A Papercrafted Pattern on a Triply Periodic Polyhedron

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Abstract

We constructed an Escher-like fish tessellation on part of the regular {4,6|4} triply periodic polyhedron. We will discuss the background and motivation for this pattern.

Introduction

The goal of this project was to use papercrafting technology to create part of the regular triply periodic polyhedron {4,6|4} decorated with an Escher-like fish pattern. We succeeded as shown in Figure 1.

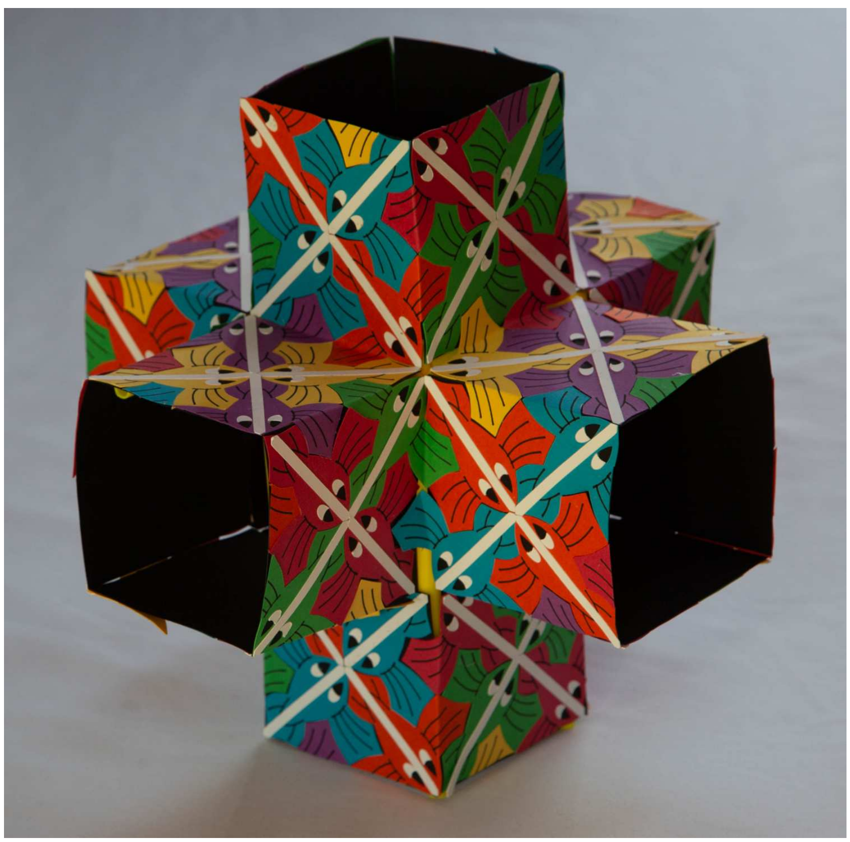

Figure 1: *Part of a papercrafted* {4,6|4} *polyhedron with a fish pattern.*

The first author had also used ordinary printing technology to create the previously published fish pattern on the {4,6|4} polyhedron as shown in Figure 2 below [2].

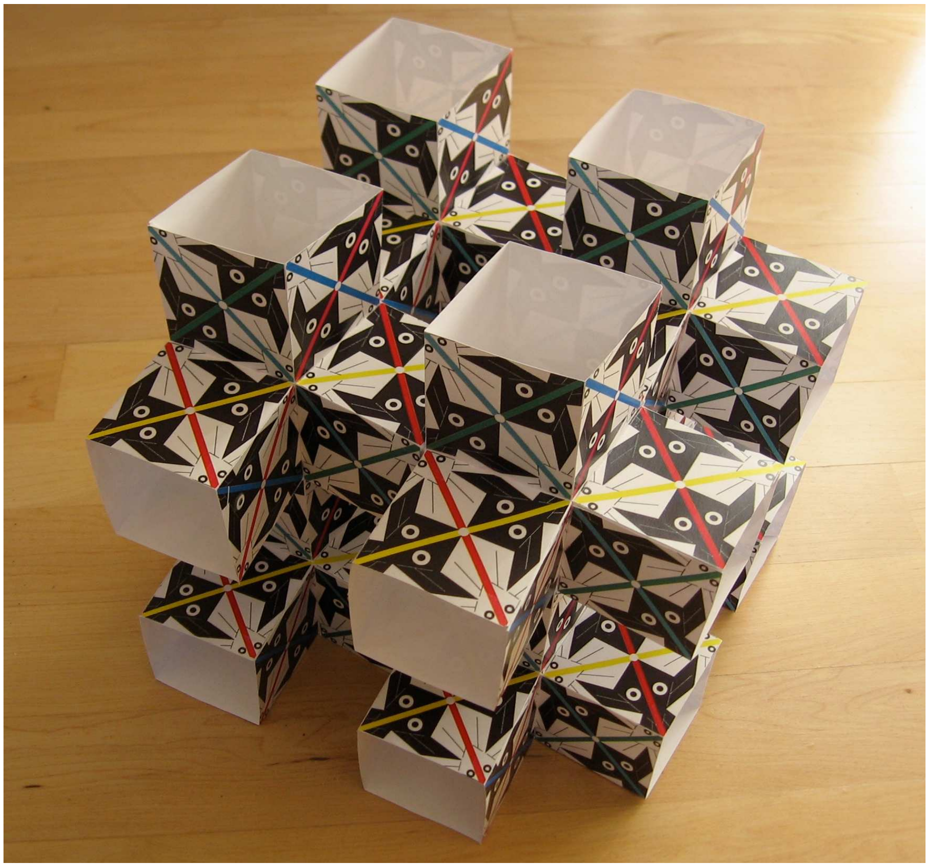

Figure 2: *Part of the previously published* {4,6 | 4} *polyhedron with a fish pattern.*

But there were problems with the pattern on that polyhedron, one of which was discovered only recently. These flaws were what motivated us to create the new art work. To build our polyhedron we decided to use papercraft technology, i.e. a computer-controlled cutter/scorer/printer. This had several advantages: we could (1) use vibrantly colored paper, (2) cut an entire fish with crisp edges, (3) print interior details in black, and (4) score the fins for precise folding across cube edges.

In the next section, we discuss regular triply periodic polyhedra. Then we talk about aesthetic problems with the previous polyhedron. We note similar problems that M.C. Escher had with his hyperbolic pattern *Circle Limit I* and how he fixed them in his woodcut *Circle Limit III*. Next we explain how we overcame all but one of the flaws in the polyhedron of Figure 2. Finally, we indicate a possible direction for future work.

Triply Periodic Polyhedra and the Previous Polyhedron

Mathematicians have considered regular tessellations for millennia. They are composed of regular p-sided polygons, or *p-gons*, that meet q at each vertex, and are denoted by the Schläfli symbol $\{p, q\}$. One can consider regular tessellations in each of the three "classical" geometries: spherical geometry, Euclidean geometry, and hyperbolic geometry. The tessellation $\{p, q\}$ is spherical, Euclidean, or hyperbolic when $(p-2)(q-2)$ is less than, equal to, or greater than 4, respectively. There are the regular spherical (Platonic) tessellations: $\{3, 3\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 3\}$, and $\{5, 3\}$. There are also the three regular Euclidean tessellations: $\{3, 6\}$, $\{4, 4\}$, and $\{6, 3\}$, and an infinite number of regular hyperbolic tessellations with $(p-2)(q-2) > 4$.

It is useful to consider polyhedra to be hyperbolic if the vertex angle sums are all greater than 2π. There exist some such polyhedra that are composed entirely of p-gons meeting q at a vertex, which are also denoted

$\{p, q\}$, by extension of the Schläfli symbol. If such a polyhedron repeats in three independent directions in Euclidean 3-space, it is called triply periodic. If its symmetry group is also transitive on vertices, edges, and faces, it is considered to be regular. H.S.M. Coxeter called these *regular skew polyhedra* [1, 6]. In 1926 he and John Flinders Petrie proved that there are exactly three of these: {4,6|4}, {6,4|4}, and {6,6|3}, where {p,q|r} also extends the Schläfli symbol, and denotes a polyhedron composed of p-gons meeting q at each vertex, with regular r-sided polygonal holes. These polyhedra are considered to be hyperbolic analogs of the Platonic solids and the regular Euclidean tessellations $\{3, 6\}$, $\{4, 4\}$, and $\{6, 3\}$. In this paper we only consider the {4,6|4} polyhedron. Previously [3] we also designed patterns on the {6,4|4} and the {6,6|3} polyhedra. The {4,6|4} polyhedron may be the easiest to understand since it is based on the cubical tessellation of Euclidean 3-space. It can be considered to be composed of (invisible) cubic "hubs" connected by cubic "struts" on each of the hub faces. The struts are hollow cubical cylinders with their open ends connecting neighboring hubs. For example, in Figure 1 we can see all six struts attached to a central hub. And in Figure 2, we see 8 hubs and 24 struts forming a 2-by-2-by-2 grid.

Problems with the Previous Polyhedron and Escher's *Circle Limit I*

In 1958 Escher created his first "hyperbolic" woodcut *Circle Limit I*, which can be considered to be a tessellation of the hyperbolic plane by angular black and white fish [4]. Though it was his first "circle limit" pattern, Escher was dissatisfied with it because it had these three shortcomings as he saw it:

1. The fish were not consistently colored along backbone lines — they alternated from black to white and back every two fish lengths.
2. The fish also changed direction every two fish lengths — thus there was no "traffic flow" (Escher's words) in a single direction along the backbone lines.
3. The fish are very angular and not "fish-like" (we only know they are fish because Escher said they were).

In 1959 Escher solved each of these problems in his pleasing woodcut *Circle Limit III* [5]. The fish are all the same color along a backbone line, they swim head to tail, and they are very fish-shaped.

The pattern on the polyhedron in Figure 2 suffers from the same problems as *Circle Limit I* — its inspiration. Figure 1 shows that we were able to cure the first problem (translation by two fish-lengths preserves color) and the third problem. To solve the second problem on a {p,q|r} polyhedron, both p and q need to be divisible by an odd number in order that all the fish swim head to tail, so the only possibility is {6,6|3}.

There is an additional problem with the pattern on the polyhedron of Figure 2 that was only discovered during the past year and was an important motivation to construct the new polyhedron of Figure 1. The backbone lines of one color on one side of the struts are not parallel to the backbone lines of that color on the opposite side. For example the yellow backbone lines that can be seen "on top" in Figure 2 run roughly left to right, but the yellow backbone lines underneath run front to back. Perhaps the best way to see this problem with the Figure 2 polyhedron is to view it on a mirror, as shown in Figure 3. We also show the Figure 1 polyhedron on a mirror in Figure 4, in order to demonstrate that the backbone lines of fish of one color are parallel (this is easiest to see with the yellow fish). We achieved this by being more careful in designing the pattern for the polyhedron. Thus there are a total of six families of parallel lines, two orthogonal families in each plane and one family for each of the six colors of fish. However, the first author was not so careful in constructing the polyhedron of Figure 2, and didn't think to use a mirror to check the results.

In Figure 3 we can see that the yellow backbone lines go from left to right on the direct image at the top, but from front to back in the mirror image at the bottom, and so aren't parallel. On the other hand, in Figure 4 the backbone lines of the yellow fish go from left to right in the direct image at the top and in the mirror image below, and are thus parallel as desired (as are the backbones of the fish of the other five colors).

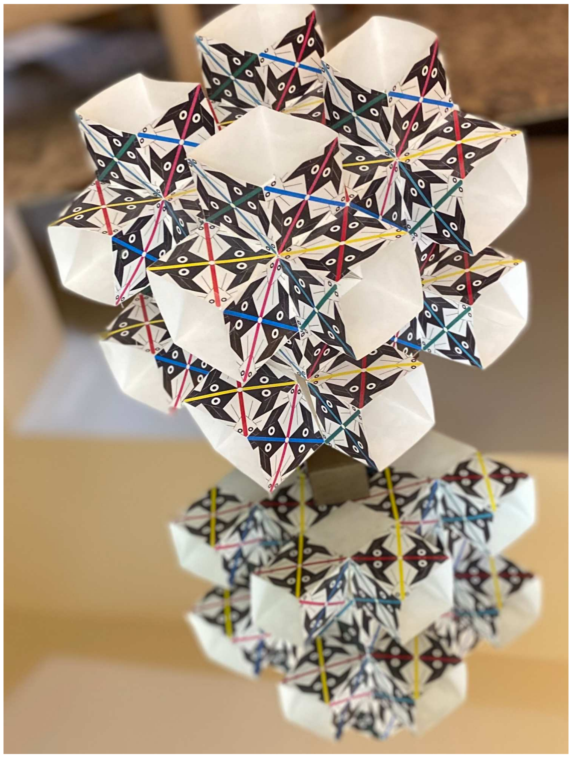

Figure 3: *The Figure 2 polyhedron on a mirror.*

Figure 4: *The Figure 1 polyhedron on a mirror .*

Conclusions and Future Work

We have shown a pattern on a new polyhedron in Figure 1 that fixes most of the problems of the polyhedron in Figure 2. The only problem that was not fixed is the "'traffic flow" problem: the fish along a backbone line change directions every two fish. As noted above, it may be possible to place an aesthetic fish pattern with "traffic flow" along each backbone line on the {6,6|3} polyhedron, a project for future work.

References

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