Bresenham’s Midpoint Algorithm

CS5600 Computer Graphics

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Line Characterizations

• Explicit: \( y = mx + B \)

• Implicit: \( F(x,y) = ax + by + c = 0 \)

• Constant slope: \( \frac{\Delta y}{\Delta x} = k \)

• Constant derivative: \( f'(x) = k \)

Line Characterizations - 2

• Parametric: \( P(t) = (1-t)P_0 + tP_1 \)
  where, \( P(0) = P_0; \quad P(1) = P_1 \)

• Intersection of 2 planes

• Shortest path between 2 points

• Convex hull of 2 discrete points

Discrete Lines

• Lines vs. Line Segments

• What is a discrete line segment?
  – This is a relatively recent problem
  – How to generate a discrete line?
“Good” Discrete Line - 1

- No gaps in adjacent pixels
- Pixels close to ideal line
- Consistent choices; same pixels in same situations

“Good” Discrete Line - 2

- Smooth looking
- Even brightness in all orientations
- Same line for \( P_0 P_1 \) as for \( P_1 P_0 \)
- Double pixels stacked up?

Incremental Fn Eval

- Recall \( f(x_{i+1}) = f(x_i) + \Delta(x_i) \)
- Characteristics
  - Fast
  - Cumulative Error
- Need to define \( f(x_o) \)

Meeting Bresenham Criteria

- \( m = 0; \ m = 1 \implies \) trivial cases
- \( (x_0, y_0) \neq (0, 0) \implies \) translate
- \( 0 > m > -1 \implies \) flip about \( x\)-axis
- \( m > 1 \implies \) flip about \( x = y \)
Case 1: Translate to Origin

- Move \((x_0, y_0)\) to the origin
  \((x'_0, y'_0) = (0,0)\);
  \((x'_1, y'_1) = (x_1 - x_0, y_1 - y_0)\)
- Need only consider lines emanating from the origin.

Case 0: Trivial Situations

- \(m = 0 \implies\) horizontal line
- \(m = 1 \implies\) line \(y = x\)
- Do not need Bresenham

Case 2: Flip about \(x\)-axis

- Need only consider lines emanating from the origin.
Case 2: Flip about $x$-axis

- Suppose, $0 > m > -1$,
- Flip about $x$-axis $(y' = -y)$:
  \[(x_0', y_0') = (x_0, -y_0);\]
  \[(x_1', y_1') = (x_1, -y_1)\]

How do slopes relate?

\[
m = \frac{y_1 - y_0}{x_1 - x_0};
\]
\[
m' = \frac{y_1' - y_0'}{x_1' - x_0'}\]

by definition

Since $y_i' = -y_i$, $m' = \frac{-y_1 - (-y_0)}{x_1 - x_0}$

\[∵ 0 > m > -1 \Rightarrow 0 < m' < 1\]

Case 3: Flip about line $y = x$

\[\begin{align*}
\text{How do slopes relate?} \\
\quad m' &= -\left(\frac{y_1 - y_0}{x_1 - x_0}\right) \\
\quad m' &= -m \\
\therefore 0 > m > -1 &\Rightarrow 0 < m' < 1
\end{align*}\]
Case 3: Flip about line $y = x$

$y = mx + B$,  
swap $x \leftrightarrow y$ and prime them,  
$x' = my' + B$,  
$my' = x' - B$

Case 3: $m' = 1$

$y' = \left(\frac{1}{m}\right)x' - B$,  
$\therefore m' = \left(\frac{1}{m}\right)$ and,  
$m > 1 \Rightarrow 0 < m' < 1$

Restricted Form

- Line segment in first octant with $0 < m < 1$
- Let us proceed

Two Line Equations

- Explicit: $y = mx + B$
- Implicit: $F(x,y) = ax + by + c = 0$
  Define: $dy = y_1 - y_0$  
  $dx = x_1 - x_0$  
  Hence, $y = \left(\frac{dy}{dx}\right)x + B$
From previous

We have, \( y = \left( \frac{dy}{dx} \right) x + B \)

Hence, \( \frac{dy}{dx} x - y + B = 0 \)

Relating Explicit to Implicit Eq’s

Recall, \( \frac{dy}{dx} x - y + B = 0 \)

Or, \( (dy)x + (-dx)y + (dx)B = 0 \)

\[ F(x, y) = (dy)x + (-dx)y + (dx)B = 0 \]

where, \( a = (dy); \quad b = -(dx); \quad c = B(dx) \)

Investigate Sign of \( F \)

Verify that

\[ F(x, y) = \begin{cases} + & \text{below line} \\ 0 & \text{on line} \\ - & \text{above line} \end{cases} \]

Look at extreme values of \( y \)

The Picture

\[ F(x, y) < 0 \quad \text{above line} \]

\[ F(x, y) > 0 \quad \text{below line} \]
Key to Bresenham Algorithm

"Reasonable assumptions" have reduced the problem to making a binary choice at each pixel:

- NE (next)
- E (next)

Decision Variable $d$ (logical)

Define a logical decision variable $d$
- linear in form
- incrementally updated (with addition)
- tells us whether to go E or NE

The Picture

Ideal line

$y' = y_p + 1$

Previous

$x = x_p + 1$

NE

Q

M \text{ midpoint}

The Picture (again)

Ideal line

Ideal line

$x_p + 1, y_p + \frac{1}{2}$
Observe the relationships

- Suppose $Q$ is above $M$, as before.
- Then $F(M) > 0$, $M$ is below the line
- So, $F(M) > 0$ means line is above $M$
- Need to move NE, increase $y$ value

The Picture (again)

$M = \text{Midpoint} = (x_p + 1, y_p + \frac{1}{2})$

- Want to evaluate at $M$
- Will use an incr decision var $d$
- Let, $d = F(x_p + 1, y_p + \frac{1}{2})$
  $d = a(x_p + 1) + b(y_p + \frac{1}{2}) + c$
How will $d$ be used?

Recall, $d = a(x_p + 1) + b(y_p + \frac{1}{2}) + c$

Therefore,

$$
\begin{align*}
&> 0 \Rightarrow \text{NE (midpoint below ideal line)} \\
&< 0 \Rightarrow \text{E (midpoint above ideal line)} \\
&= 0 \Rightarrow \text{E (arbitrary)}
\end{align*}
$$

Case 1: Suppose $E$ is chosen

Recall $d_{old} = a(x_p + 1) + b(y_p + \frac{1}{2}) + c$

$E \Rightarrow x \leftarrow x + 1; \ y \leftarrow y,$

$\therefore d_{new} = F(x_p + 2, y_p + \frac{1}{2})$

$$= a(x_p + 2) + b(y_p + \frac{1}{2}) + c$$

Case 1: Suppose $E$ is chosen

$$\begin{align*}
d_{new} - d_{old} &= \left[a(x_p + 2) + b(y_p + \frac{1}{2}) + c\right] \\
&\quad - \left[a(x_p + 1) + b(y_p + \frac{1}{2}) + c\right] \\
\therefore d_{new} &= d_{old} + a
\end{align*}$$

Review of Explicit to Implicit

Recall, $\frac{dy}{dx} x - y + B = 0$

Or, $(dy)x + (-dx)y + (dx)B = 0$

$\therefore F(x, y) = (dy)x + (-dx)y + (dx)B = 0$

where, $a = (dy); \ b = -(dx); \ c = B(dx)$
Case 1: $d_{new} = d_{old} + a$

$\Delta_E \equiv \text{increment we add if } E \text{ is chosen.}$

So, $\Delta_E = a$. But remember that $a = dy$ (from line equations).

Hence, $F(M)$ is not evaluated explicitly.

We simply add $\Delta_E = a$ to update $d$ for $E$.

Case 2: Suppose $NE$ chosen

Recall $d_{old} = a(x_p + 1) + b(y_p + \frac{1}{2}) + c$

and, $NE \Rightarrow$ $x \leftarrow x + 1$; $y \leftarrow y + 1$,

$\therefore d_{new} = F(x_p + 2, y_p + \frac{3}{2})$

$= a(x_p + 2) + b(y_p + \frac{3}{2}) + c$

Case 2: Suppose $NE$

$d_{new} - d_{old} =$

$= \left( a(x_p + 2) + b(y_p + \frac{3}{2}) + c \right)$

$\quad - \left( a(x_p + 1) + b(y_p + \frac{1}{2}) + c \right)$

$\therefore d_{new} = d_{old} + a + b$

$\Delta_{NE} \equiv \text{increment that we add if } NE \text{ is chosen.}$

So, $\Delta_{NE} = a + b$. But remember that $a = dy$, and $b = -dx$ (from line equations).

Hence, $F(M)$ is not evaluated explicitly.

We simply add $\Delta_{NE} = a + b$ to update $d$ for $NE$. 
Case 2: \[ d_{\text{new}} = d_{\text{old}} + a + b. \]

\[ \Delta_{NE} = a + b, \text{ where } a = dy, \text{ and } b = -dx \]

means, we simply add \[ \Delta_{NE} = a + b, \text{ i.e.,} \]

\[ \Delta_{NE} = dy - dx \]

to update \( d \) for \( NE \).

Summary

• At each step of the procedure, we must choose between moving \( E \) or \( NE \) based on the sign of the decision variable \( d \).

• Then update according to

\[ d \leftarrow \begin{cases} 
\Delta_E, & \text{where } \Delta_E = dy, \text{ or} \\
\Delta_{NE}, & \text{where } \Delta_{NE} = dy - dx 
\end{cases} \]

What is initial value of \( d \)?

• First point is \( (x_0, y_0) \)

• First midpoint is \( (x_0 + 1, y_0 + \frac{1}{2}) \)

• What is initial midpoint value?

\[ d(x_0 + 1, y_0 + \frac{1}{2}) = F(x_0 + 1, y_0 + \frac{1}{2}) \]

What is initial value of \( d \)?

\[ F(x_0 + 1, y_0 + \frac{1}{2}) = a(x_0 + 1) + b(y_0 + \frac{1}{2}) + c \]

\[ = (ax_0 + by_0 + c) + (a + b) \frac{1}{2} \]

\[ = F(x_0, y_0) + (a + b) \frac{1}{2} \]
What is initial value of \( d \) ?

Note, \( F(x_0, y_0) = 0 \), since \((x_0, y_0)\) is on line.

Hence,
\[
F(x_0 + 1, y_0 + \frac{1}{2}) = 0 + a + \frac{b}{2}
\]
\[
= (dy) - \left(\frac{dx}{2}\right)
\]

What Does “2 x” Do?

- Has the same 0-set
  \[
  2F(x, y) = 2(ax + by + c) = 0
  \]
- Changes the slope of the plane
- Rotates plane about the 0-set line

What is initial value of \( d \)?

Note, \( F(x_0, y_0) = 0 \), since \((x_0, y_0)\) is on line.

Hence,
\[
F(x_0 + 1, y_0 + \frac{1}{2}) = 0 + a + \frac{b}{2}
\]
\[
= (dy) - \left(\frac{dx}{2}\right)
\]

Multiplyin g \( F(x_0 + 1, y_0 + \frac{1}{2}) = (dy) - \left(\frac{dx}{2}\right) \)

by 2 gives,
\[
2F(x_0 + 1, y_0 + \frac{1}{2}) = 2(dy) - dx
\]
What is initial value of \( d \)?

\[
2F(x, y) = 2(ax + by + c) = 0
\]

So, first value of

\[
d = 2(dy) - (dx)
\]

More Summary

- Initial value \(2(dy) - (dx)\)
- Case 1: \(d \leftarrow d + \Delta_E\), where \(\Delta_E = 2(dy)\)
- Case 2: \(d \leftarrow d + \Delta_{NE}\), where \(\Delta_{NE} = 2((dy) - (dx))\)

More Summary

Choose

\[
\begin{cases} 
E & \text{if } d \leq 0 \\
NE & \text{otherwise}
\end{cases}
\]

Example

- Line end points:
  \((x_0, y_0) = (5, 8); \quad (x_1, y_1) = (9, 11)\)
- Deltas: \(dx = 4; dy = 3\)
**Example (dx = 4; dy = 3)**

- Initial value of 

  \[ d(5,8) = 2(2y) - (dx) \]
  \[ = 6 - 4 = 2 > 0 \]

  \[ d = 2 \implies NE \]

- Update value of \( d \)

- Last move was \( NE \), so

  \[ 2d(6,9) = 2(dy - dx) \]
  \[ = 2(3 - 4) = -2 \]

  \[ d = 2 - 2 = 0 \implies E \]
Example \((dx=4; \ dy=3\)) -2

- Update value of \(d\)
- Last move was NE, so
  
  \[
  2d(6,9) = 2d(y - dy)
  \]
  
  \[
  = 2(4 - 3) = -2
  \]
  
  \[
  d = 2 - 2 = 0 \Rightarrow E
  \]

Example \((dx=4; \ dy=3\))

- Previous move was \(E\)
  
  \[
  d(7,9) = 2(dy)
  \]
  
  \[
  = 2(3) = 6
  \]
  
  \[
  d = 0 + 6 > 0 \Rightarrow NE
  \]
Example \((dx=4; \; dy=3)\)

- Previous move was \(NE\), so

\[
2d(8,10) = 2(dy - dx) = 2(3 - 4) = -2
\]

\[
d = 6 - 2 = 4 \implies NE
\]
More Raster Line Issues

- Fat lines with multiple pixel width
- Symmetric lines
- How should end pt geometry look?
- Generating curves, e.g., circles, etc.
- Jaggies, staircase effect, aliasing...

Pixel Space

Example

Example