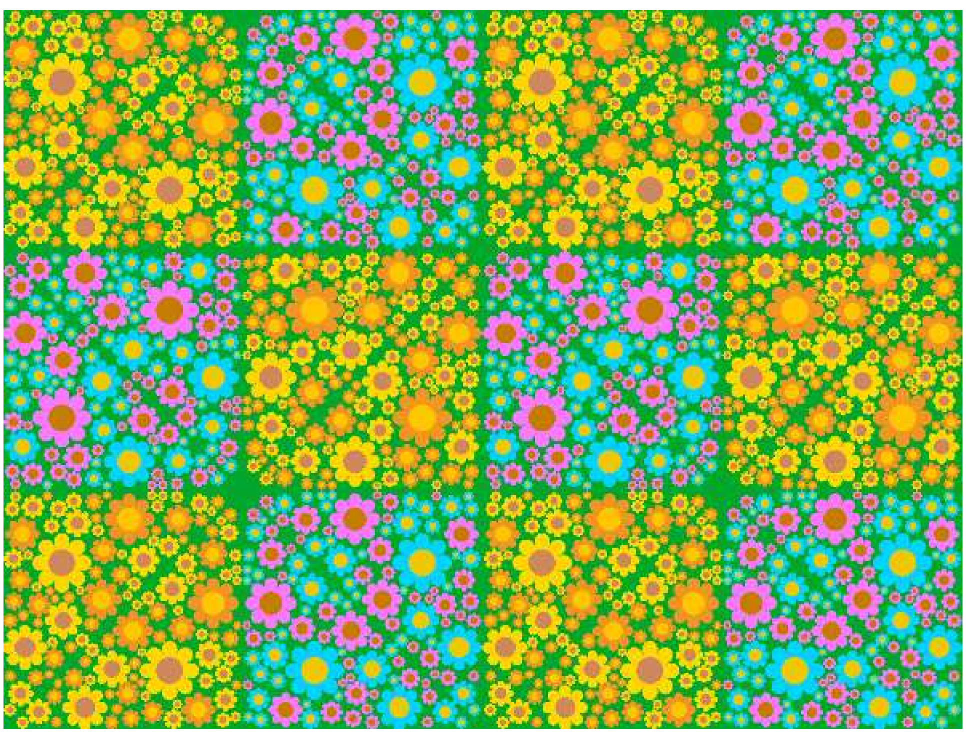

Figure 4: *A random flower pattern with $p4mm$ symmetry with none on mirror boundaries.*

Another solution if the motif has mirror symmetry itself, is that we move the motif (perpendicularly) onto the boundary so that the mirror of the motif aligns with the boundary mirror. Figure 5 shows such a pattern with $p6mm$ symmetry.

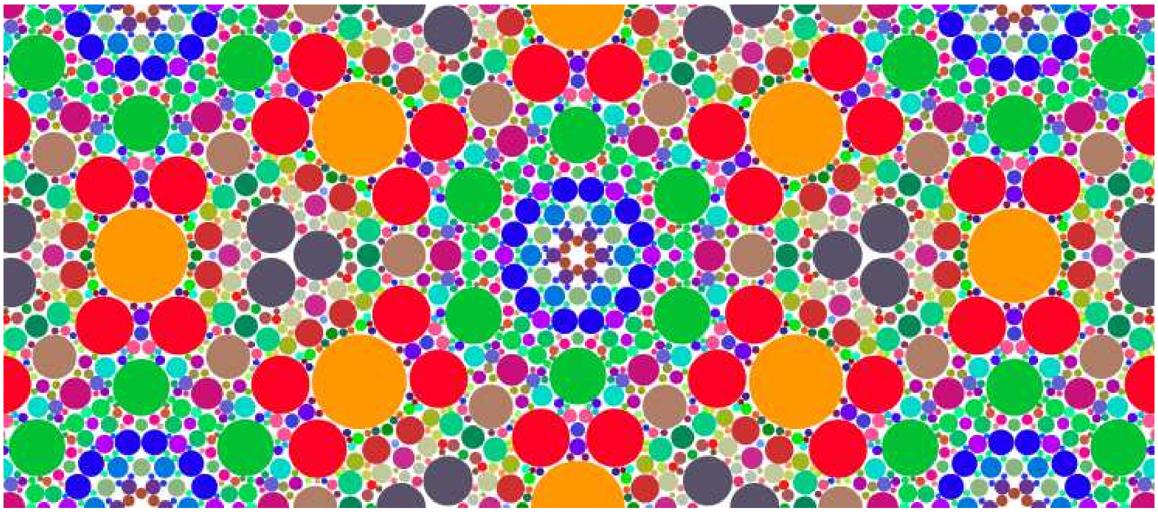

Figure 5: *A random circle pattern with $p6mm$ symmetry with some on mirror boundaries.*

Using this method, the mirror lines are more subtle, producing arguably more interesting patterns. Also, the area rule calculation needs to be adjusted each time this happens since only half of the motif is placed within the fundamental region.

A similar issue that arises with the groups $p4, p4mm$, and $p4mg$ is what to do if a motif overlaps a 2-fold or 4-fold rotation point of the pattern. Again, a simple solution is to avoid such a point, either by rejecting that trial placement or moving it away from the rotation point. Another solution, if the motif has 4-fold rotational symmetry is to move the motif so that it is centered on the rotation point. If the motif only has 2-fold rotational symmetry, it can be centered on a 2-fold rotation point of the whole pattern, but must be rejected (or moved away) if it overlaps a 4-fold rotation point of the pattern. This solution is discussed and illustrated in the next section about $p4$ patterns.

Patterns with Symmetry Group $\mathrm{p4}$

Figure 6 shows a $p4$ pattern of circles that were alternately colored red and blue in the generation process. There are two different kinds of 4-fold rotation points, one with a small blue circle centered on it, and the other not containing any circles but surrounded by four small red circles. The first kind of 4-fold rotation point can be seen at the center of the figure, the corners, and the centers of the edges. The second kind 4-fold rotation point can be seen half way between the center and the corners of the figure. There are no motifs on the 2-fold rotation points. This pattern is composed of a 2×2 grid of translation units, each translation unit being comprised of 4 rotated fundamental regions for $p4$,

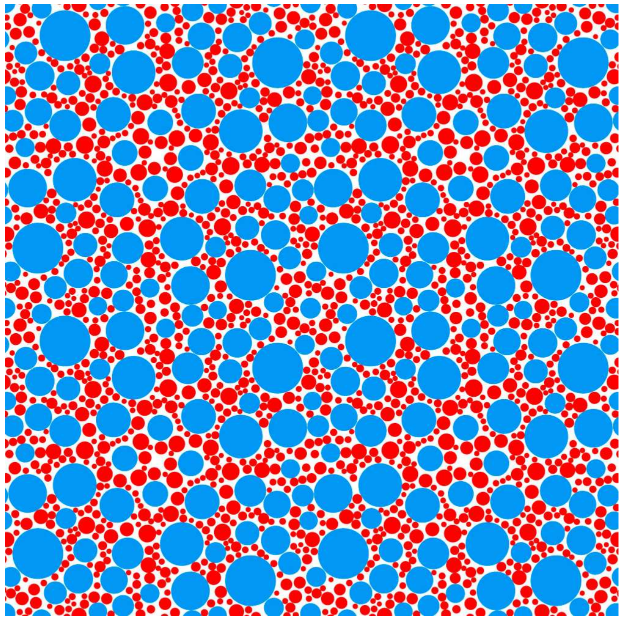

Figure 6: *A pattern of red and blue circles with $p4$ symmetry.*

Figure 7 shows the title page pattern with $p4$ symmetry in which the circles are colored according to the spatial position of their centers. A 2D Fourier series was created with a small number of non-zero coefficients and this was used to find the color based on the x-y position. This technique has the feature that nearby circles are also close together in RGB color space. This pattern is also composed of a 2×2 grid of translation units. There is a magenta circle at one of the kinds of 4-fold rotation points, one at the center of the figure and pieces of it at the corners and centers of the sides. As with Figure 6, there are no circles at the other kind of 4-fold point and those are half way from the center to the corners. There are also no circles at the 2-fold rotation points. The rotation points are somewhat harder to spot in $p4$ patterns than in $p4mm$ and $p4mg$ patterns in which the motifs avoid the mirror lines.

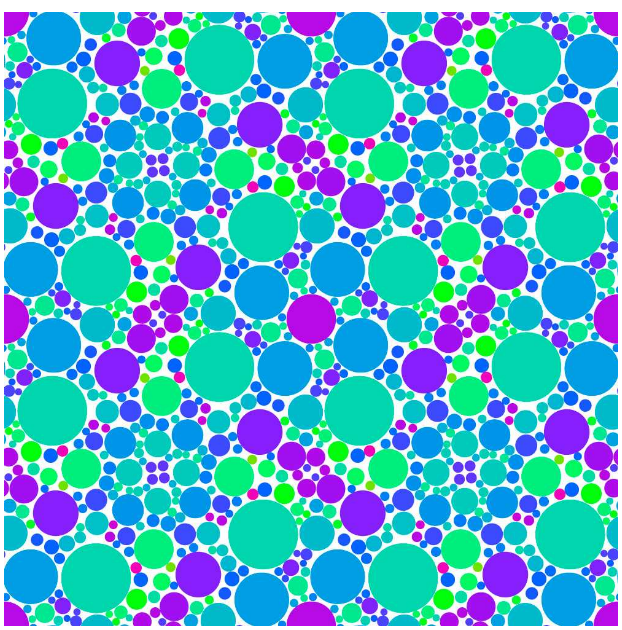

Figure 7: *The title page pattern of various colored circles with $p4$ symmetry.*

Patterns with Symmetry Group p4mm

As noted above, Figure 5 above shows a pattern with $p4mm$ symmetry. Figure 5 is interesting in that the horizontal and vertical reflections induce 2-multicolor symmetry which exchanges the multicolors $\{$ yellow, dark magenta, orange, dark yellow $\}$ and $\{$ cyan, yellow, magenta, brown $\}$. This pattern is composed of a 2×1.5 grid of translation units, each translation unit consisting of eight 45-45-90 triangles.

Figure 8 shows a $p4mm$ pattern of black and white triangle motifs on a blue background. This pattern is composed of a 2×2 grid of translation units, each one again being comprised of eight 45-45-90 triangles. It is easy to see the mirror lines, but it is not quite as easy to see that the motif triangles avoid them as it is that the flowers avoid them in Figure 5.

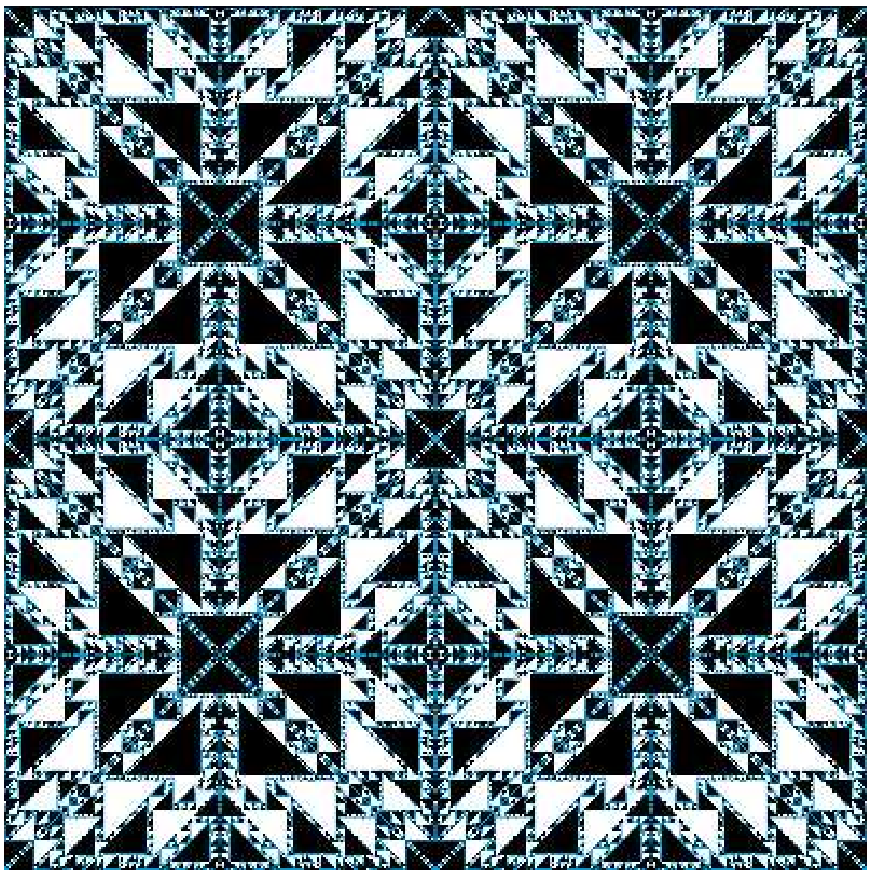

Figure 8: *A random triangle pattern with* $p4mm$ *symmetry.*

Patterns with Symmetry Group p4mg

Patterns with symmetry group $p4mg$ are interesting in that they involve both reflection symmetry and pure rotation symmetry (the rotation symmetries in $p4mm$ are generated by products of reflection symmetries). If the motif of a trial placement overlaps the 4-fold rotation point or a mirror boundary of the fundamental region, we can apply the techniques mentioned above: (1) avoid the mirror boundary or 4-fold rotation axis or (2) center the motif on the mirror boundary or 4-fold rotation point (for a given motif, only one can apply). Of course it is possible to center a motif on a mirror boundary and not on the 4-fold rotation point, or vice versa.

Summary and Future Work

We have presented methods for creating patterns that generate global wallpaper patterns with 4-fold symmetries but are locally fractal in nature. Our goal was to make pleasing patterns with this kind of global symmetry but maintain local randomness. Previously we also showed how to create patterns with $p1, p2mm, p3m1$, and $p6mm$ symmetry. The methods presented here, involving symmetry groups generated by rotations and/or reflections, should also work for the groups $p2, p3, p31m$, and $p6$. In the future, in addition to implementing algorithms for these groups, the future we would like to devise methods that would create locally fractal patterns for the remaining 17 wallpaper groups.

We have used *ad hoc* methods to create a few fractal wallpaper patterns with color symmetry, but it would nice to incorporate color symmetry into our algorithms in a more systematic way.

It would also be interesting to create corresponding spherical or hyperbolic patterns that are locally random, but have global symmetries.

References

[1] J. Conway, H. Burgiel, C. Goodman-Strauss, *The Symmetries of Things,* A.K. Peters, Ltd., Wellesley, MA, 2008. ISBN 1-56881-220-5. Wikipedia site for orbifold notation: `http://en.wikipedia.org/wiki/Orbifold_notation` (accessed Jan. 24, 2016)

[2] Doug Dunham and John Shier, The Art of Random Fractals, in *Bridges Seoul,* (eds. Gary Greenfield, George Hart, and Reza Sarhangi), Seoul, Korea, 2014, pp. 79–86. Also online at: `http://archive.bridgesmathart.org/2014/bridges2014-79.html`

[3] Doug Dunham and John Shier, Fractal Wallpaper Patterns, in *Bridges 2015,* Baltimore, Maryland, 2015, pp. 183–190. Also online at: `http://archive.bridgesmathart.org/2015/bridges2015-183.html`

[4] Christopher Ennis, *(Always) Room for One More,* MAA Math Horizons, February, 2016.

[5] John Shier, Filling Space with Random Fractal Non-Overlapping Simple Shapes *ISAMA 2011 Conference Proceedings,* page 131, June 13–17, 2011.

[6] Doris Schattschneider, The Plane Symmetry Groups: Their Recognition and Notation, *American Mathematical Monthly*, 85, 6, 439-450, July, 1978. Wikipedia site for wallpaper groups: `http://en.wikipedia.org/wiki/Wallpaper_group` (accessed Jan. 24, 2016)

[7] John Shier and Paul Bourke, An Algorithm for Random Fractal Filling of Space, *Computer Graphics Forum,* Vol. 32, Issue 8, pp. 89-97, December 2013. Also available on Shier's web site: `http://www.john-art.com/` (accessed Jan. 24, 2016)

[8] John Shier web site: `http://www.john-art.com/stat_geom_linkpage.html` (accessed Jan. 24, 2016)