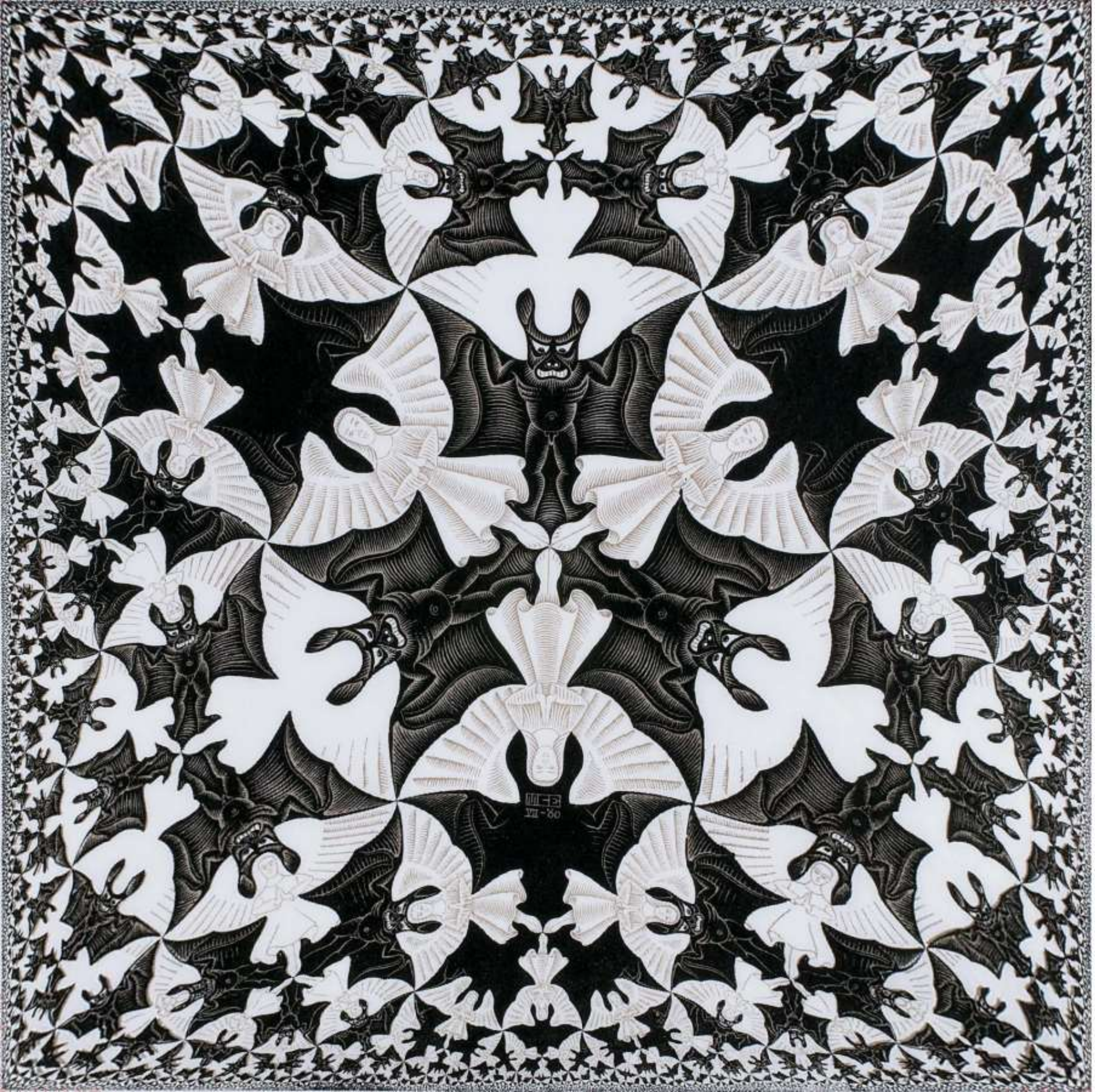


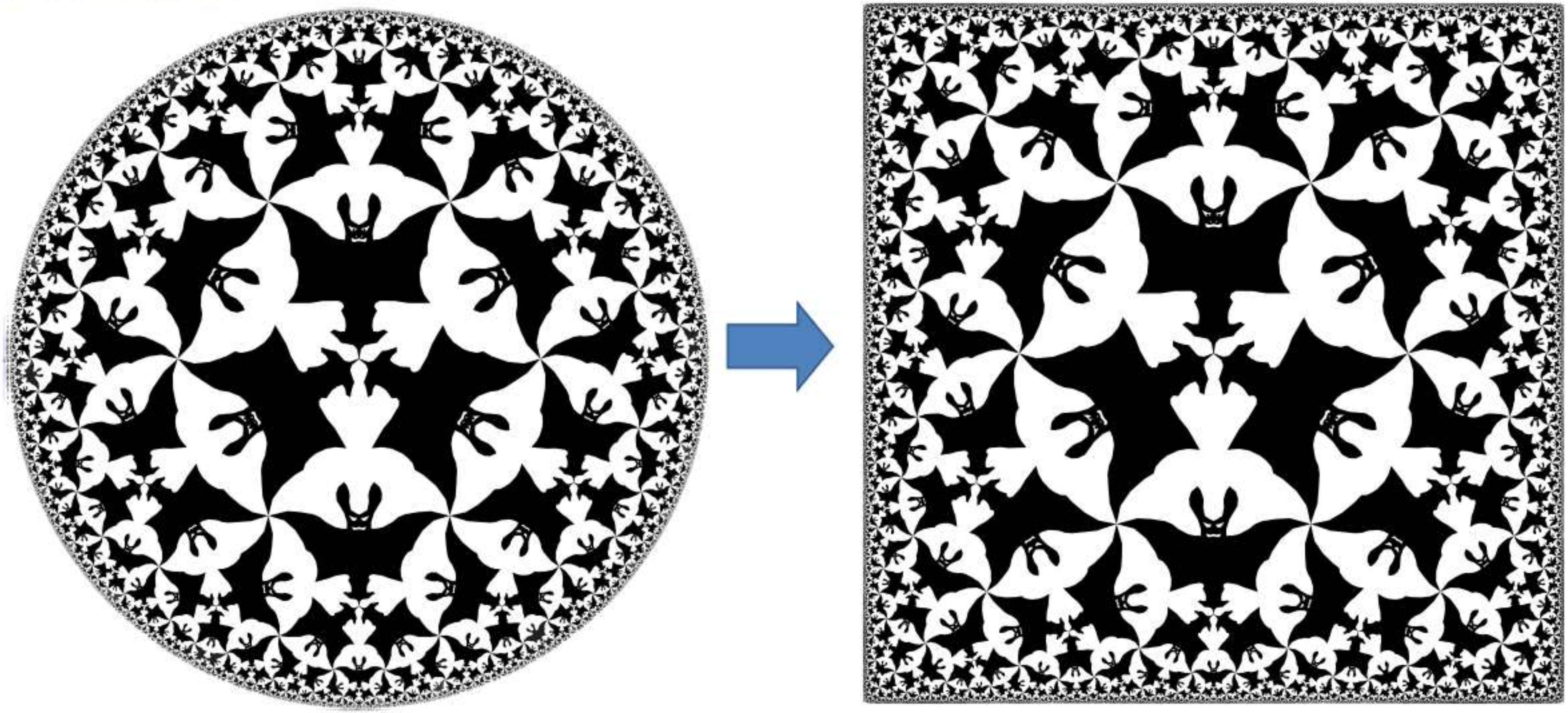
A Poor Man's Hyperbolic Square Mapping

Chamberlain Fong

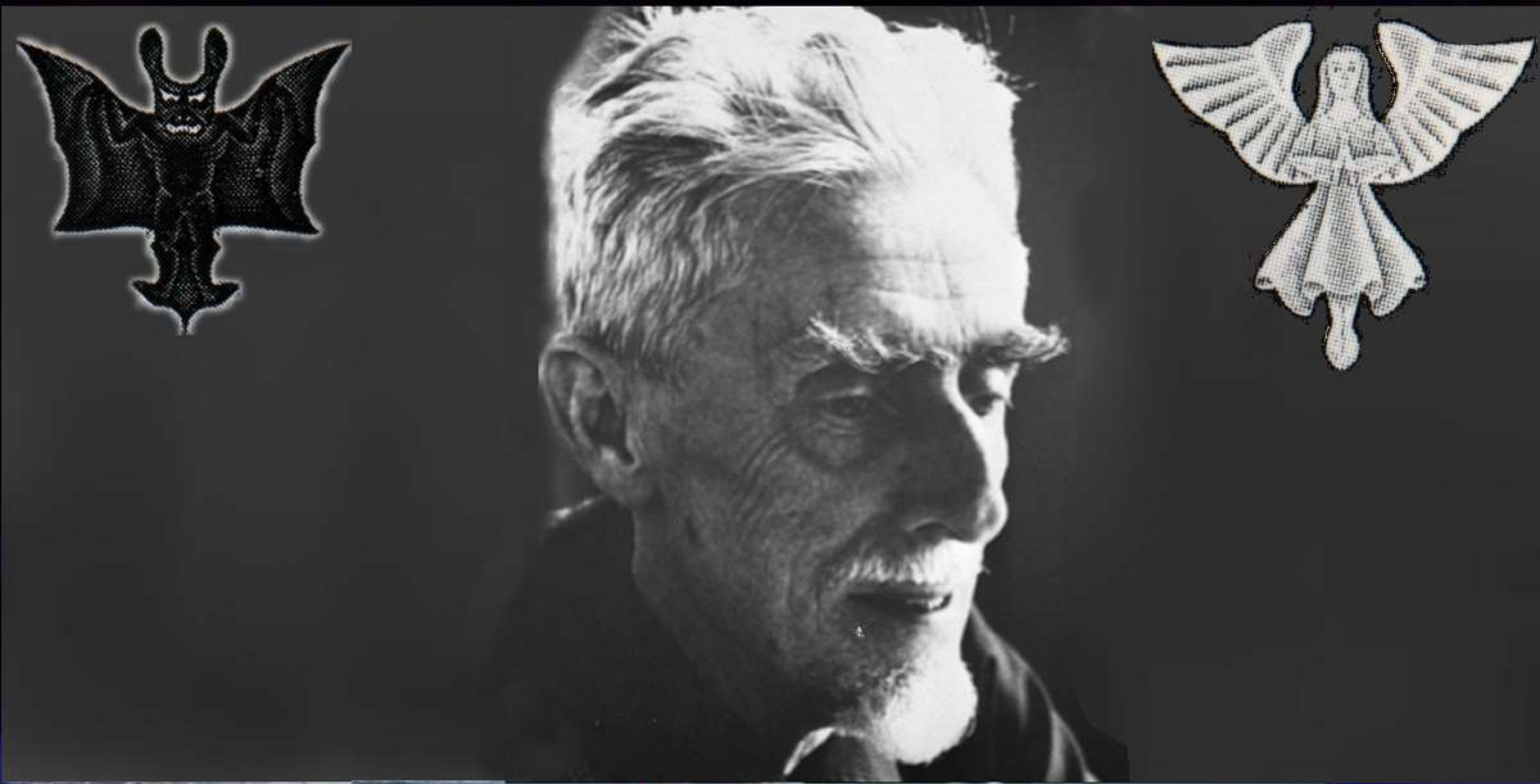
Douglas Dunham

BRIDGES STOCKHOLM 2018

MC Escher
angels & devils



circle limit IV (1960)



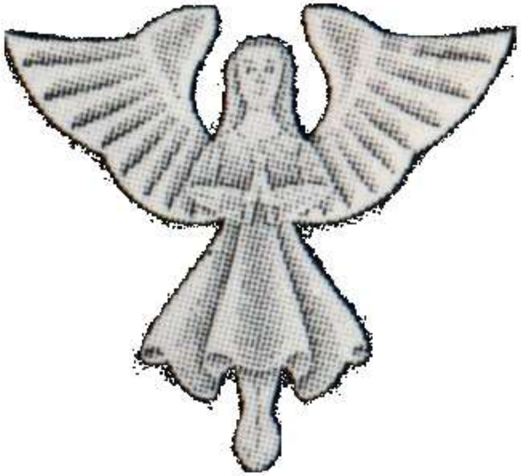
How many devils are there in this rendition of Circle Limit IV ?

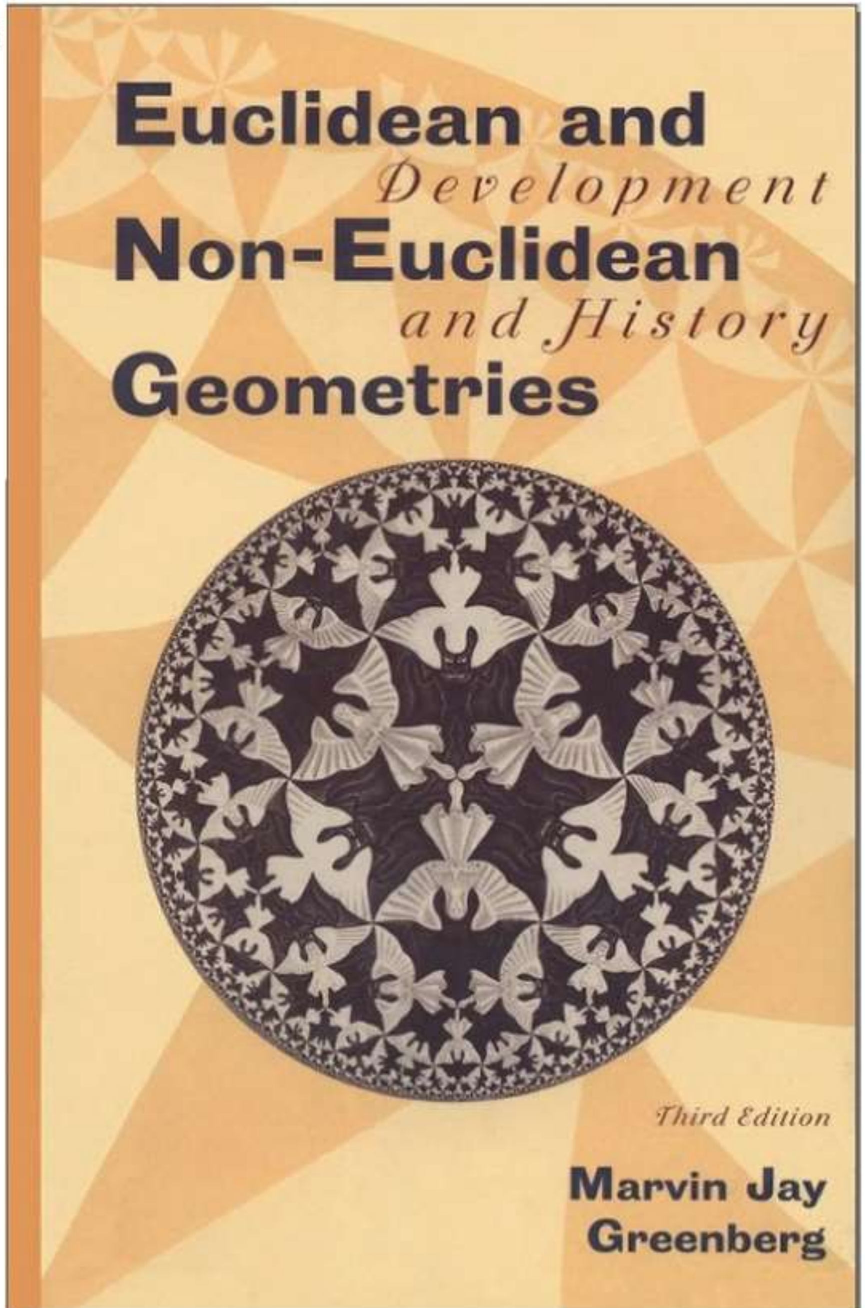
A: 1729

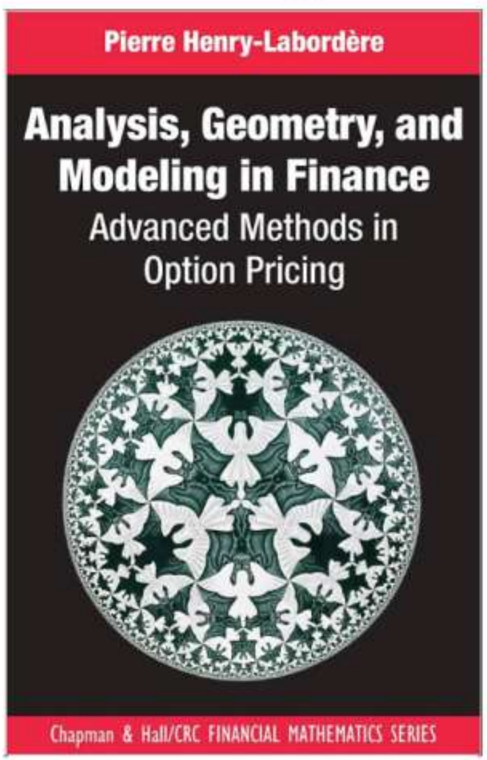
B: 393,213

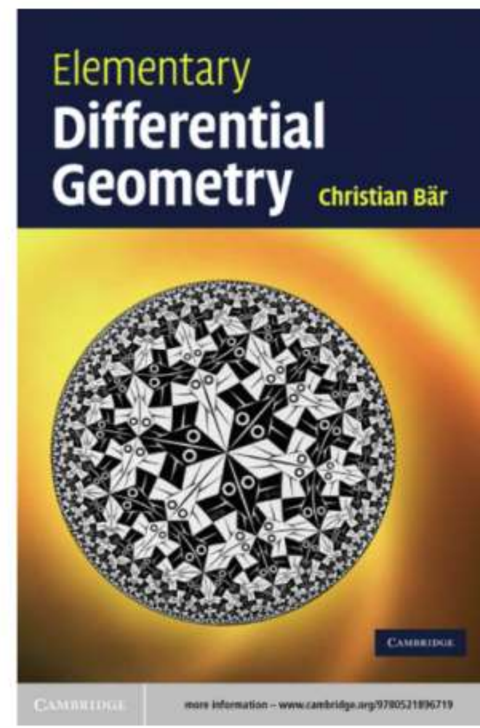
C: 196,883

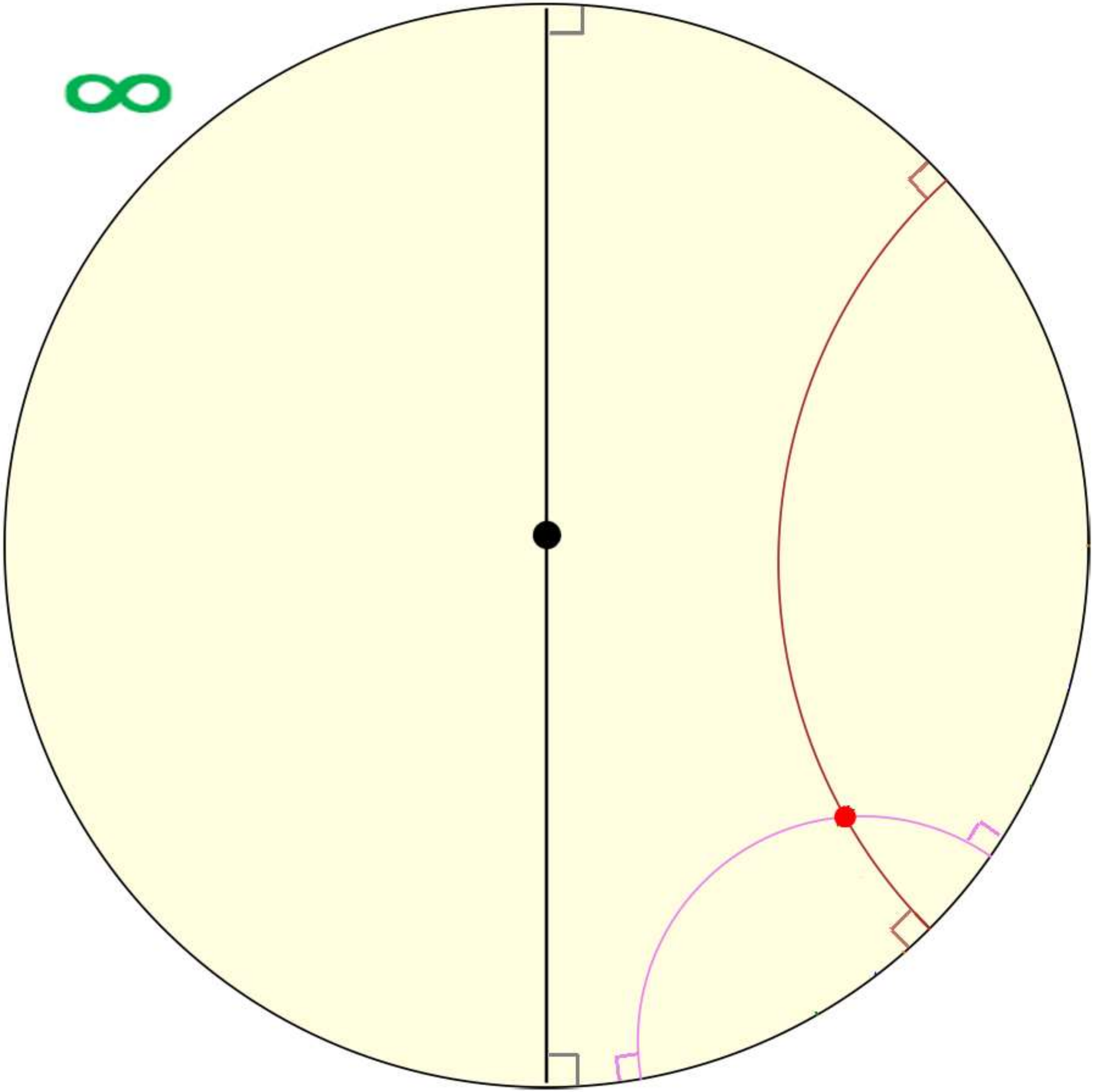
D: infinite




Euclidean and
Development
Non-Euclidean
and History
Geometries
Third Edition
Marvin Jay
Greenberg

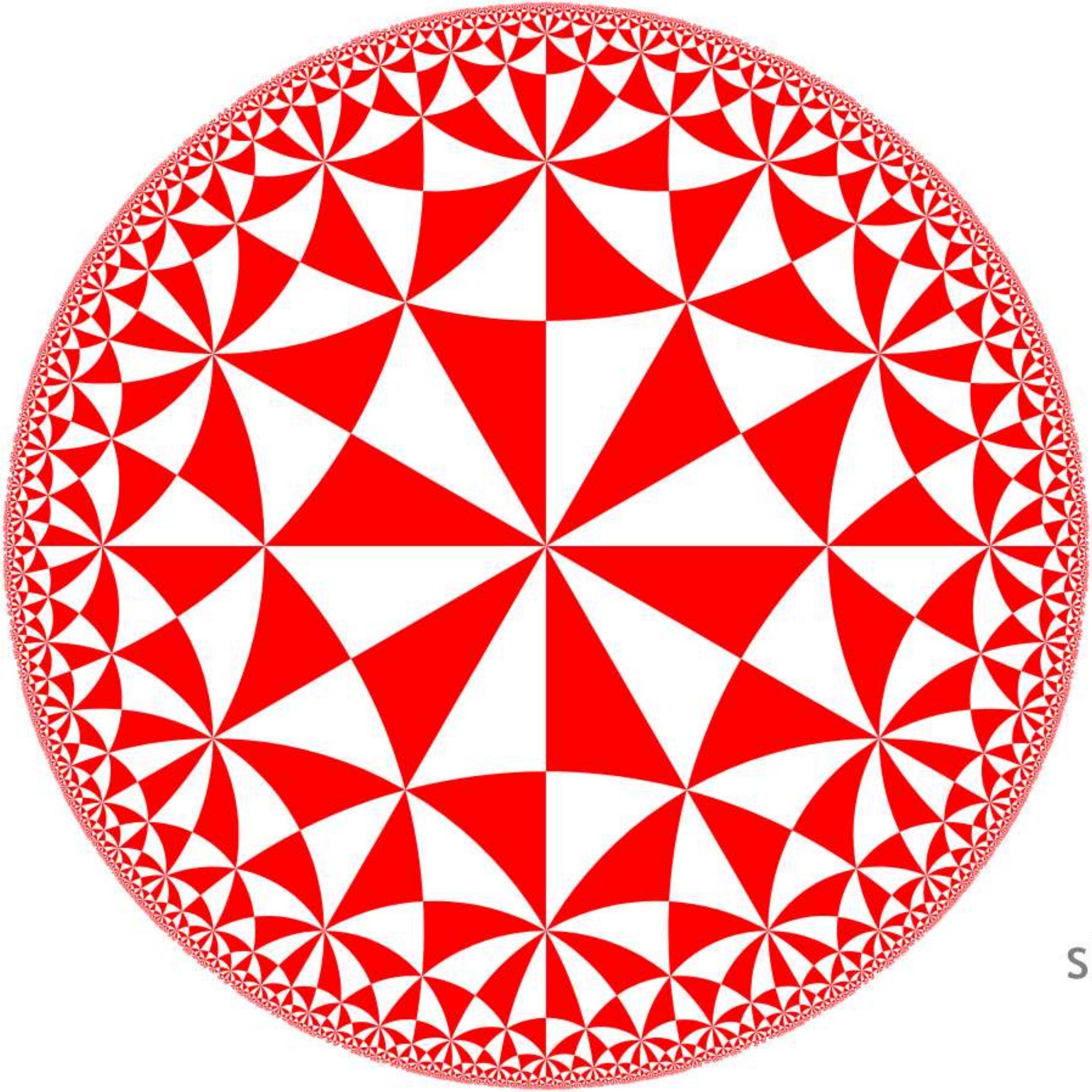

Pierre Henry-Labordère
Analysis, Geometry, and
Modeling in Finance
Advanced Methods in
Option Pricing
Chapman & Hall/CRC FINANCIAL MATHEMATICS SERIES

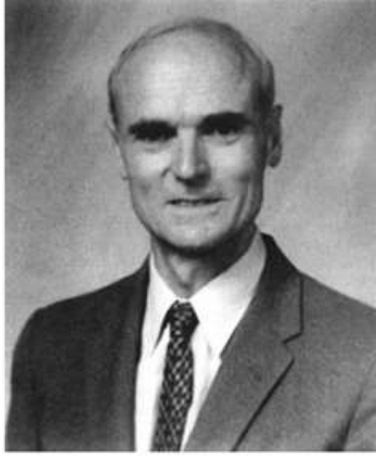

Elementary
Differential
Geometry
Christian Bär



Henri
Poincare
1882

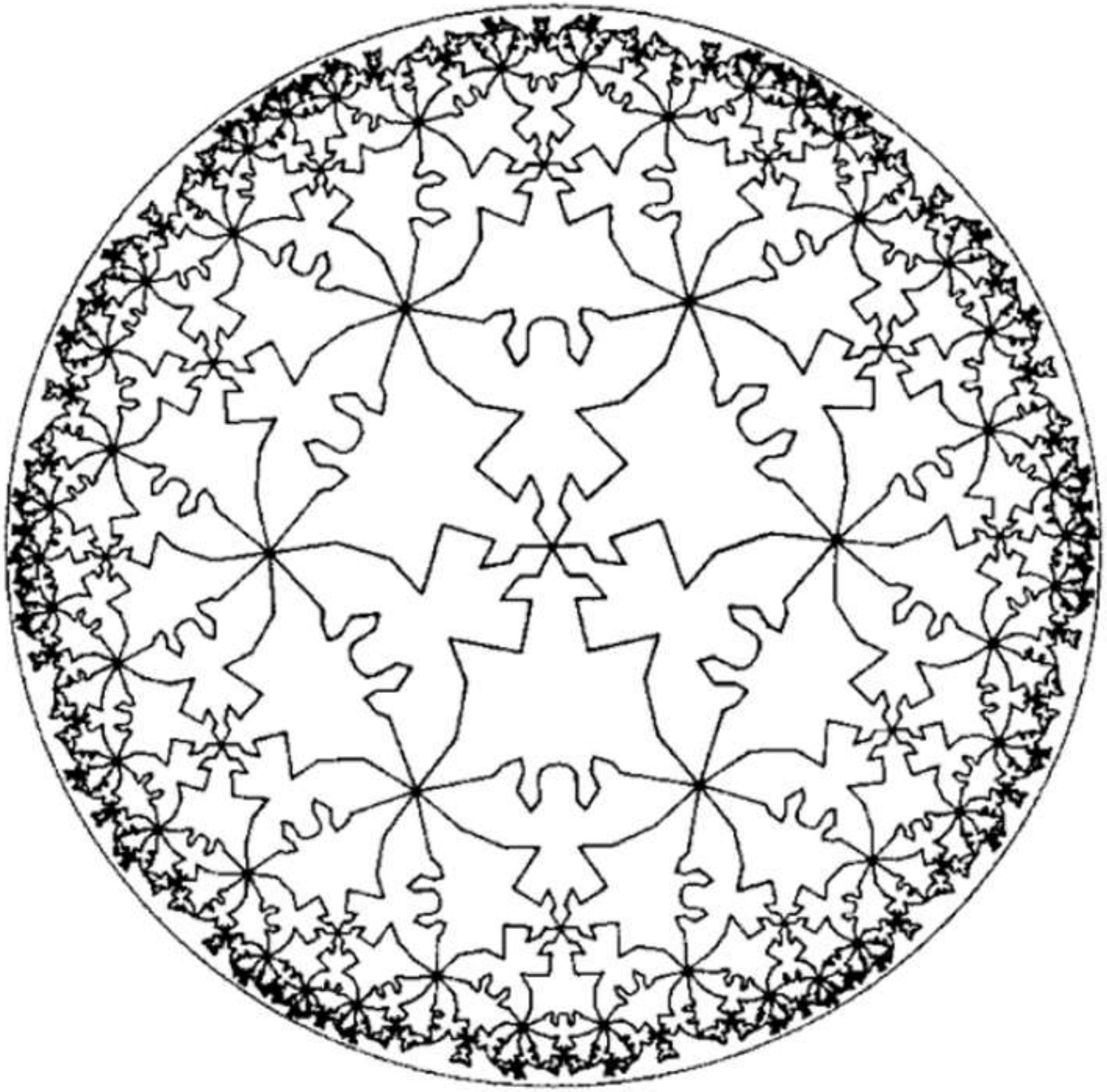
Poincare
disk
model
(conformal)





H.S.M. Coxeter 1957

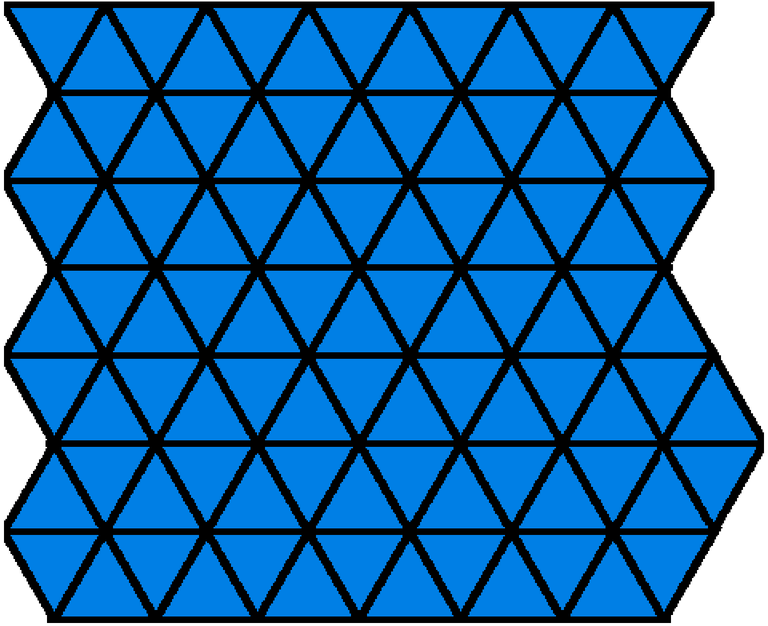
shape vs. size distortion



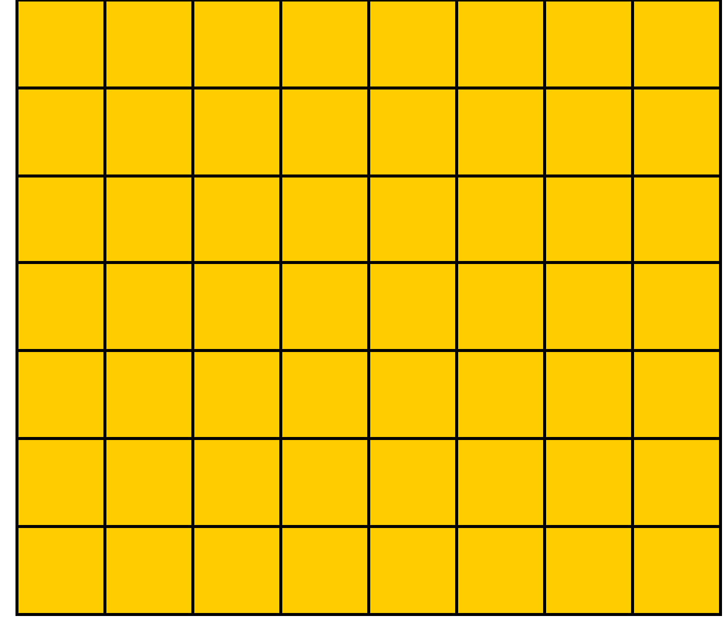
Doug Dunham

and computer-generated hyperbolic art

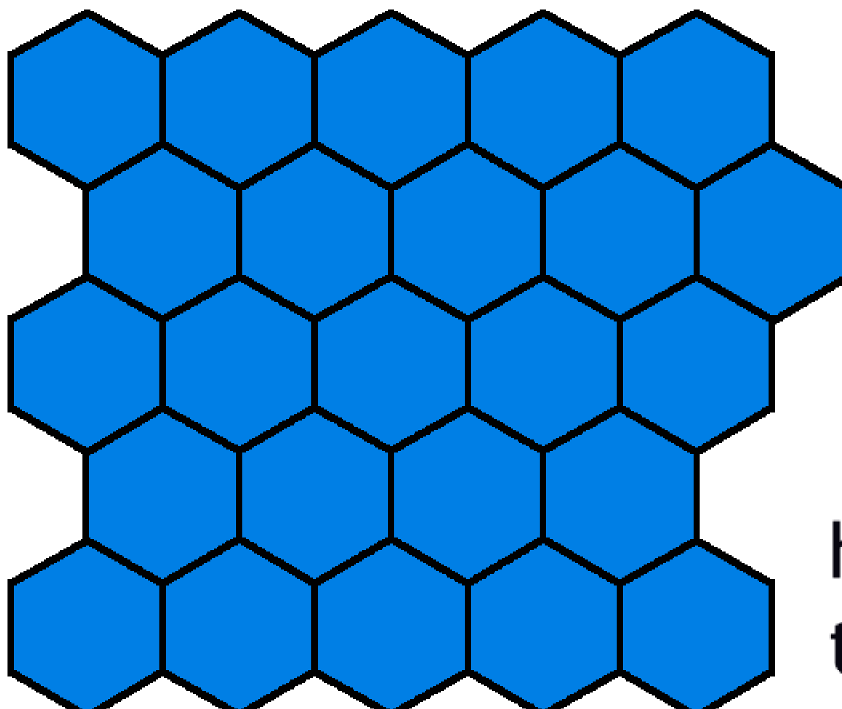
Dunham, Lindgren, Witte. "Creating Repeating Hyperbolic Patterns" Proceedings of Siggraph 1981



triangular
tiling

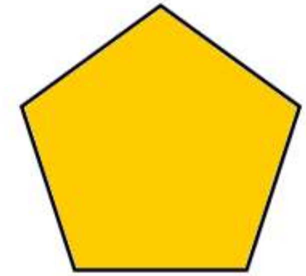


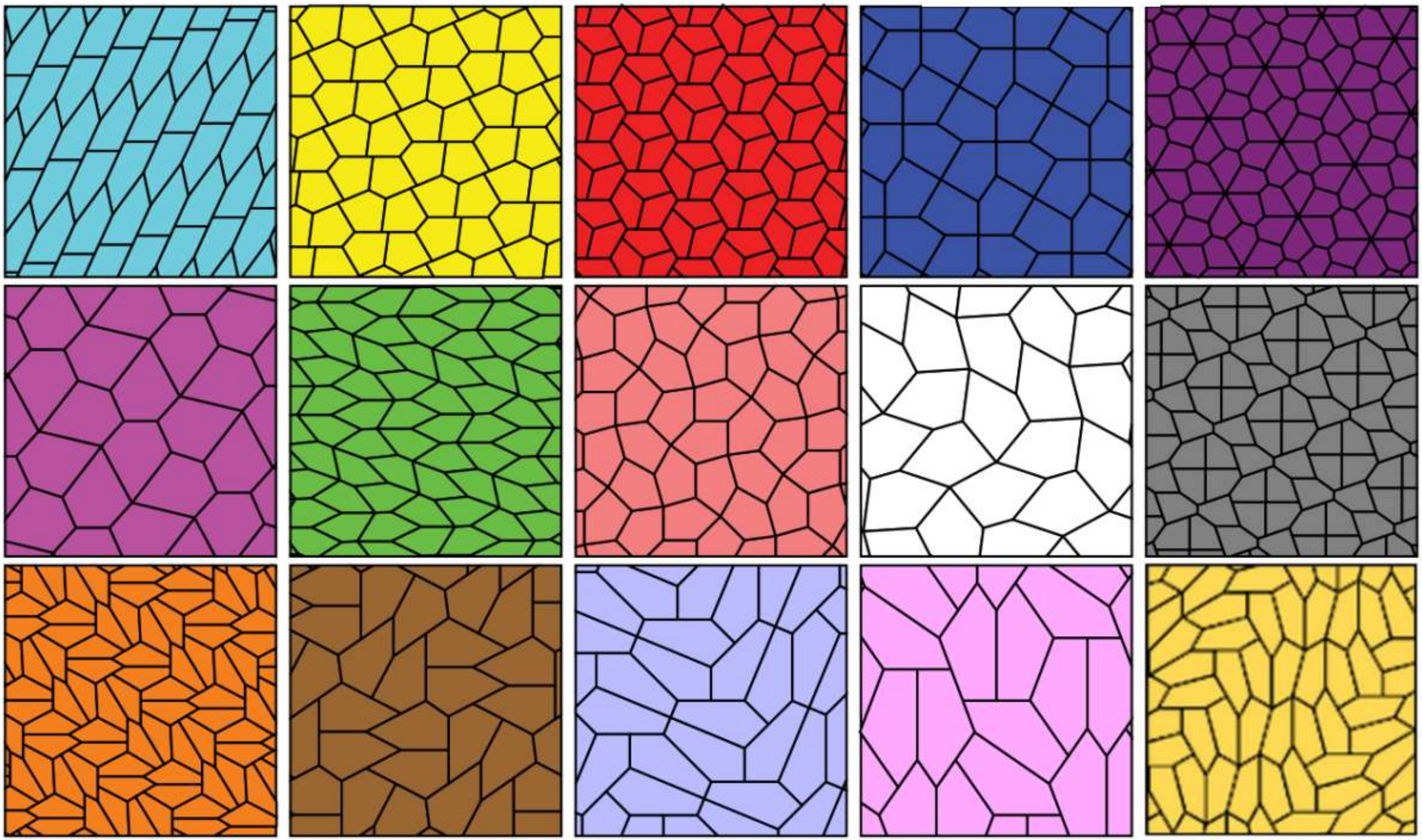
square
tiling



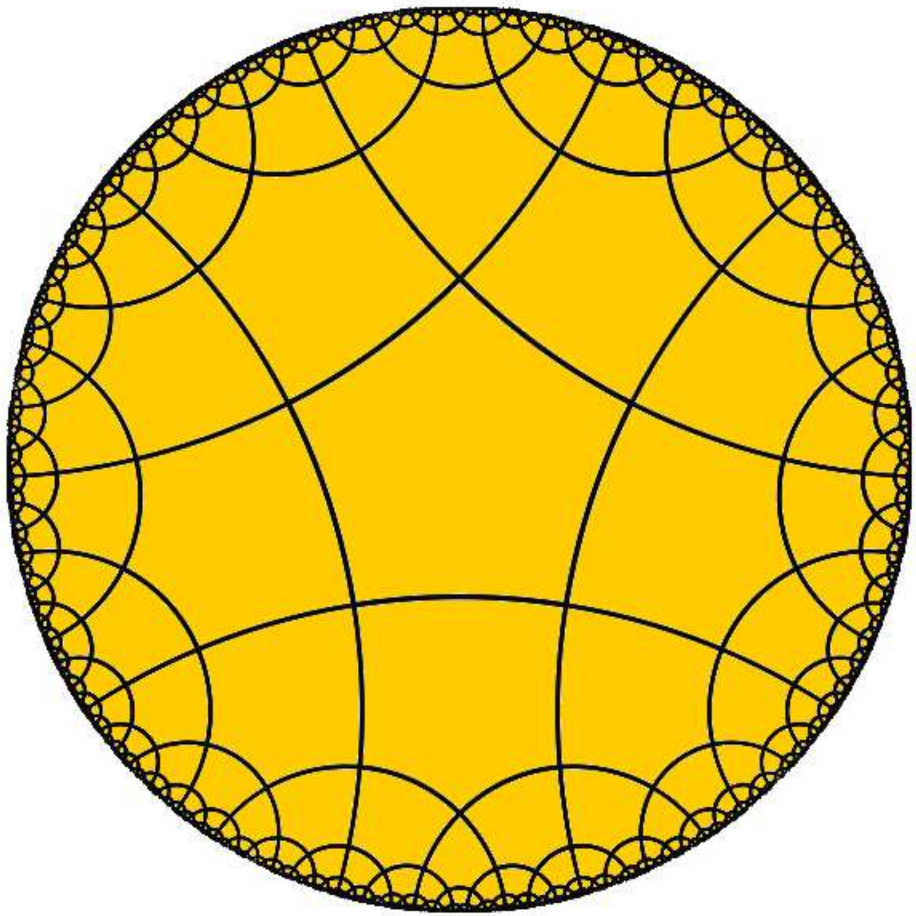
hexagonal
tiling

Euclidean pentagonal tiling

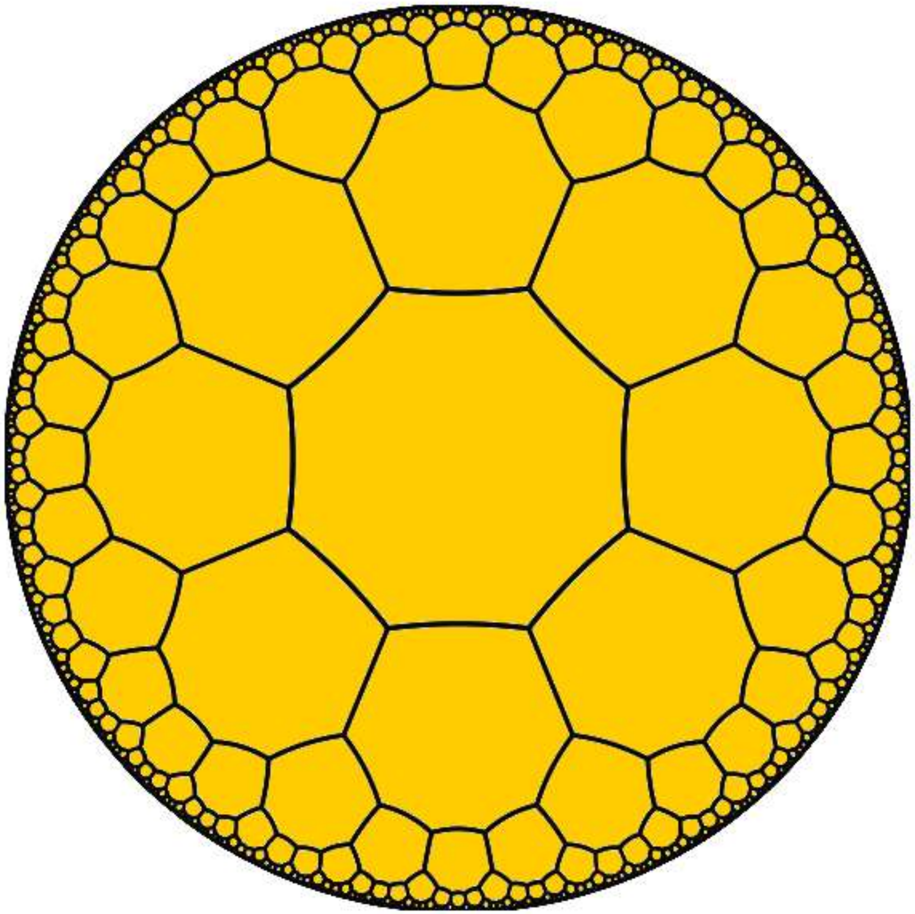




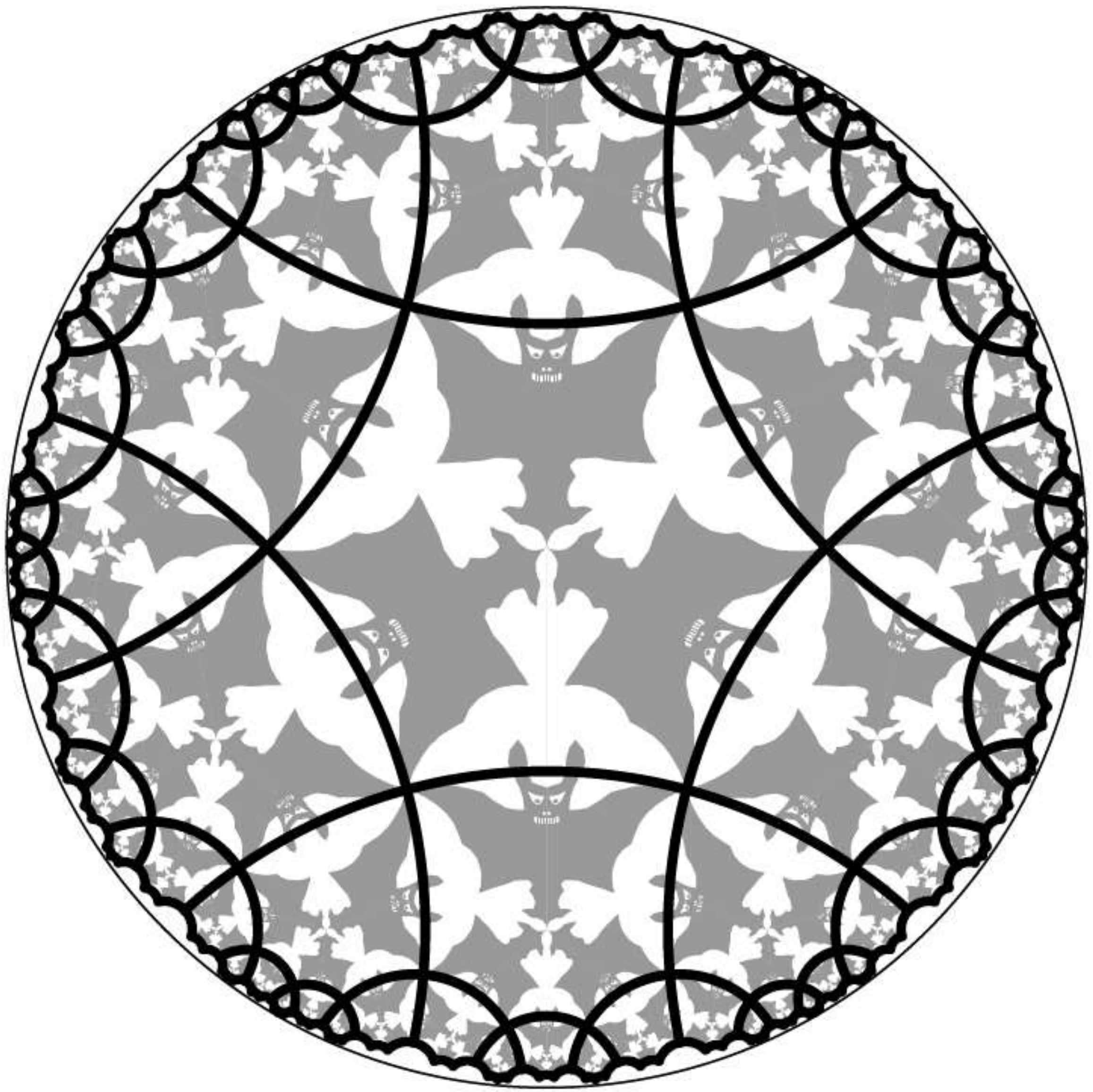
hyperbolic pentagonal tiling



hyperbolic octagonal tiling



hyperbolic hexagonal tiling



Octagonal STOP signs

United States

Mexico

Sweden

China

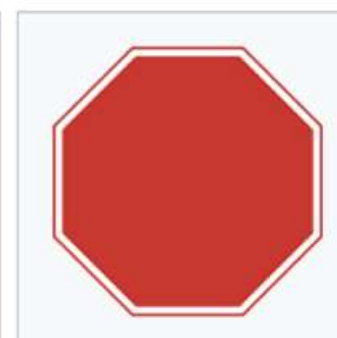
Peru

Egypt

Malaysia

Israel

Canada (Quebec)



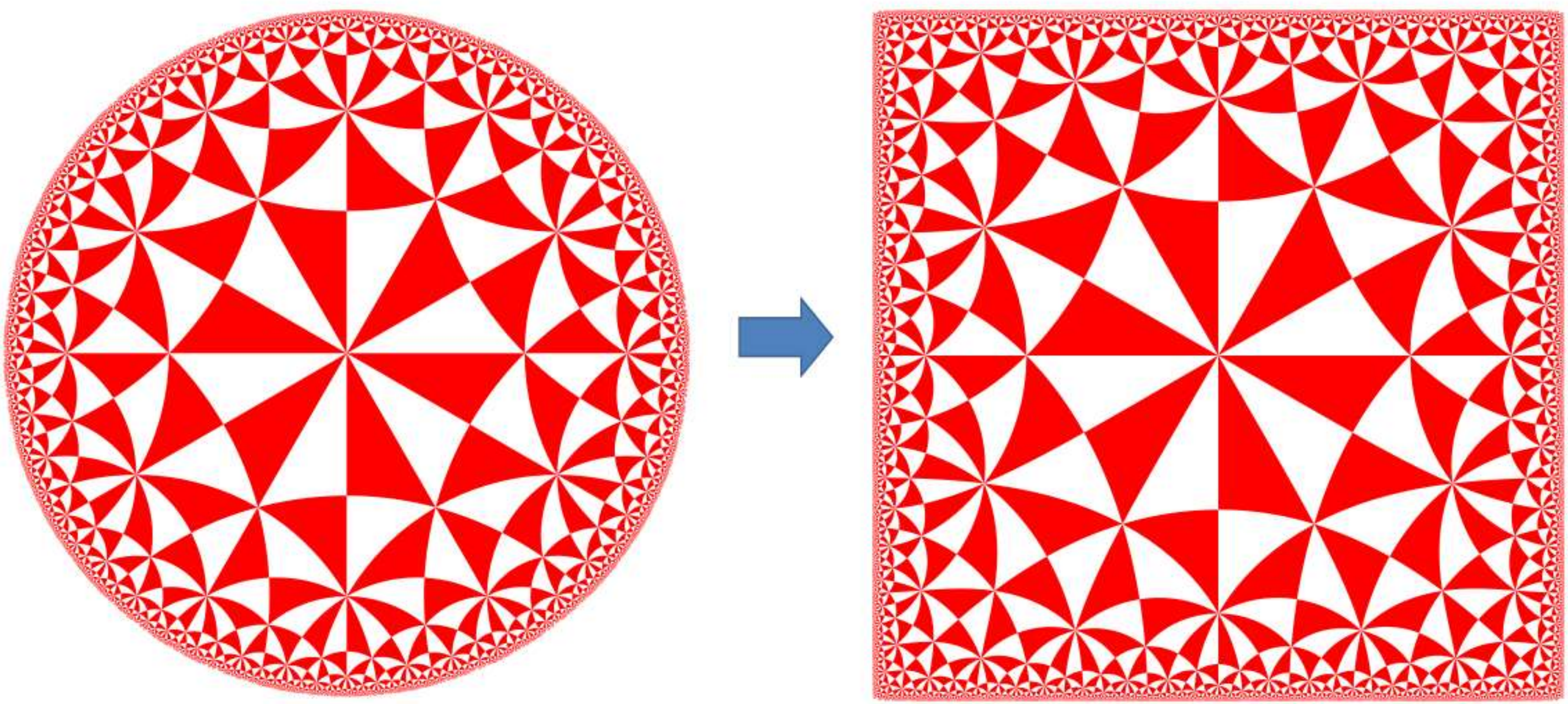
Nepal


STOP
STOP
STOP
STOP
STOP
STOP
STOP
STOP
STOP

2 explicit mappings

1) Schwarz-Christoffel

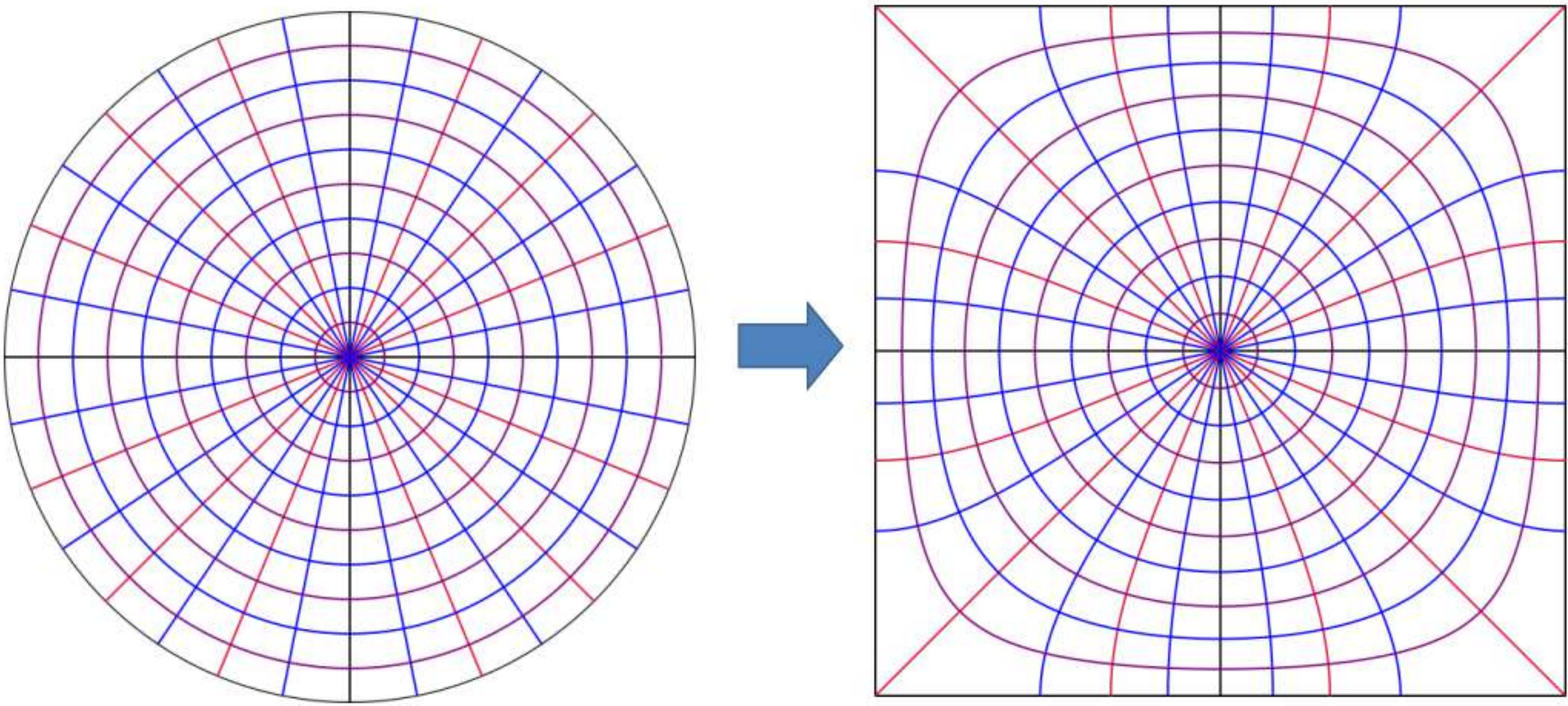
2) poor man's



mapping #1

disc-to-square

SCHWARZ - CHRISTOFFEL



$$z = \frac{1-i}{-K_e} F\left(\cos^{-1}\left(\frac{1+i}{\sqrt{2}}\boldsymbol{w}\right), \frac{1}{\sqrt{2}}\right) + 1 - i$$

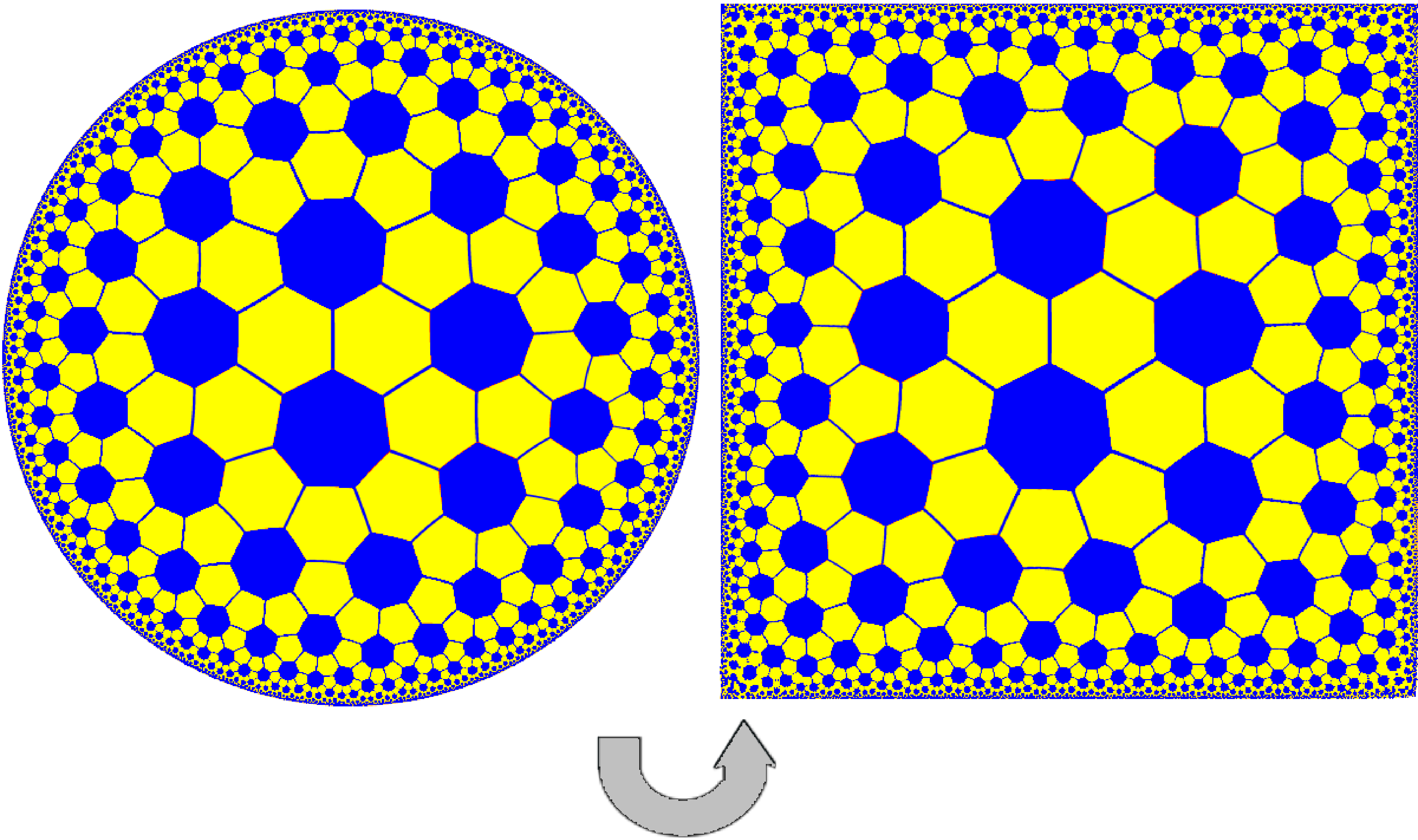
$$\boldsymbol{w} = u + v\,i$$
$$\boldsymbol{z} = x + y\,i$$

Stockholms
stad

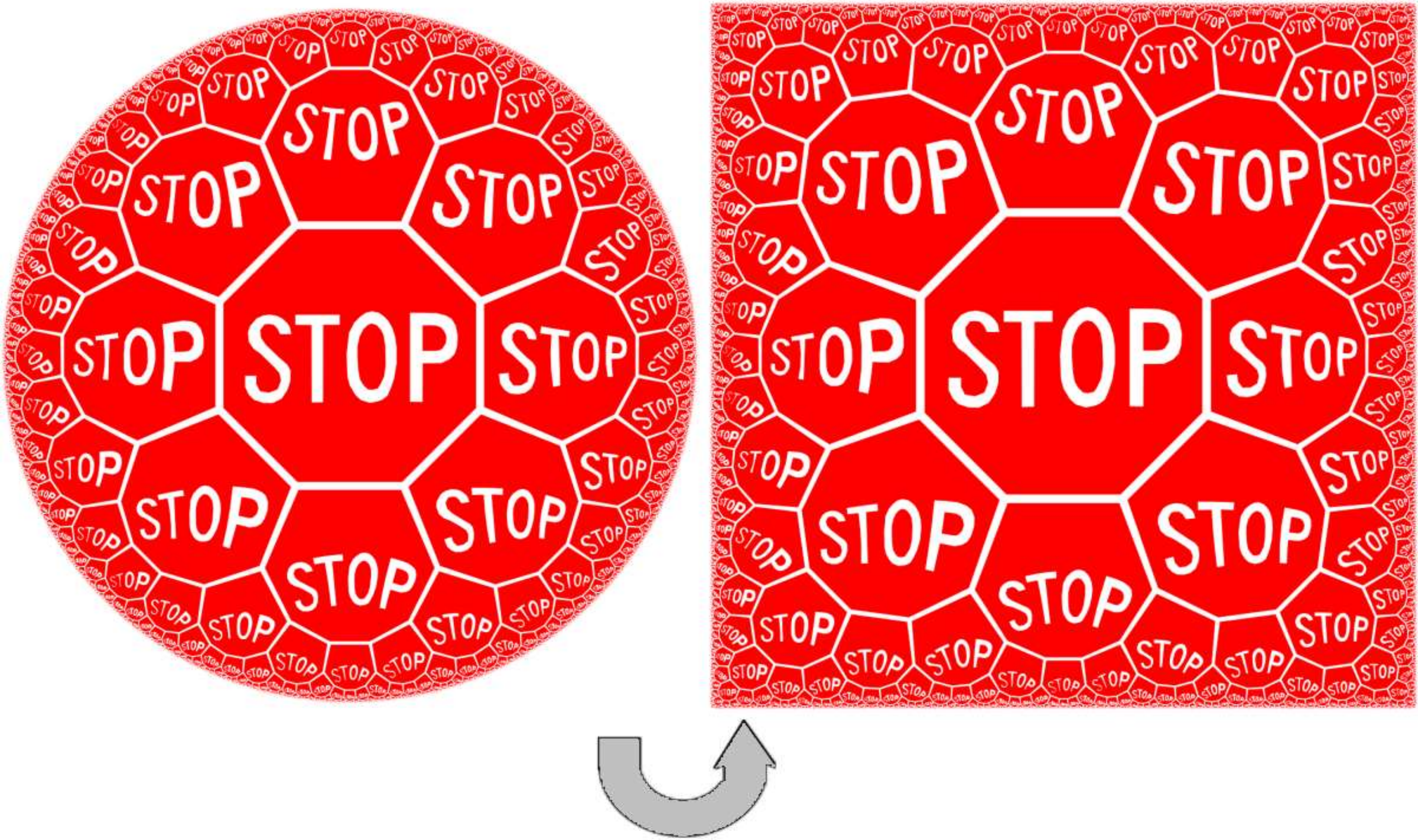
STOCKHOLMS · STAD ·

STOCKHOLMS · STAD ·

hyperbolic soccer ball



hyperbolic stop signs

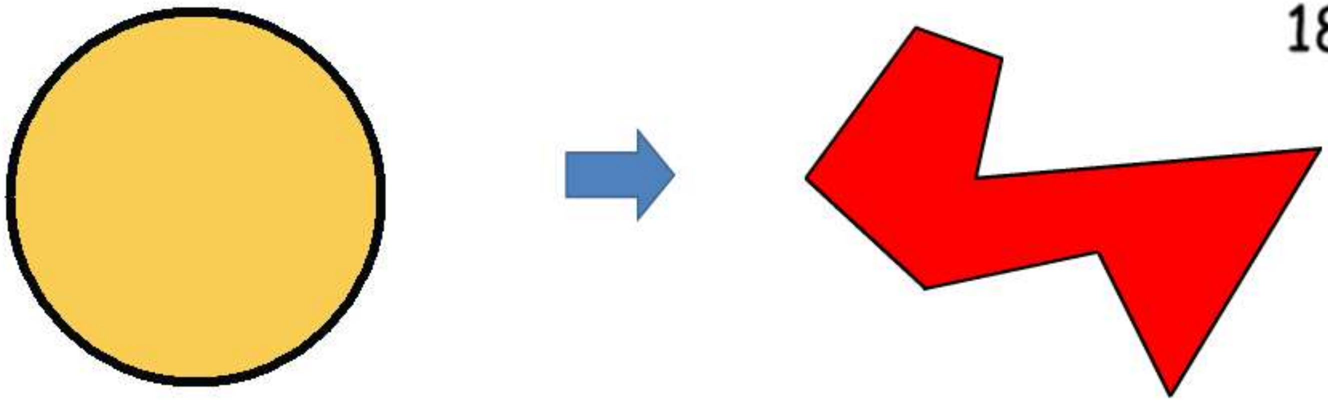


SCHWARZ-CHRISTOFFEL MAPPING

(general polygon)

Elwin Christoffel
1867

Hermann Schwarz
1869



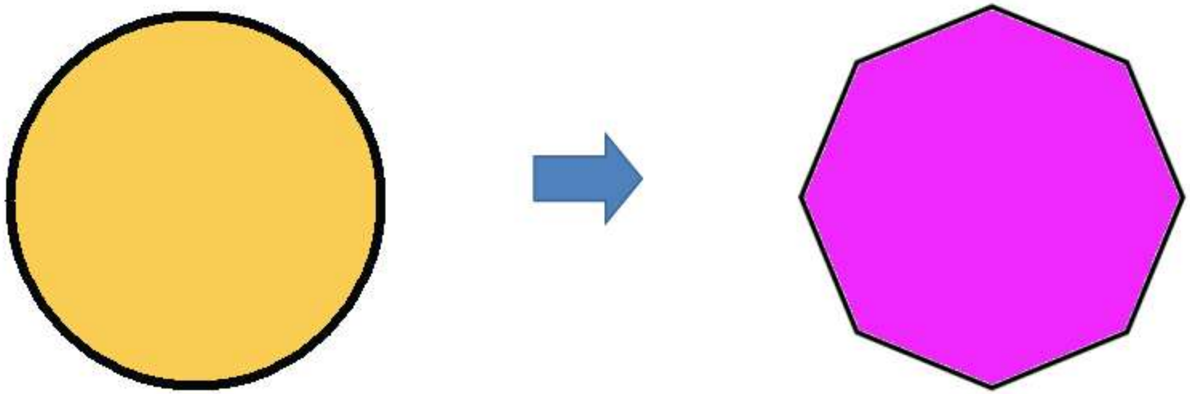
$$f(z) = A + C \int^{z} \prod_{k=1}^{n} \left(1 - \frac{\zeta}{z_k}\right)^{\alpha_k - 1} d\zeta$$

SCHWARZ-CHRISTOFFEL MAPPING

(regular polygon)

Elwin Christoffel

Hermann Schwarz



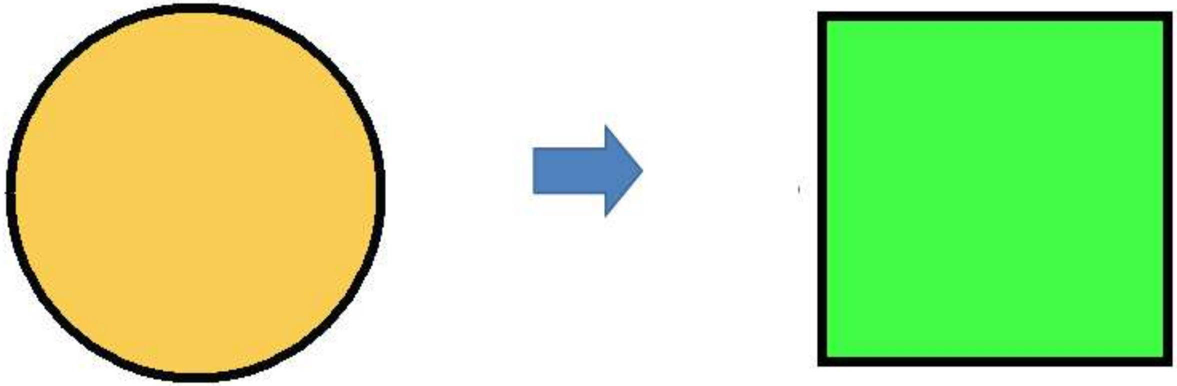
$$f(z) = \int_0^z \frac{d\tau}{(1-\tau^n)^{\frac{2}{n}}}$$

SCHWARZ-CHRISTOFFEL MAPPING

(square, n=4)

Elwin Christoffel

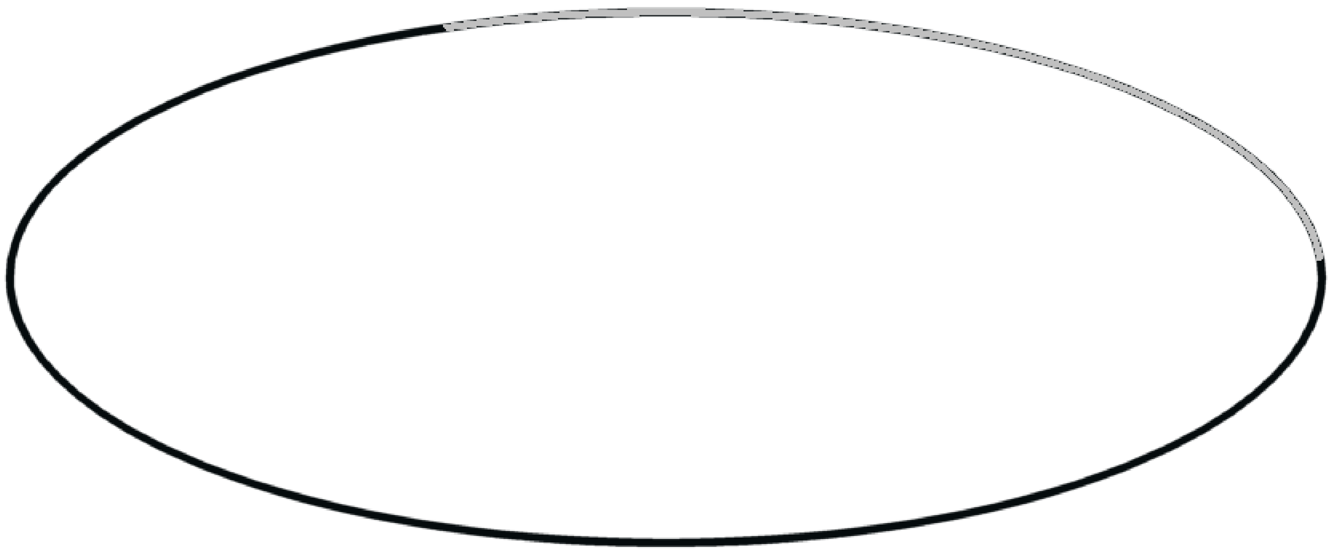
Hermann Schwarz



$$f(z) = \int_0^z \frac{d\tau}{\sqrt{1-\tau^4}} = F(\sin^{-1} z, i)$$

Legendre Elliptic Integrals

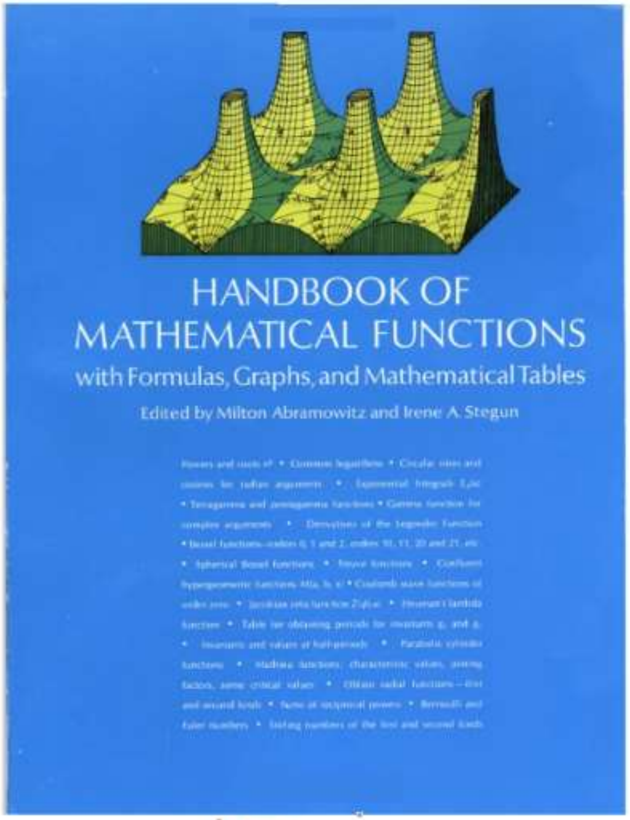
complete and incomplete

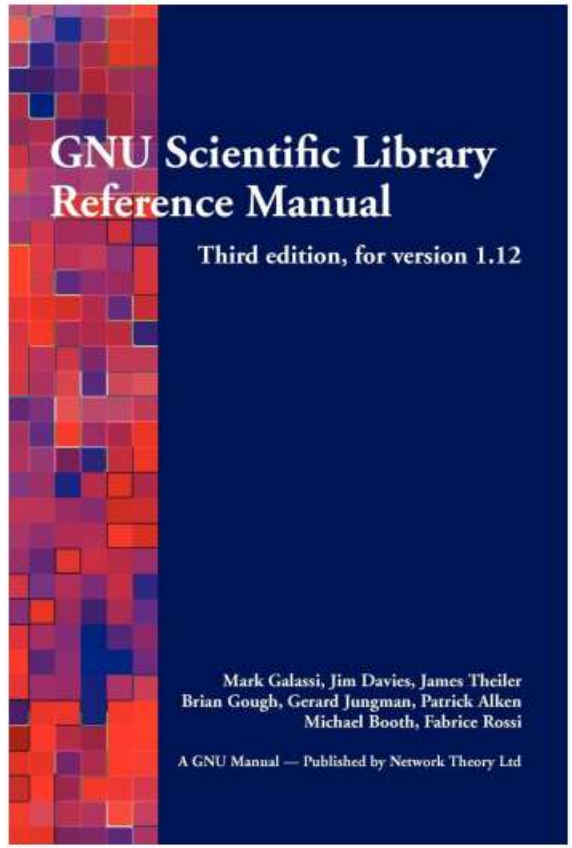


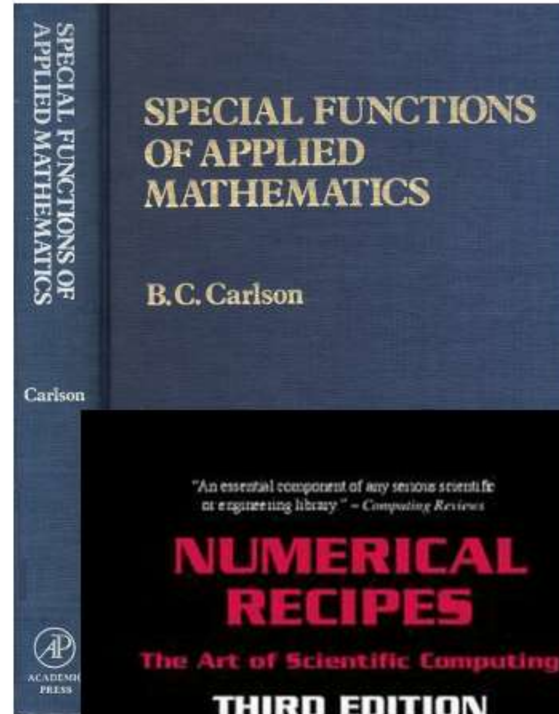
1st kind:

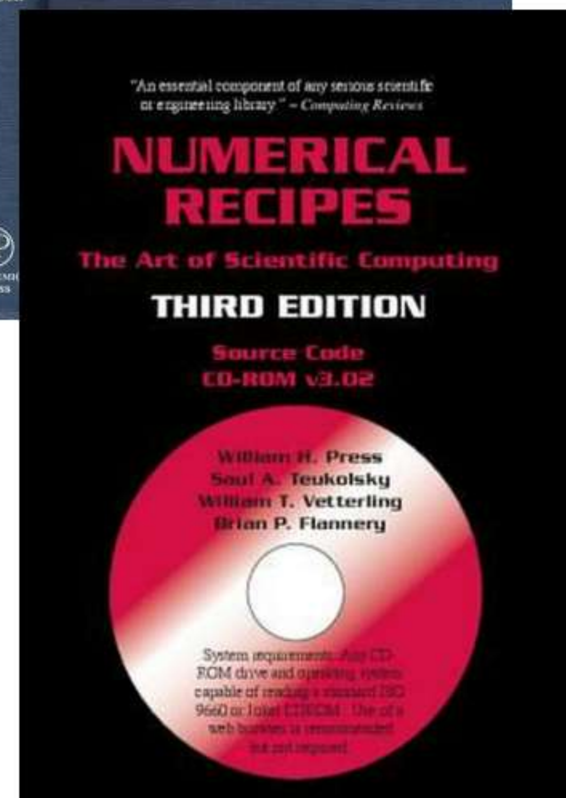
$$F(\phi, k) = \int_0^{\phi} \frac{1}{\sqrt{1 - k^2 \sin^2 t}} dt$$

$$F^{-1} = \cos^{-1} \mathrm{cn}(w,k)$$



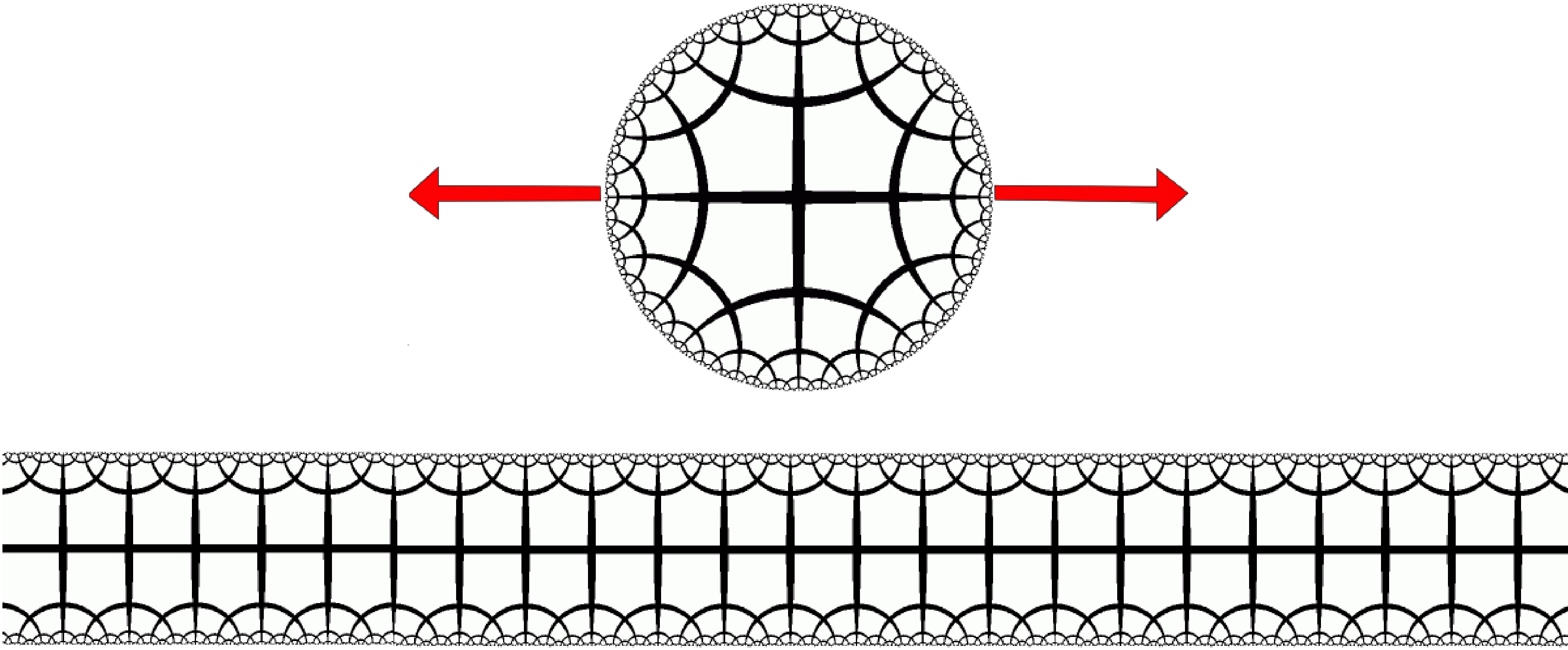






Bulatov band model

$$f(z) = 2\pi \tanh^{-1} z$$



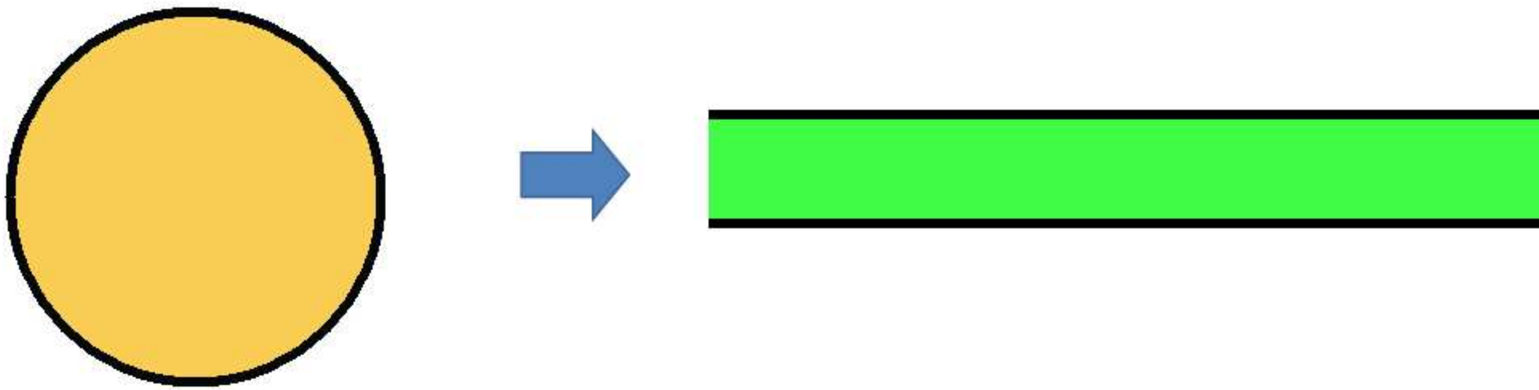
Vladimir Bulatov, "Conformal Models of the Hyperbolic Geometry"
MAA-AMS Joint Mathematics Meeting 2010

Elwin Christoffel

SCHWARZ-CHRISTOFFEL MAPPING

(infinite band, n = 2)

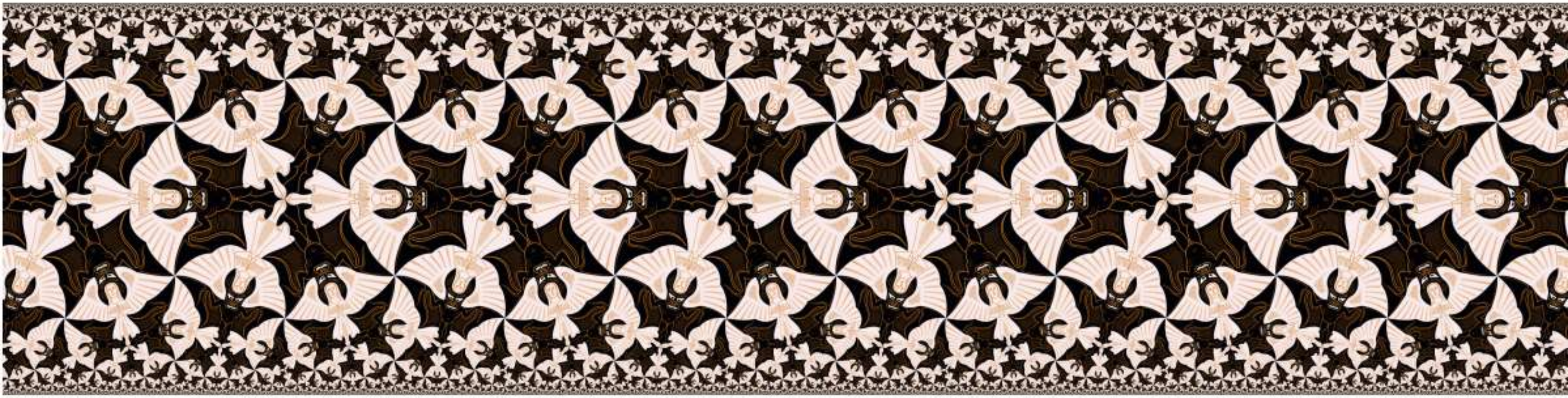
Hermann Schwarz



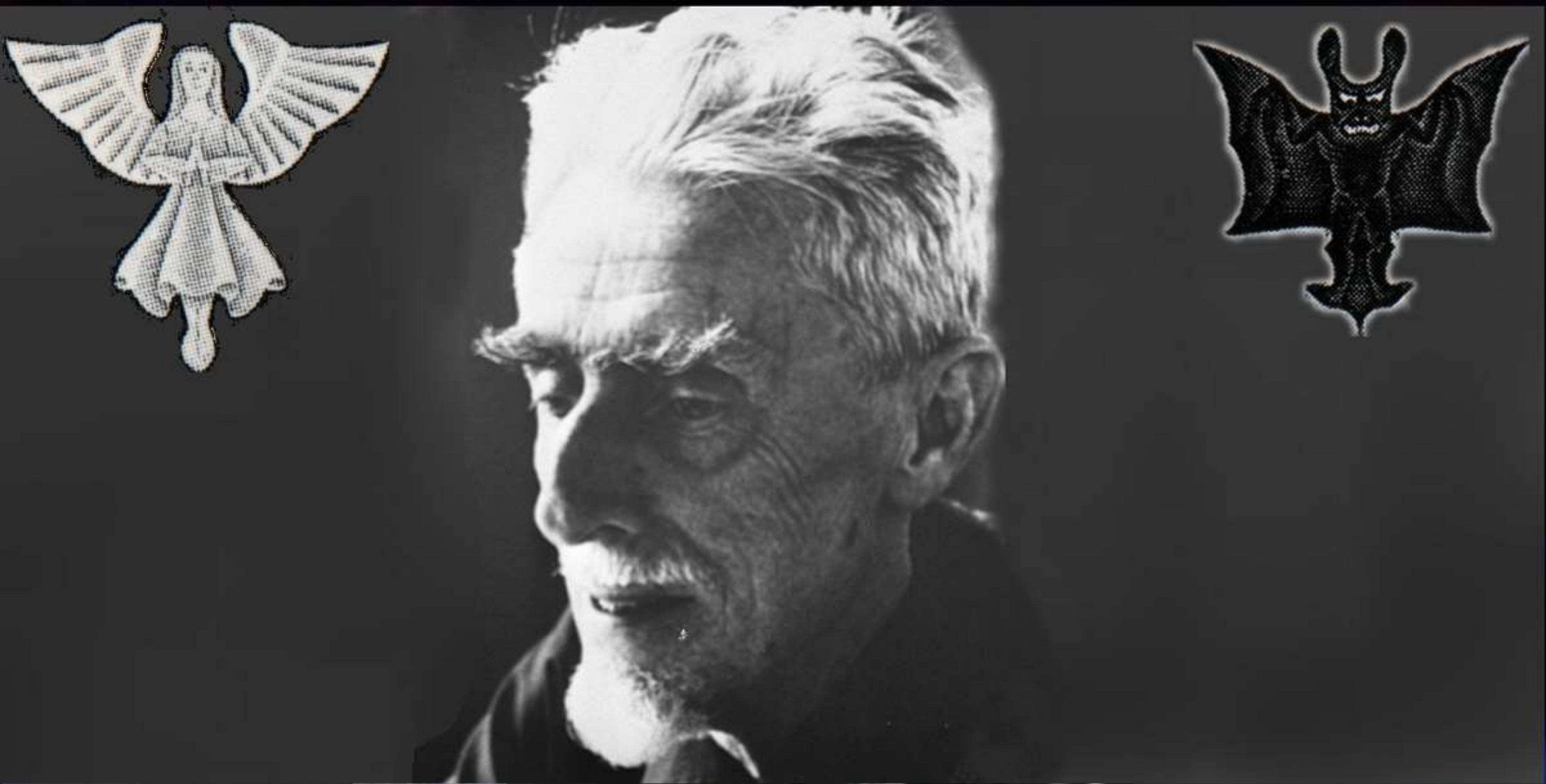
$$f(z) = \int_0^z \frac{d\tau}{1-\tau^2} = \tanh^{-1} z$$

Bulatov band model

is Schwarz-Christoffel for n=2



Vladimir Bulatov, "Conformal Models of the Hyperbolic Geometry"
MAA-AMS Joint Mathematics Meeting 2010



How many devils are there in this rendition of Circle Limit IV ?

A: 1729

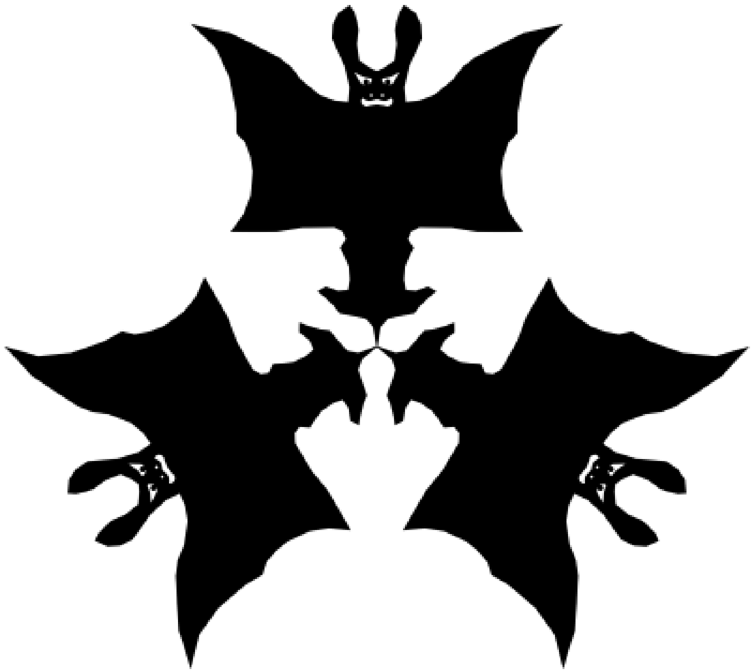
B: 393,213

C: 196,883

D: infinite

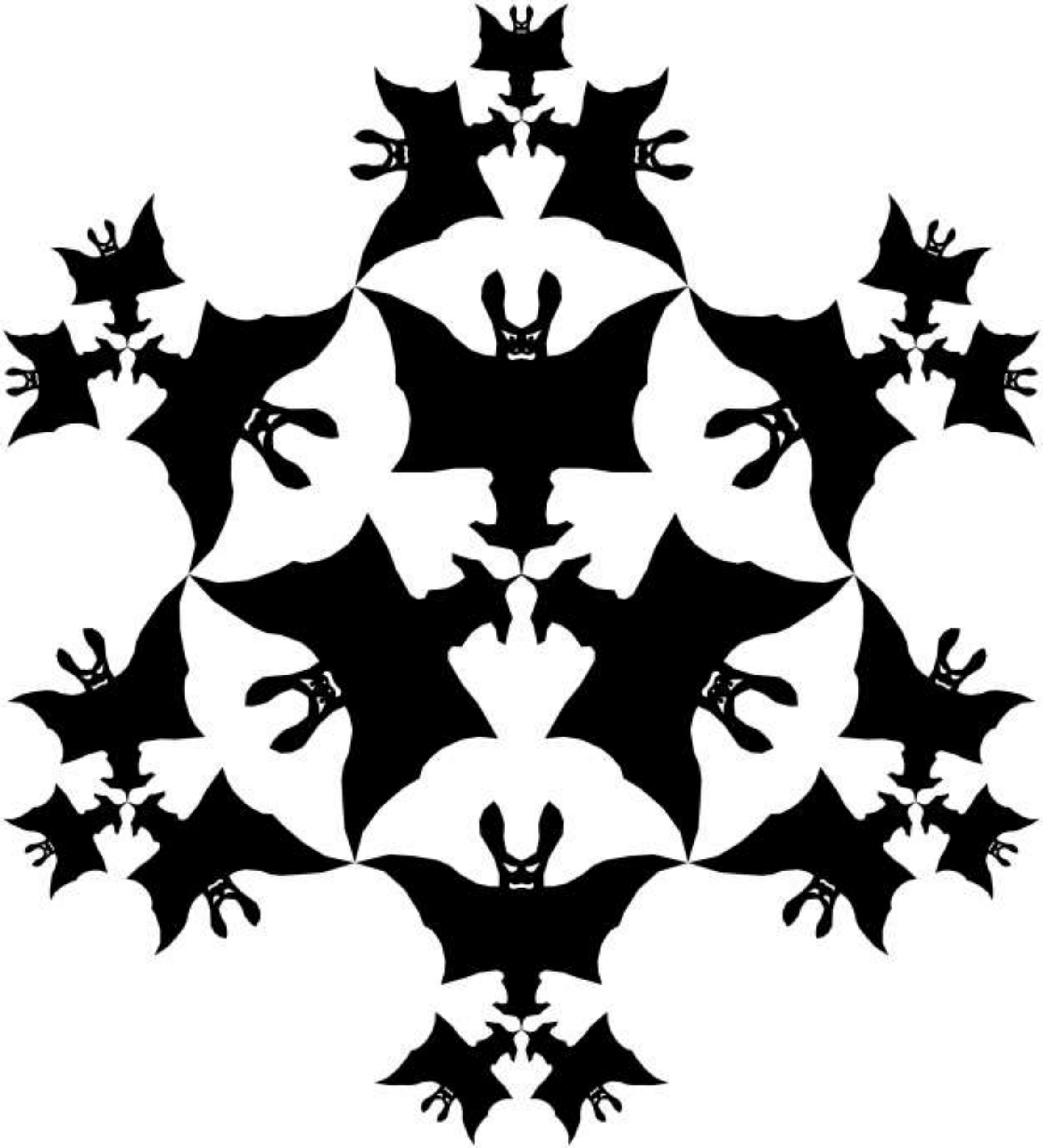
level

1



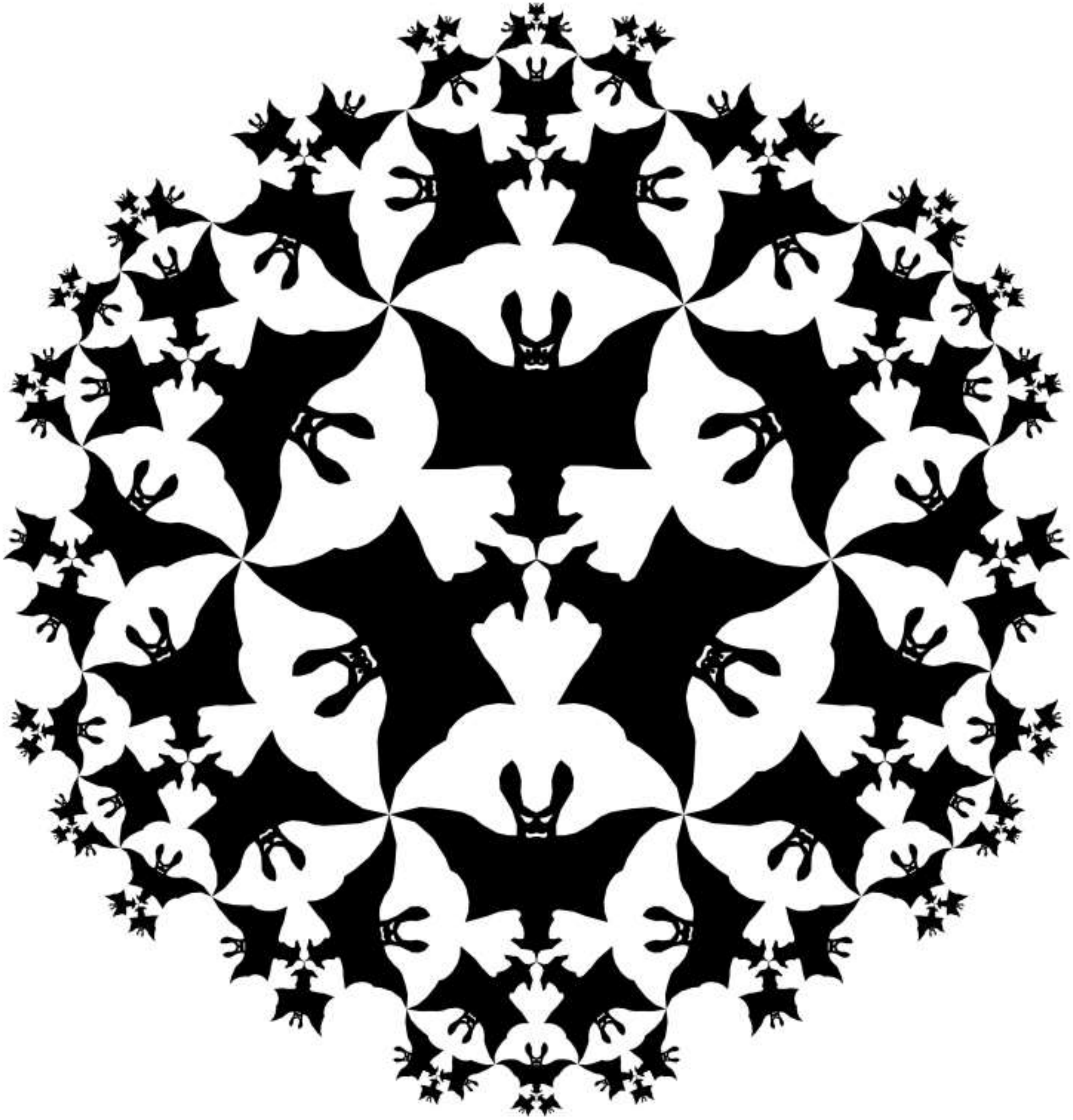
3

devils



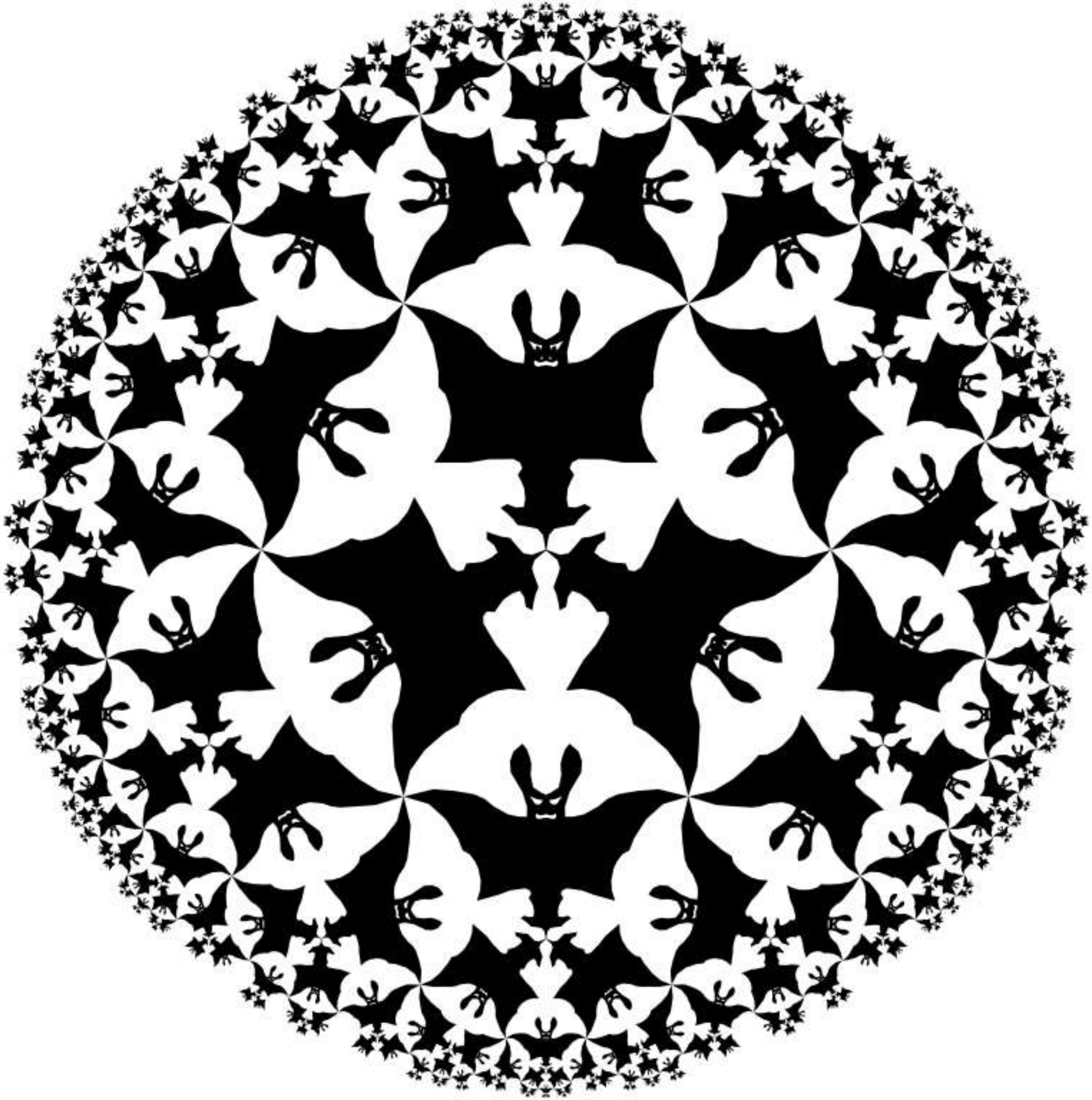
level
2

21
devils



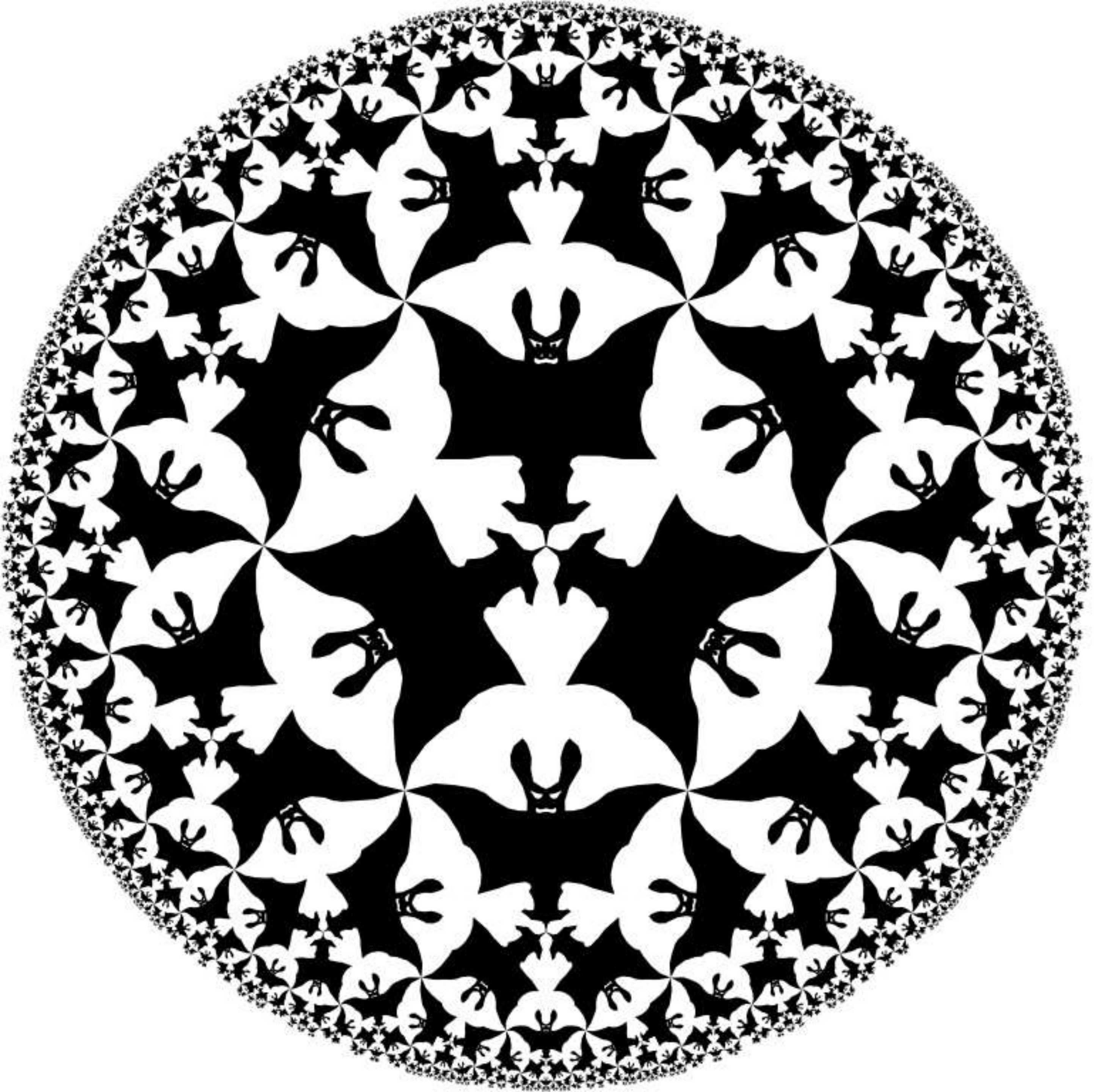
level
3

93
devils



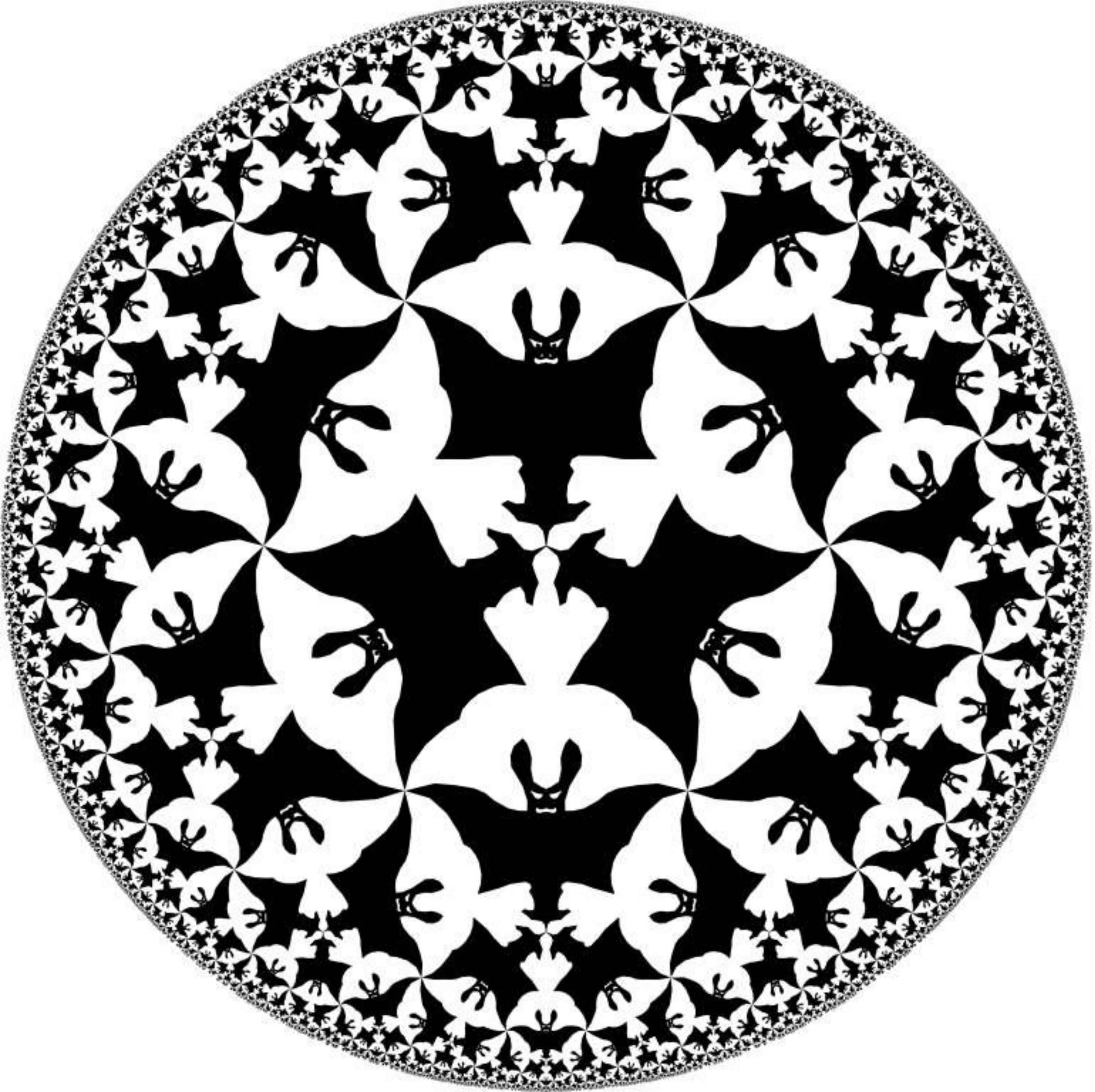
level
4

381
devils



level
5

1533
devils

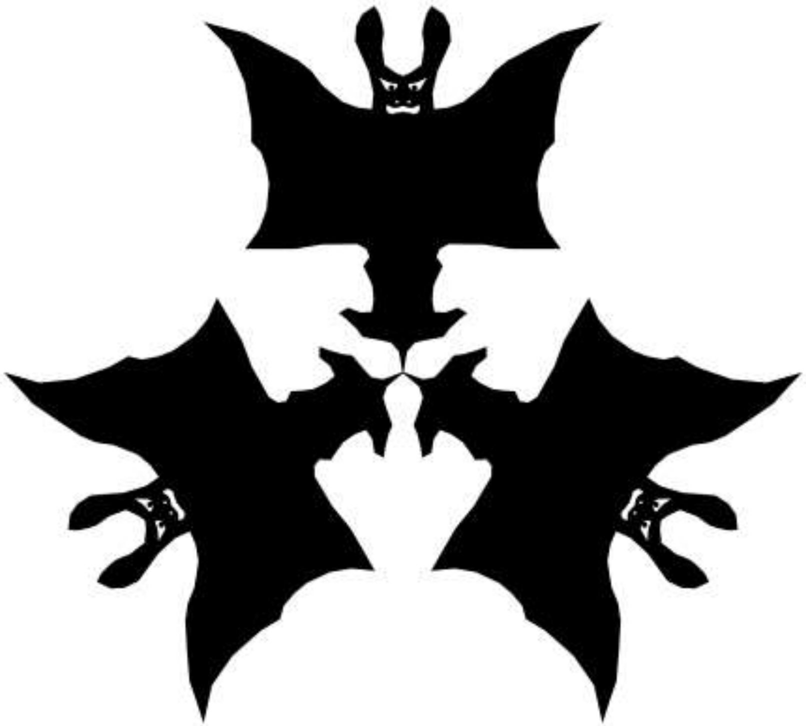


level
6

6141
devils

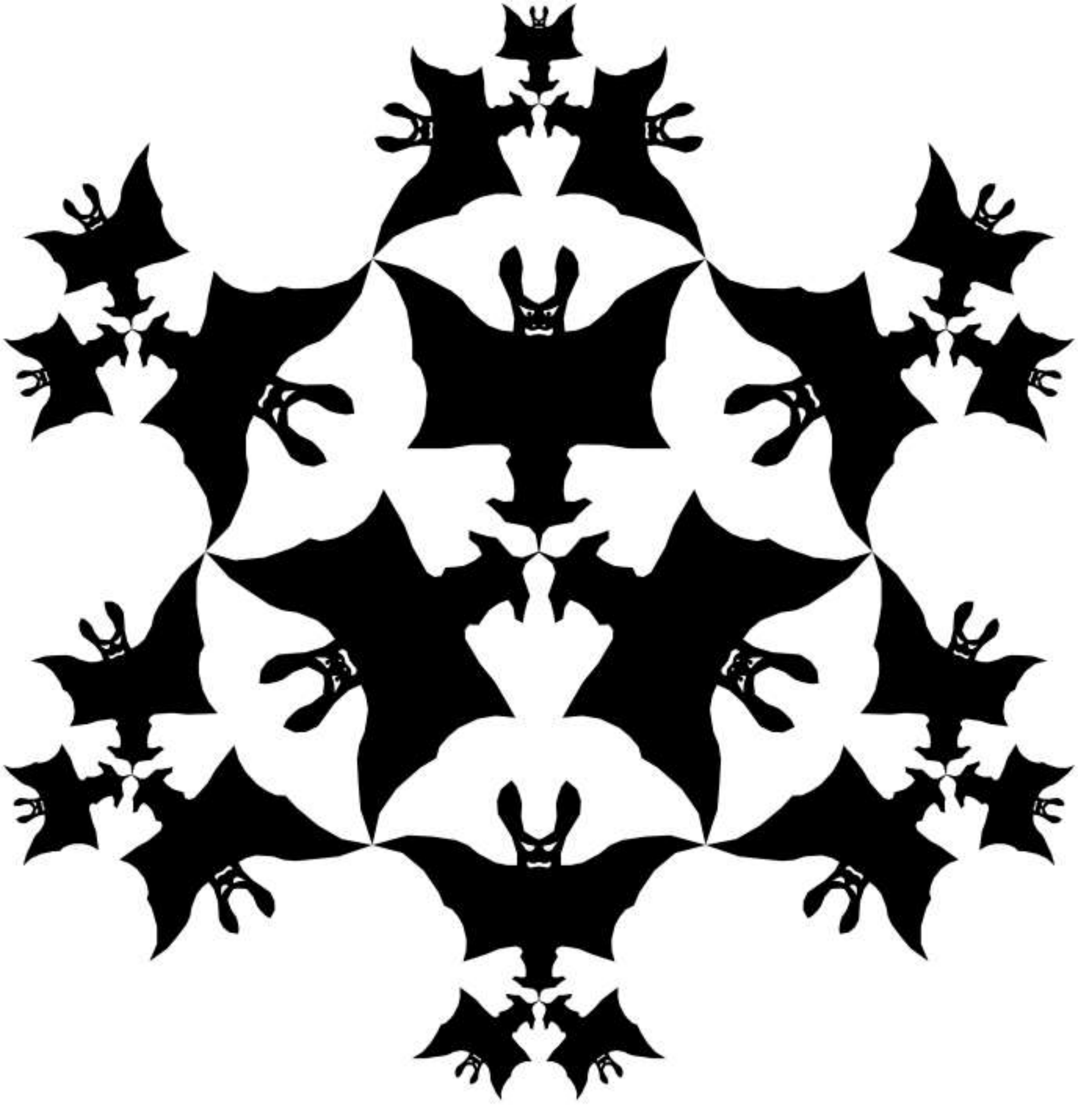
level

1



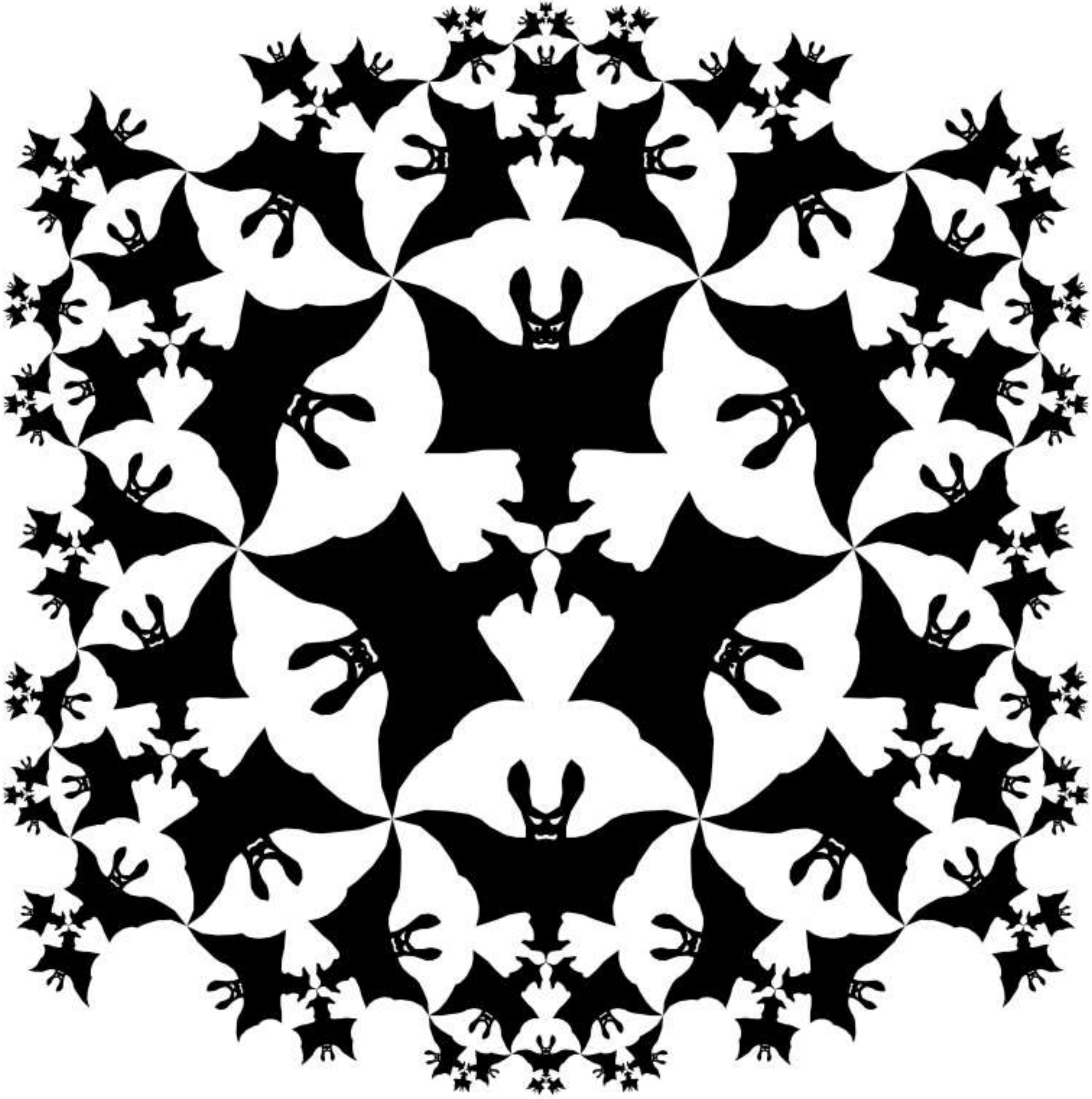
3

devils



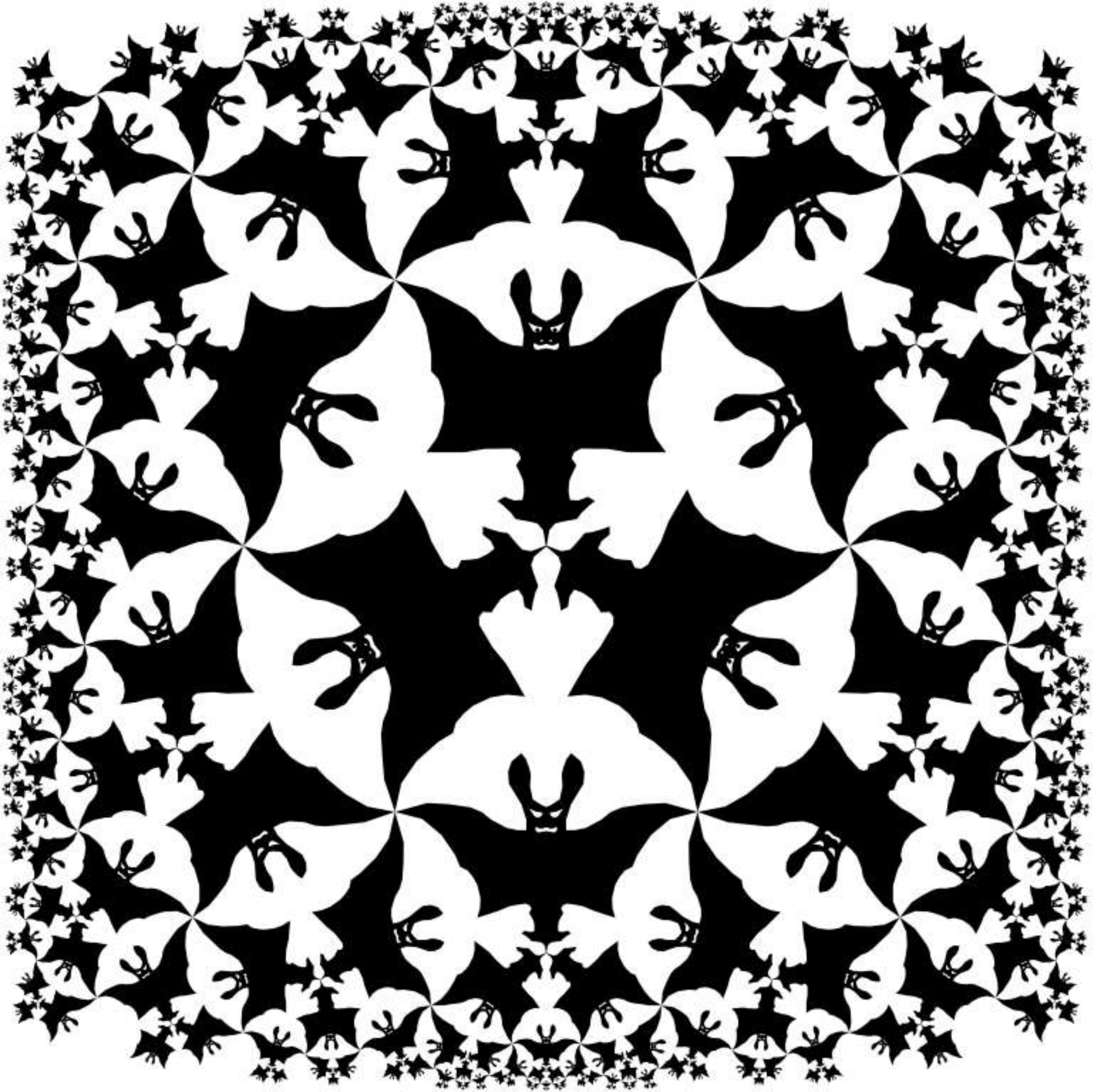
level
2

21
devils



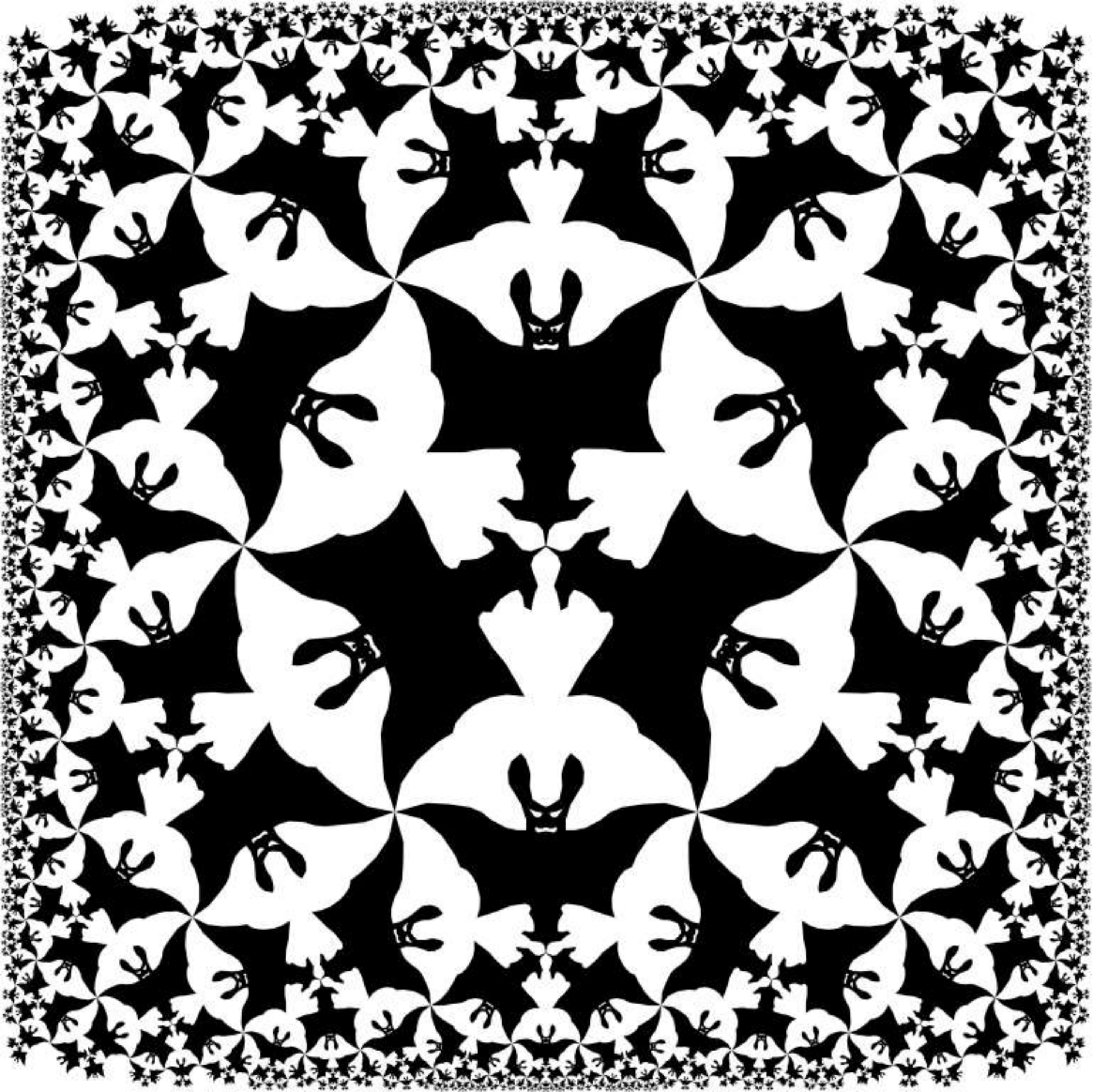
level
3

93
devils



level
4

381
devils

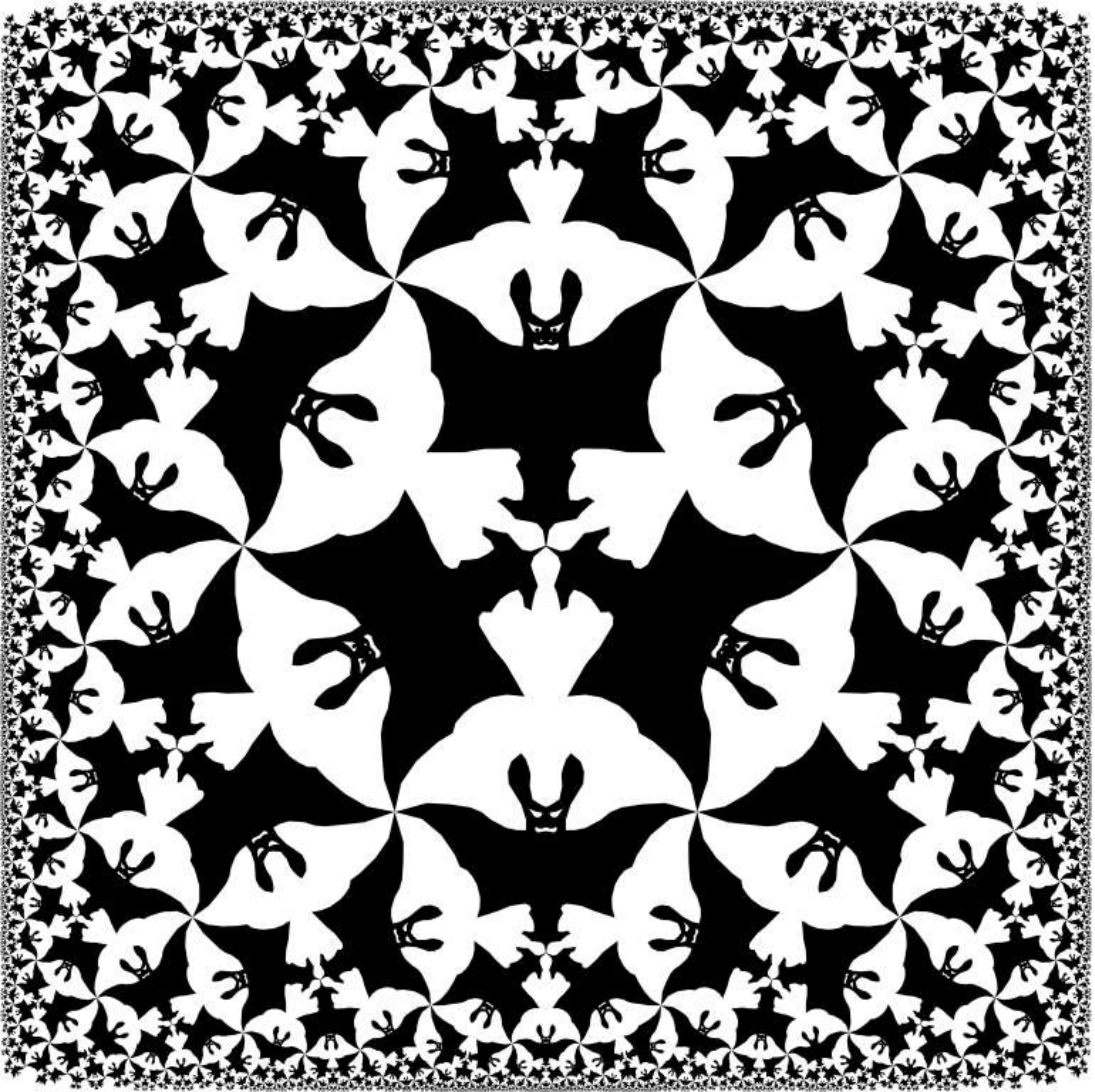


level

5

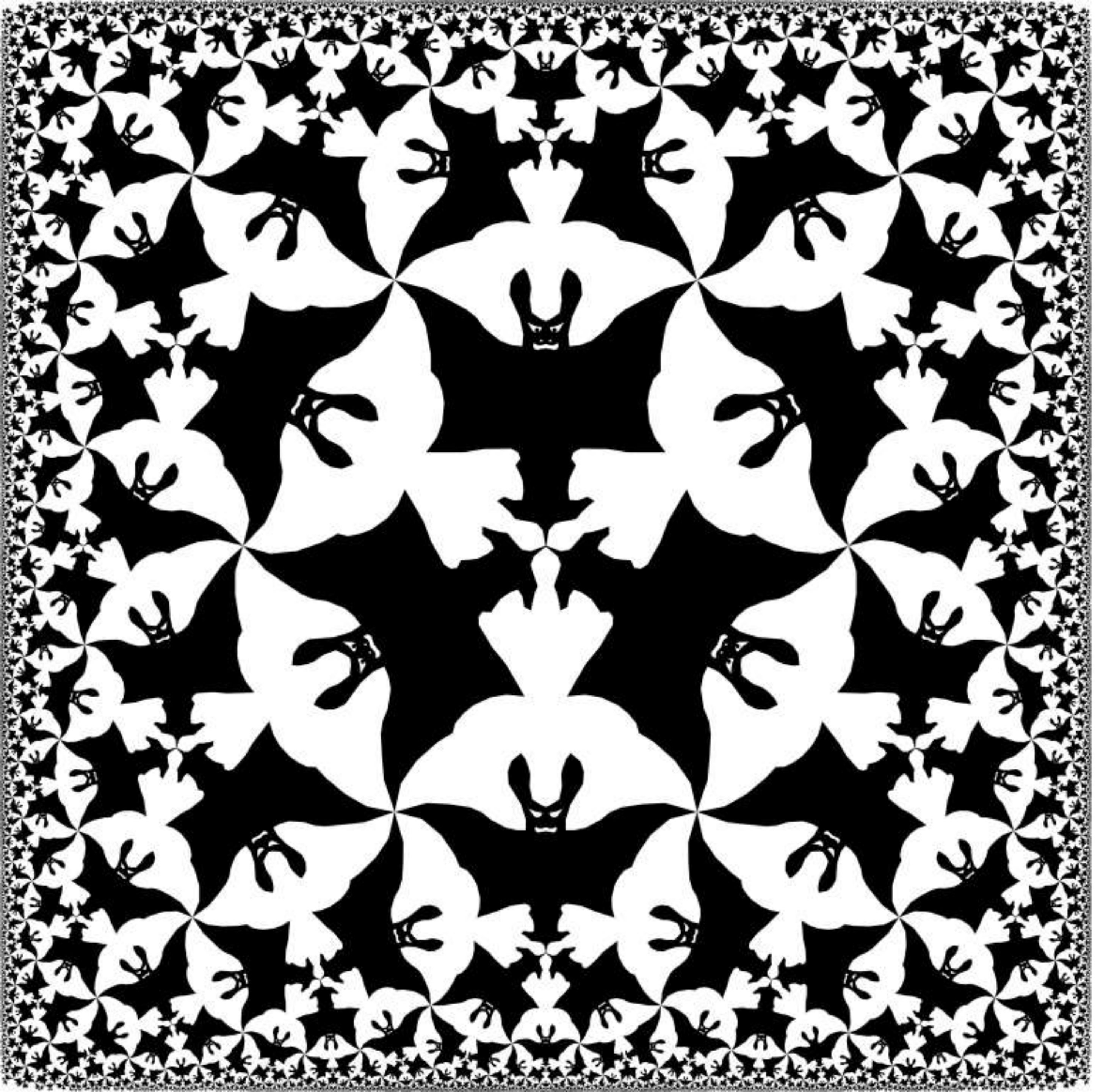
1533

devils



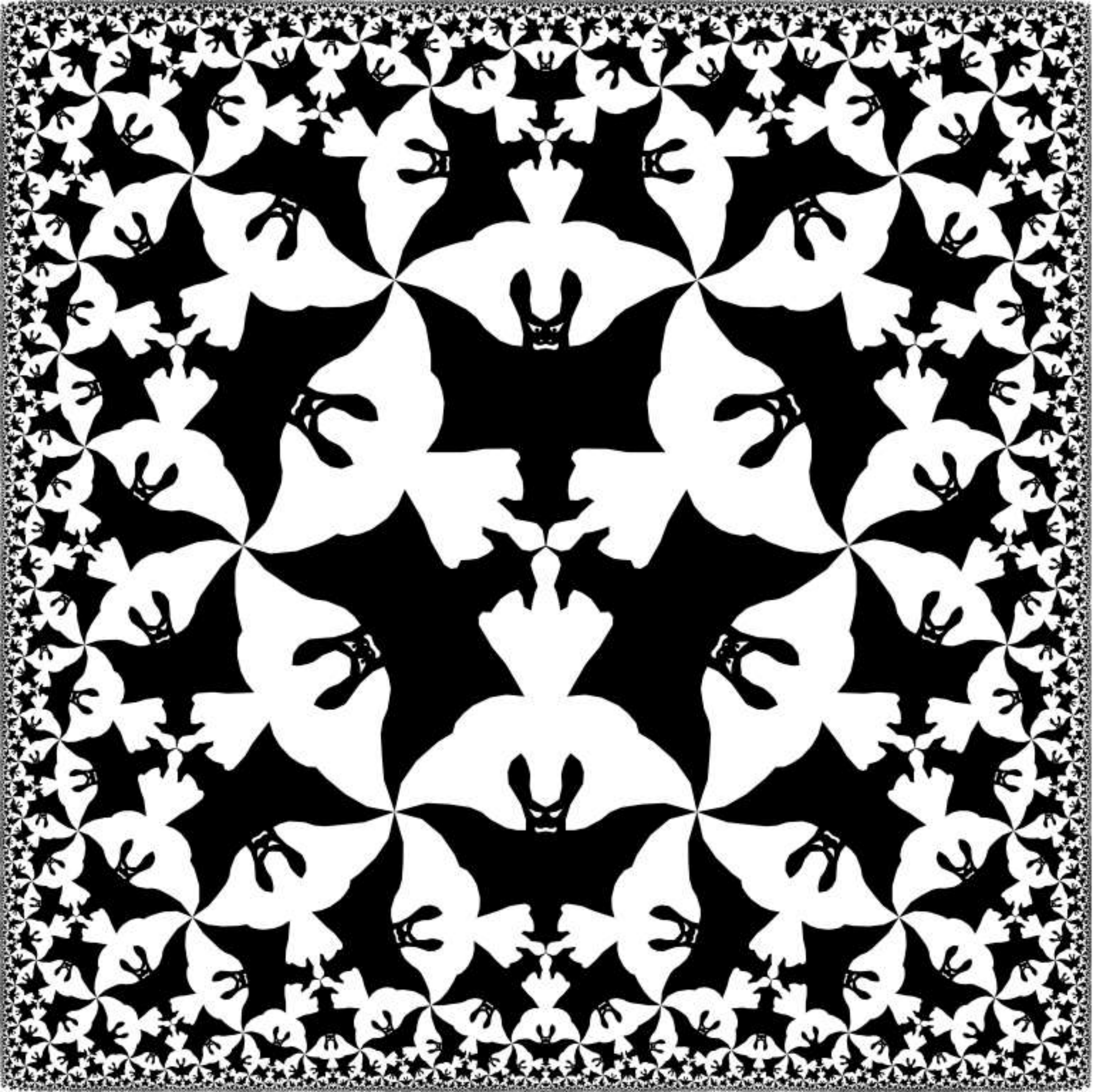
level
6

6141
devils



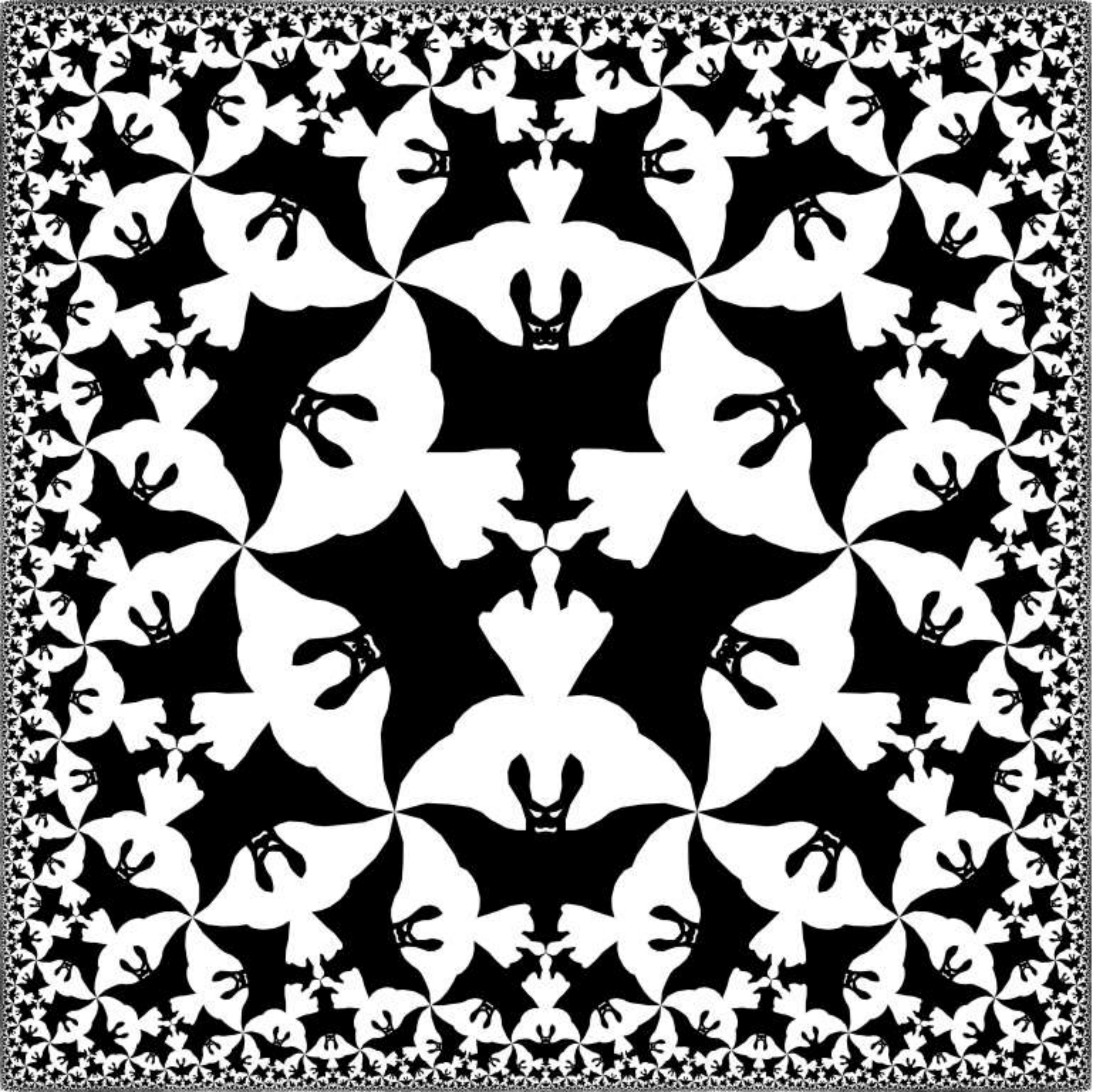
level
7

24573
devils



level
8

98301
devils



level
9

393213
devils

How many devils are there in this rendition of Escher's "Circle Limit IV" ?

A: 1729

B: 393,213

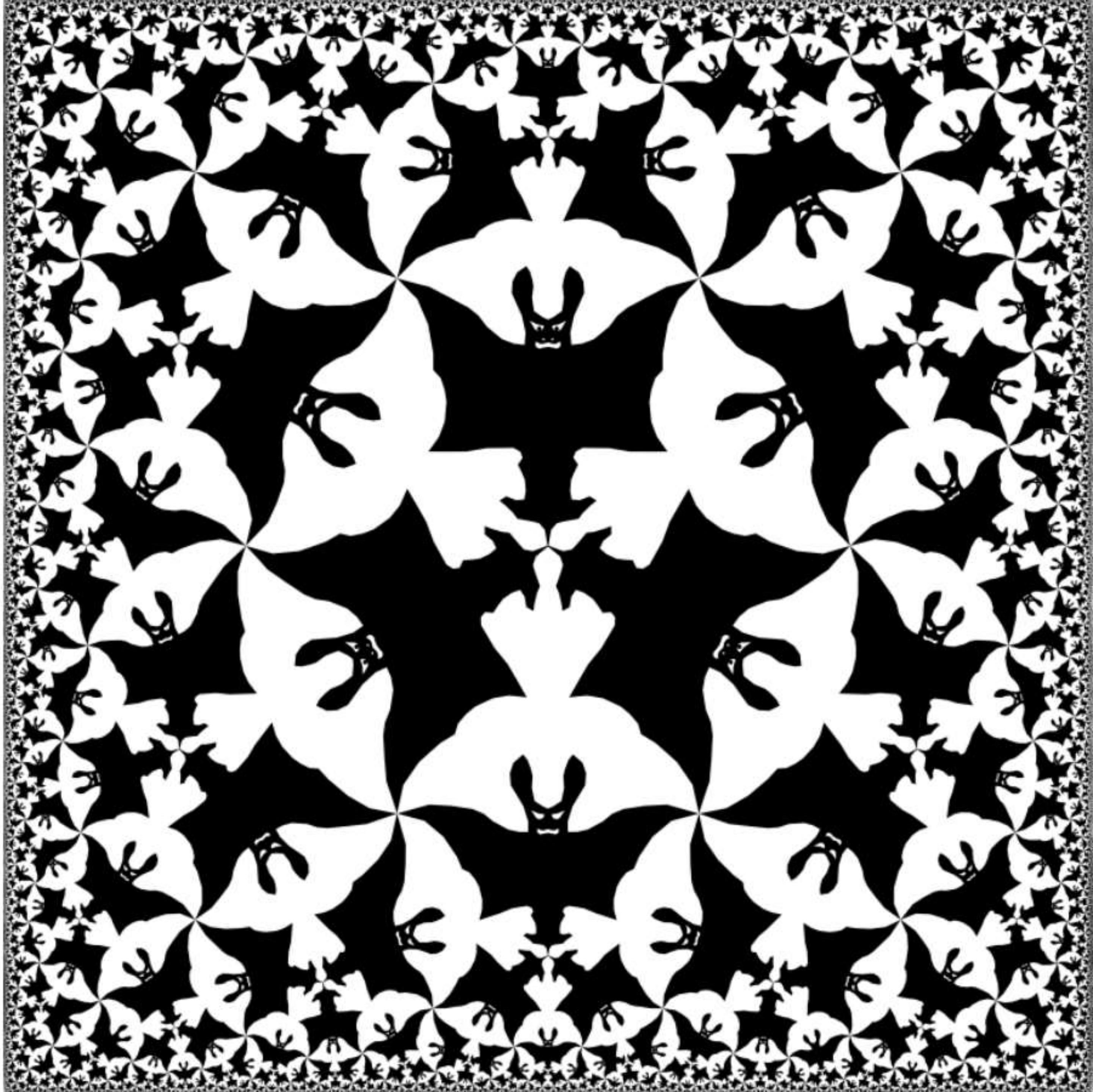
C: 196,883

D: infinite

Points

versus

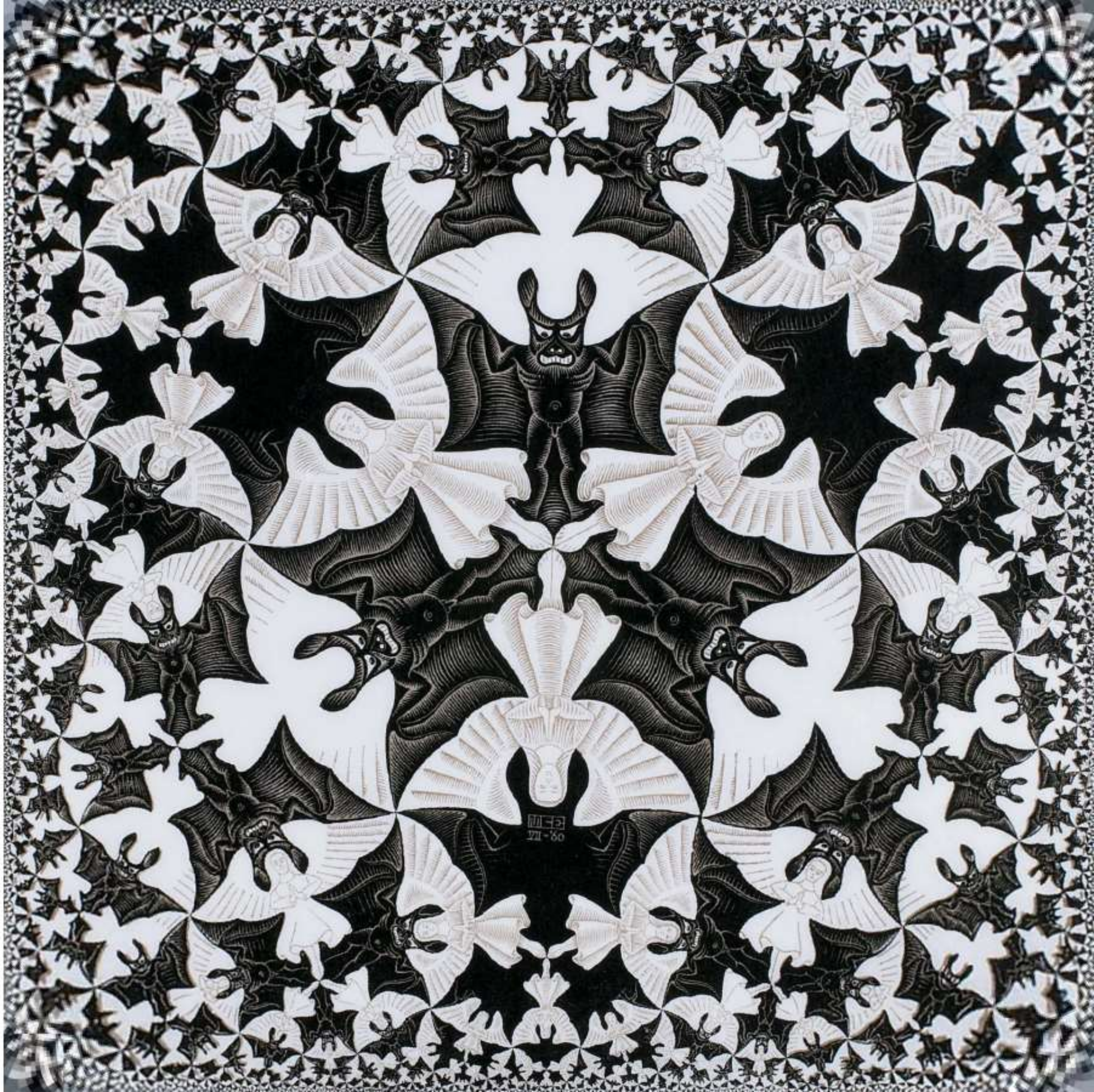
Pixels



Points

versus

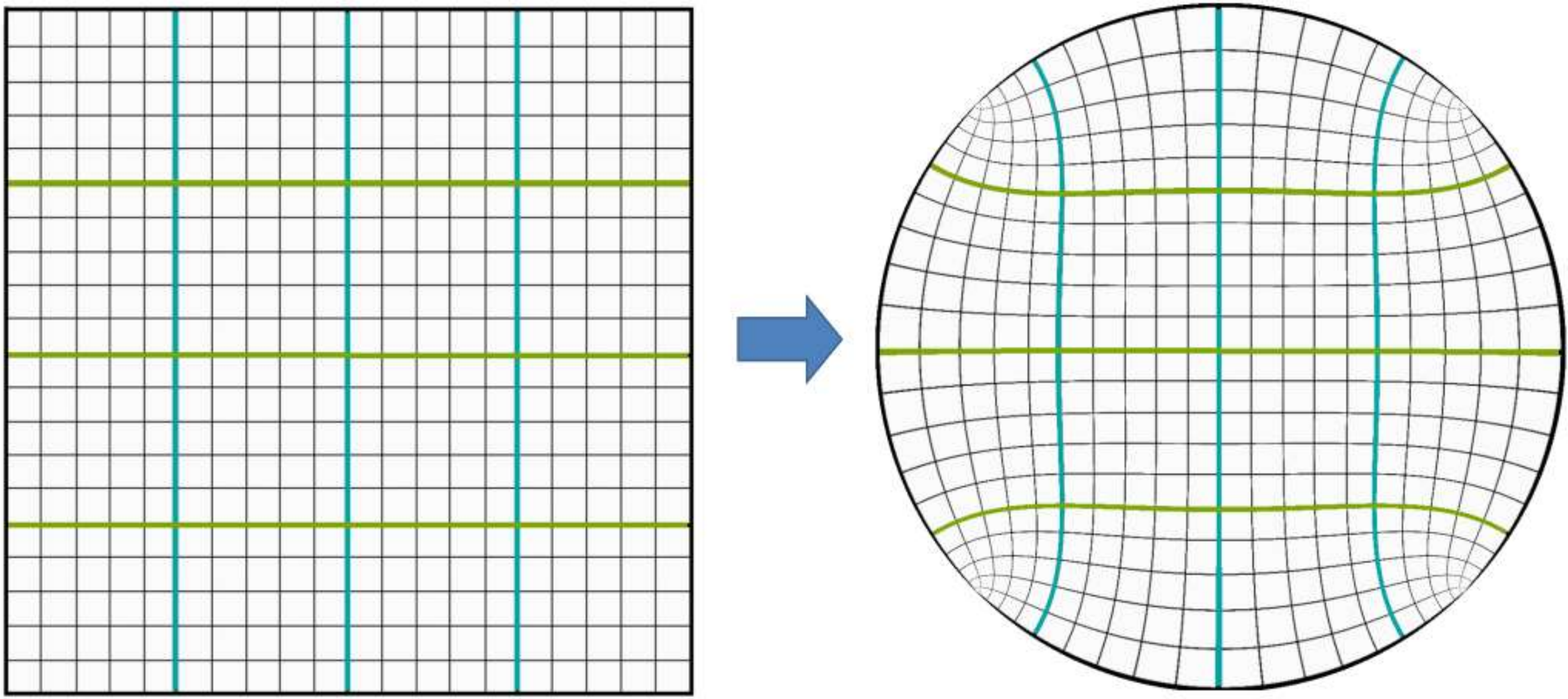
Pixels



mapping #1

square-to-disc

SCHWARZ - CHRISTOFFEL



$$\boldsymbol{w} = \frac{1-i}{\sqrt{2}}\, cn\left(K_e \frac{1+i}{2}\boldsymbol{z} - K_e, \frac{1}{\sqrt{2}}\right)$$

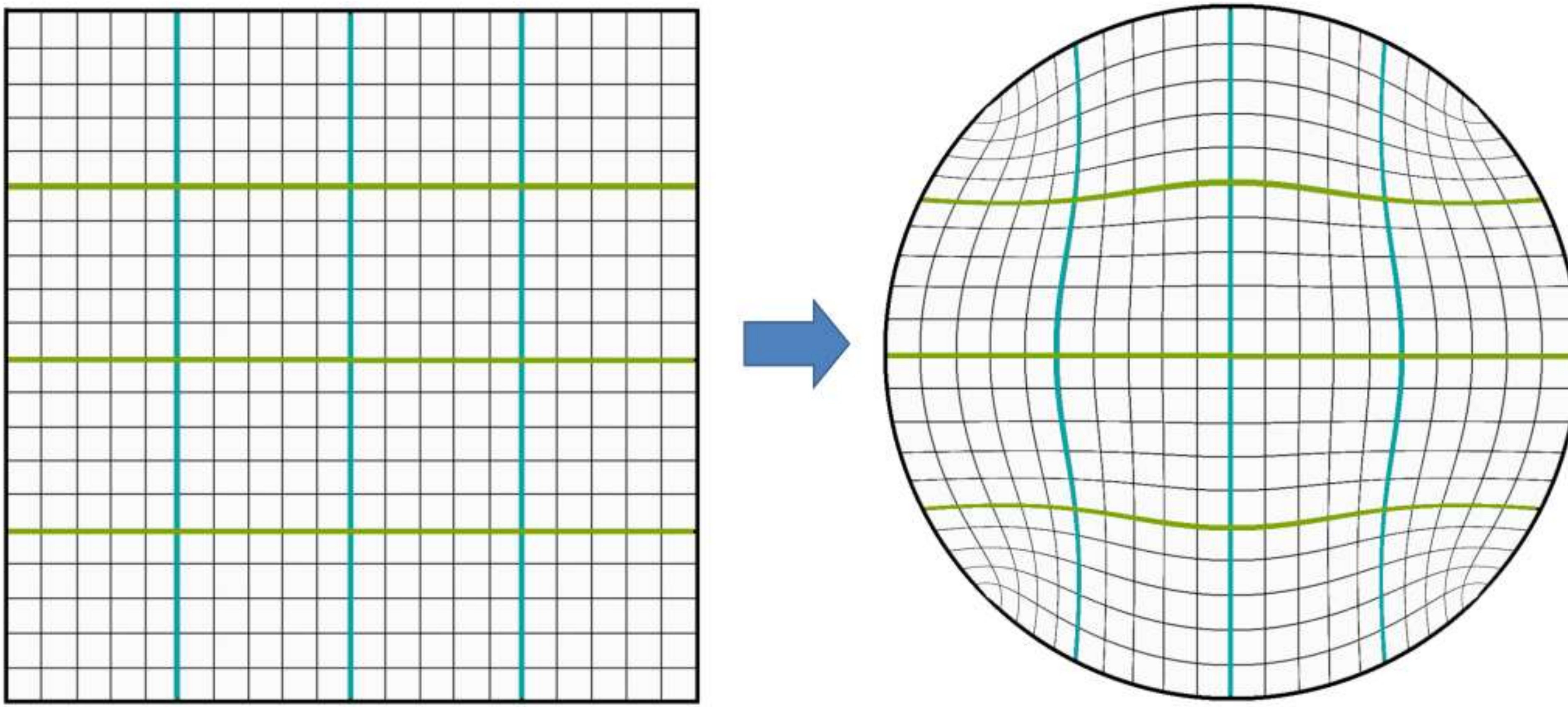
$$\boldsymbol{w} = u + v\, i$$

$$\boldsymbol{z} = x + y\, i$$

mapping # 2

square-to-disc

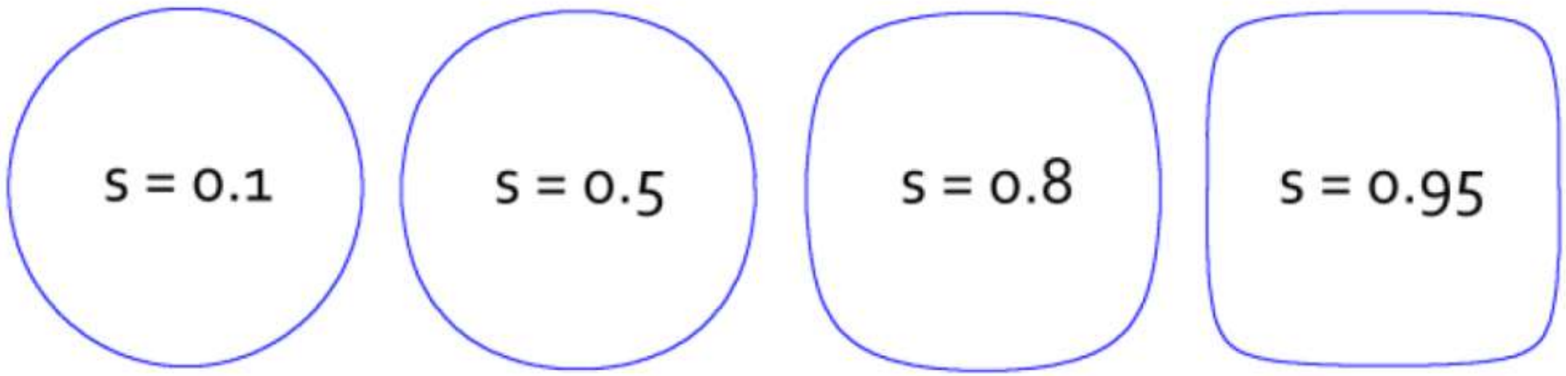
POOR MAN'S



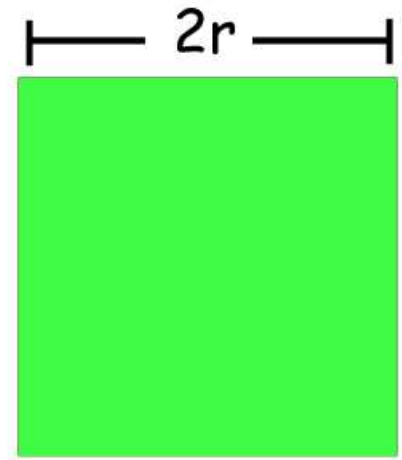
$$\begin{bmatrix} u \\ v \end{bmatrix} = \sqrt{\frac{x^2+y^2-2x^2y^2}{(x^2+y^2)(1-x^2y^2)}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Fernandez-Guasti's squircle

$$x^2 + y^2 - \frac{s^2}{r^2}\, x^2 y^2 = r^2 \qquad 0 \leq s \leq 1$$

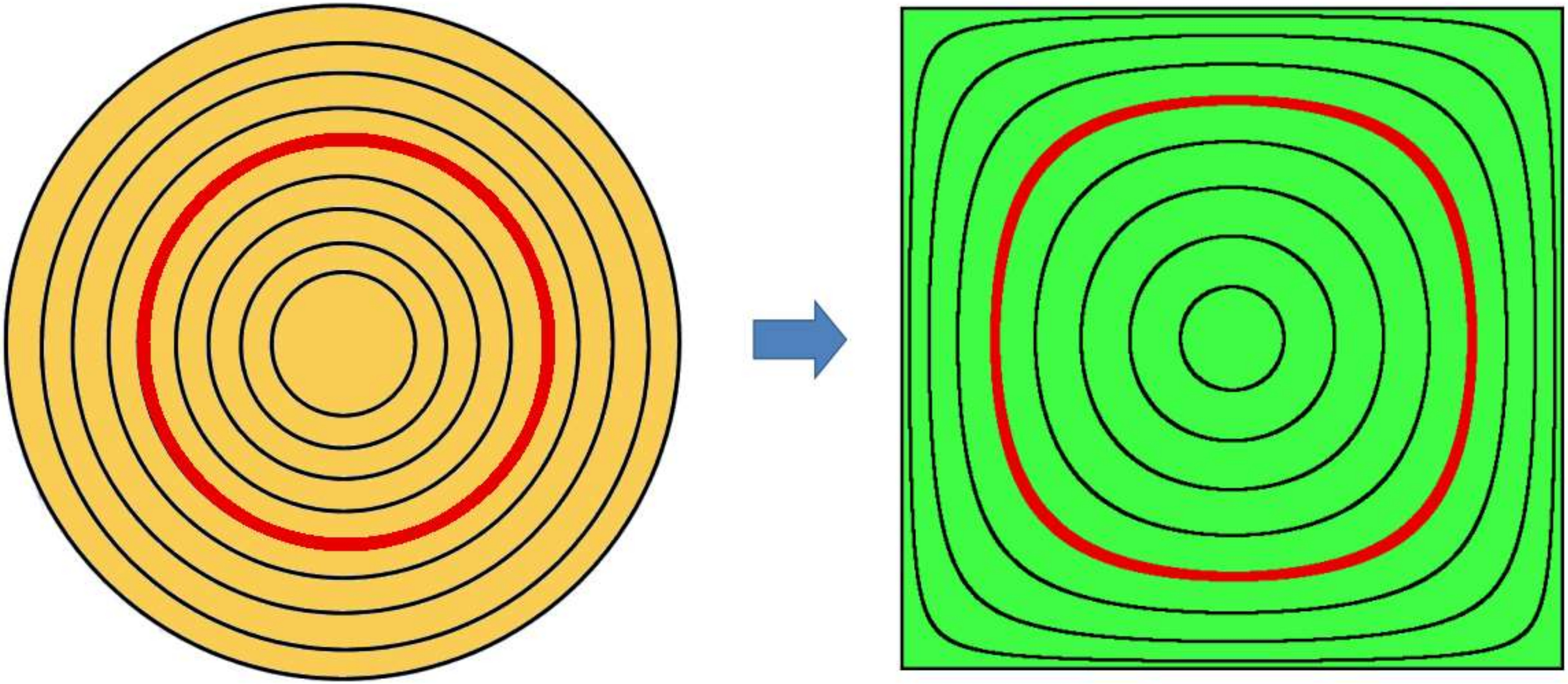


circle when s=0



square when s=1

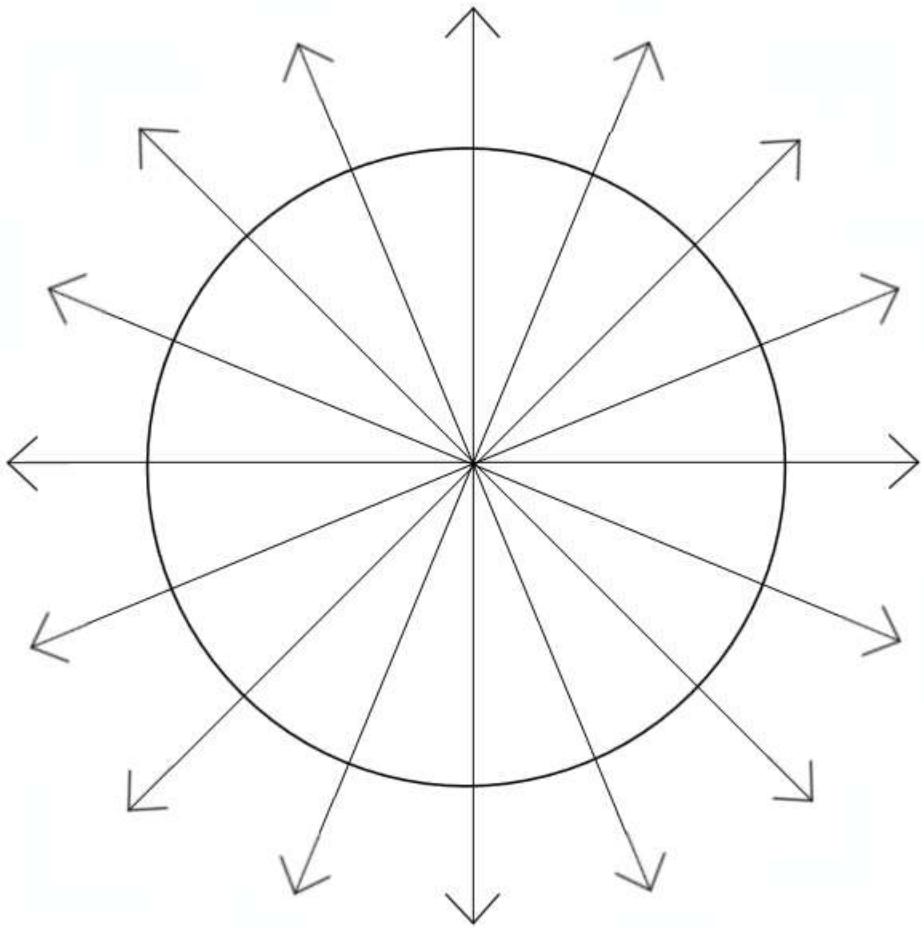
Concentric Continuum

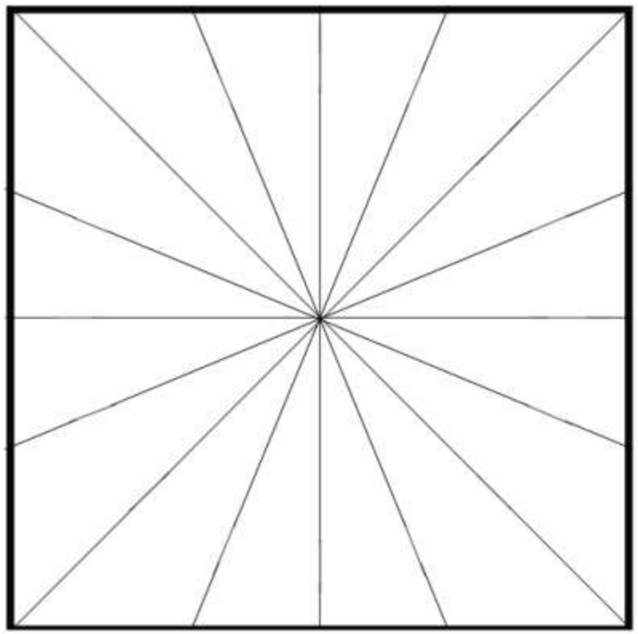


continuum of concentric circles inside the disc

continuum of concentric squircles inside the square

radial constraint



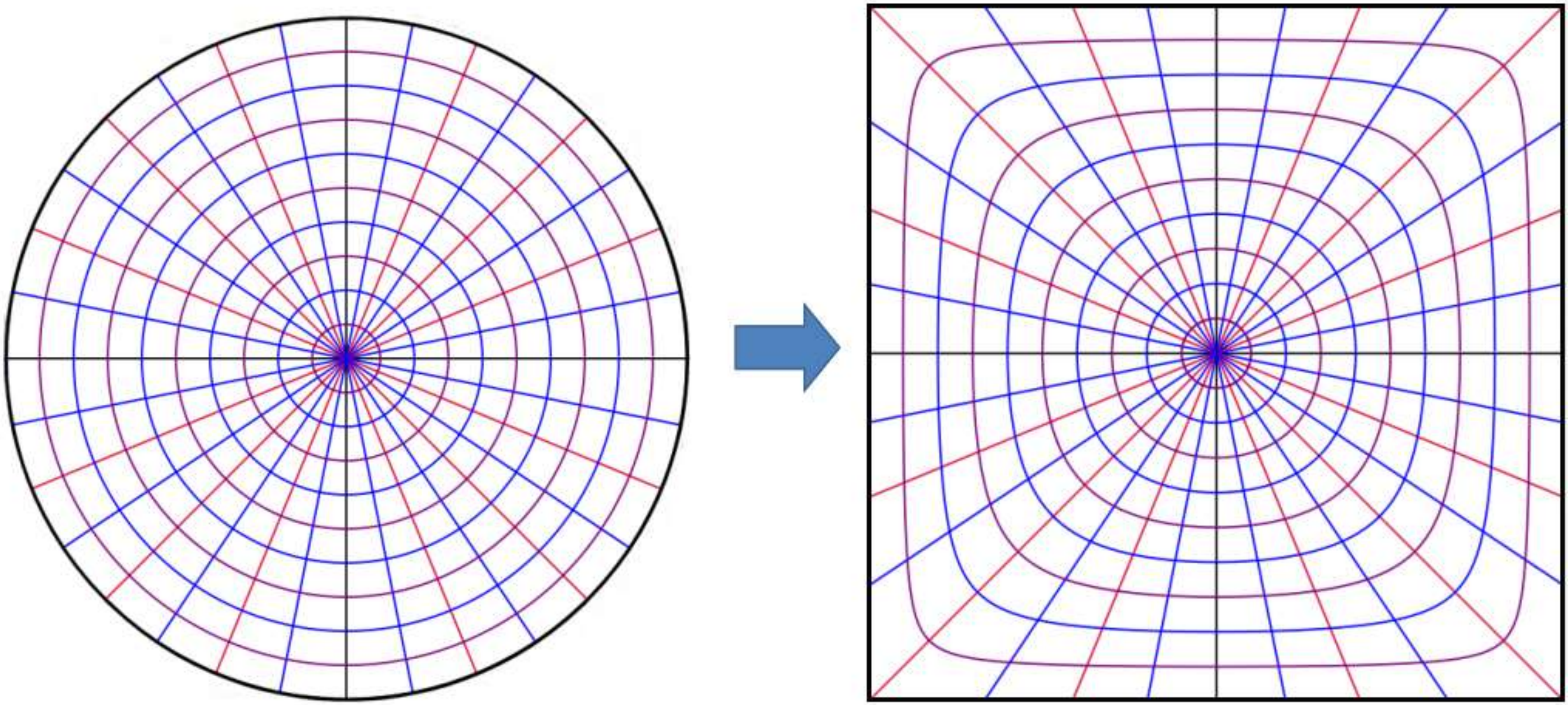


points move radially from the center

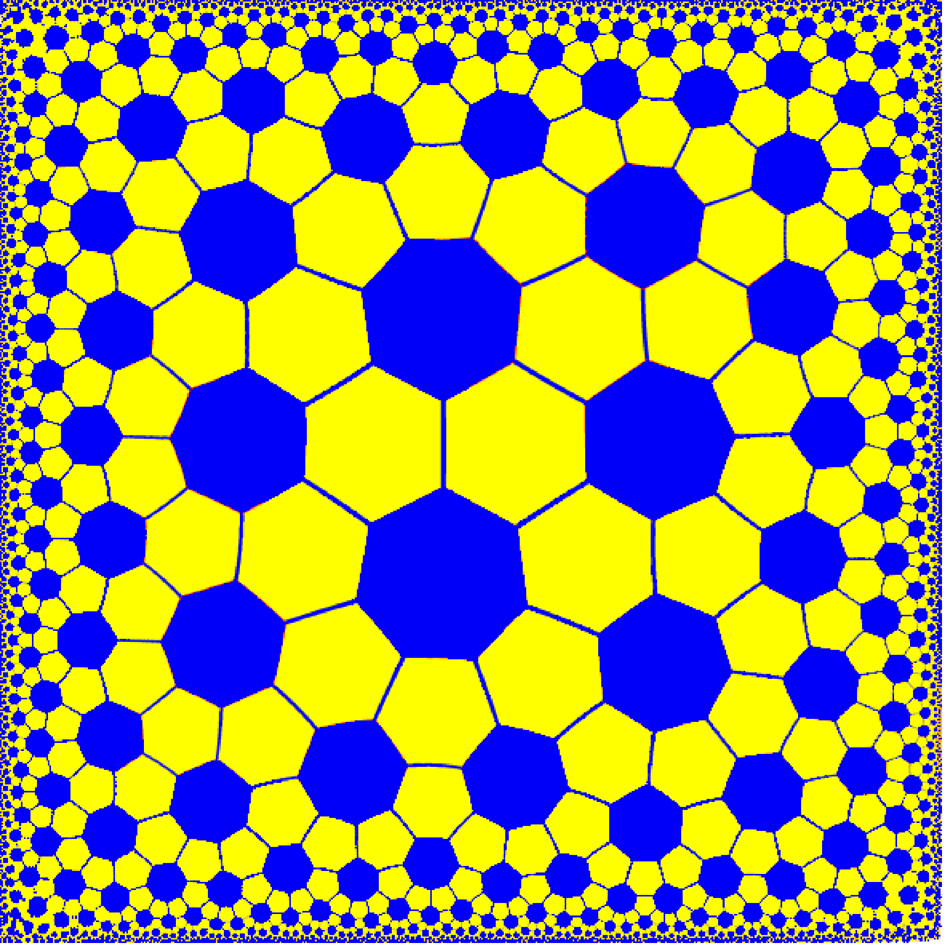
mapping # 2

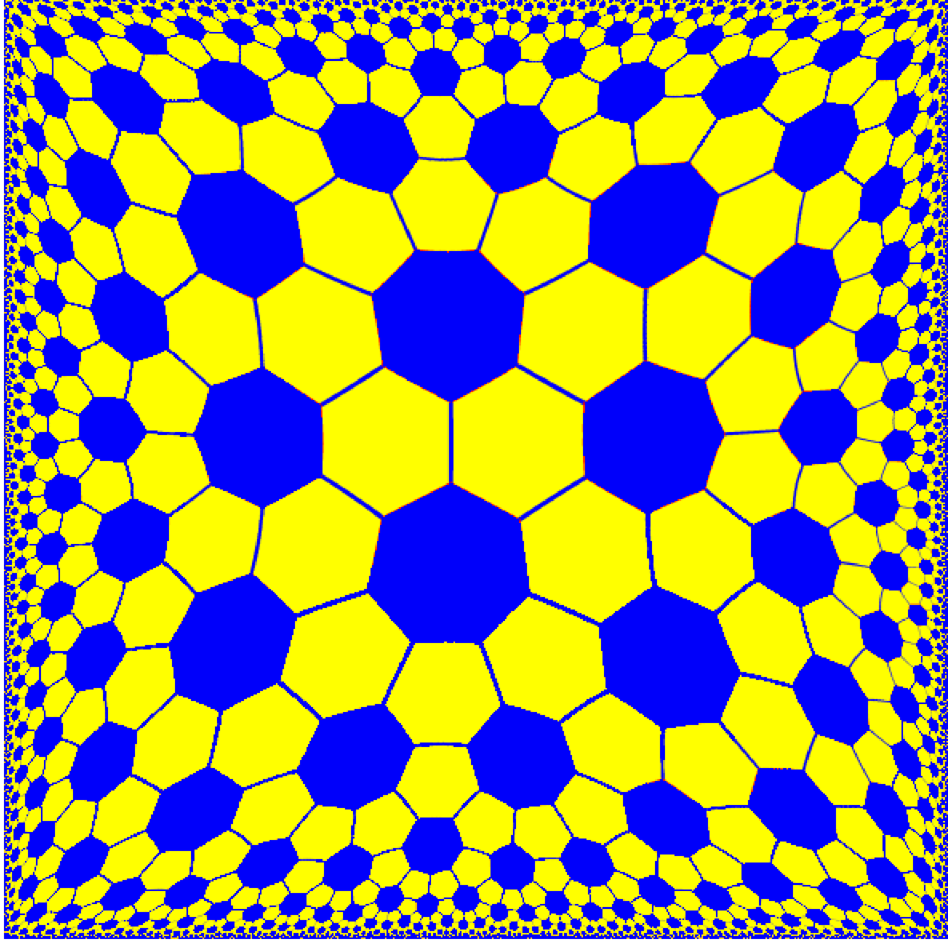
disc-to-square

POOR MAN'S



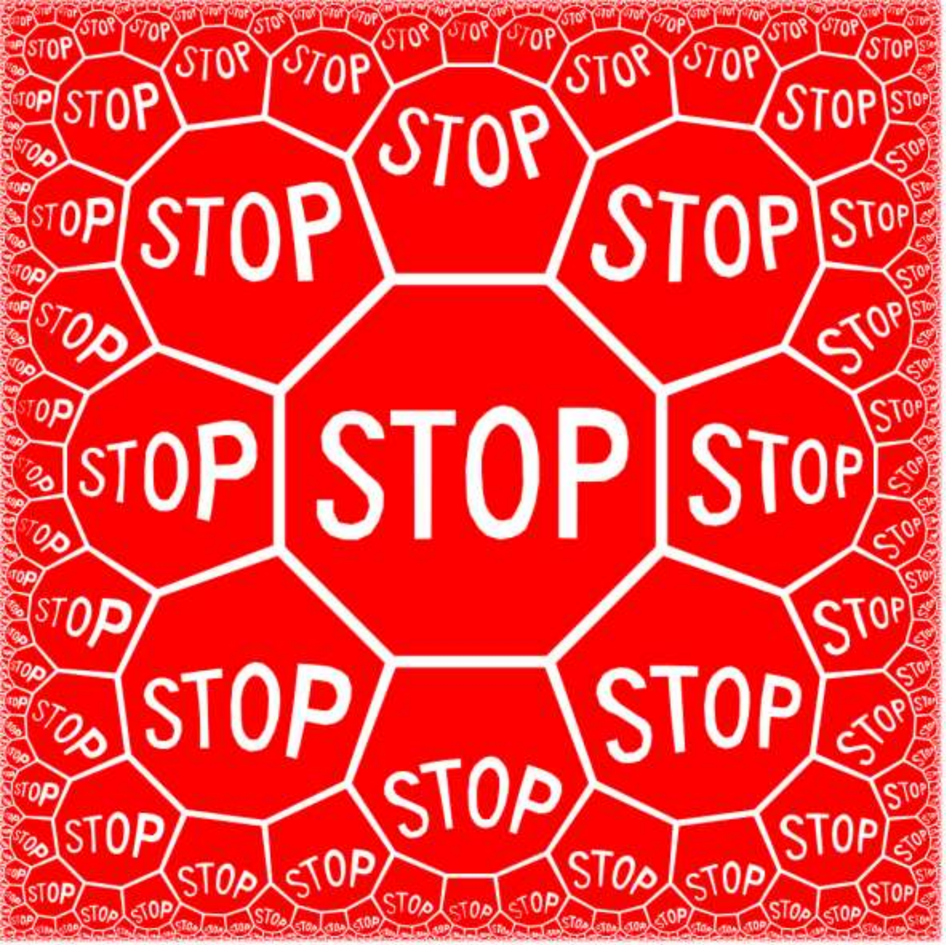
$$\begin{bmatrix} x \\ y \end{bmatrix} = sgn(uv)\sqrt{\frac{-u^2-v^2+\sqrt{(u^2+v^2)[u^2+v^2+4u^2v^2(u^2+v^2-2)]}}{2(u^2+v^2-2)}} \begin{bmatrix} 1/\mathrm{u} \\ 1/\mathrm{v} \end{bmatrix}$$

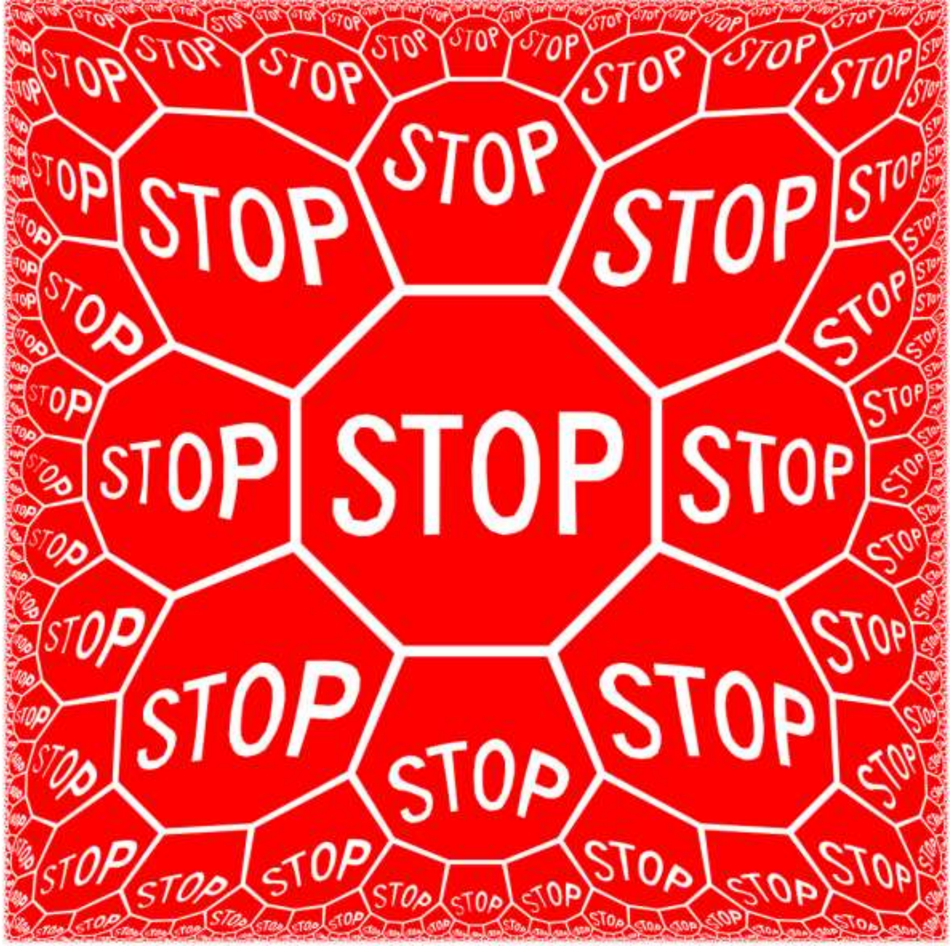




Schwarz Christoffel (conformal)

poor man's squircular mapping





Schwarz Christoffel (conformal)

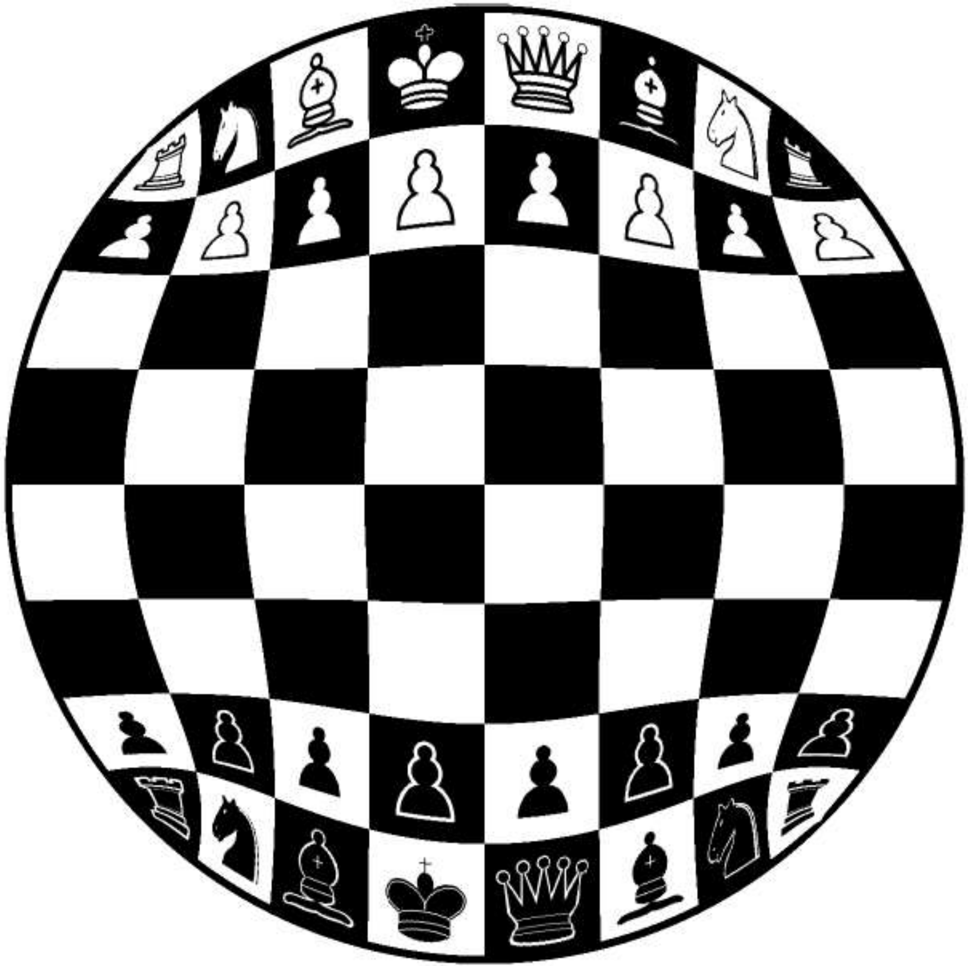
poor man's squircular mapping

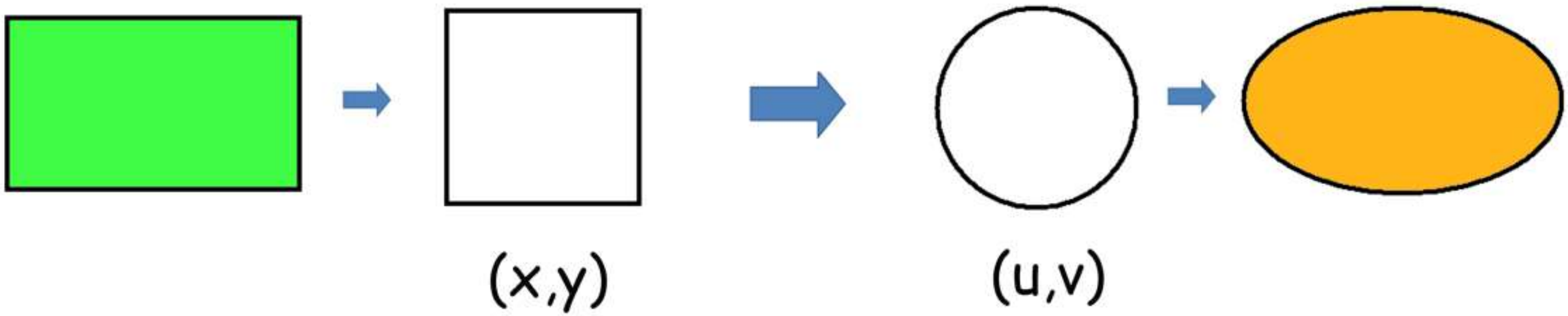
Schwarz Christoffel

poor man's (squircular)

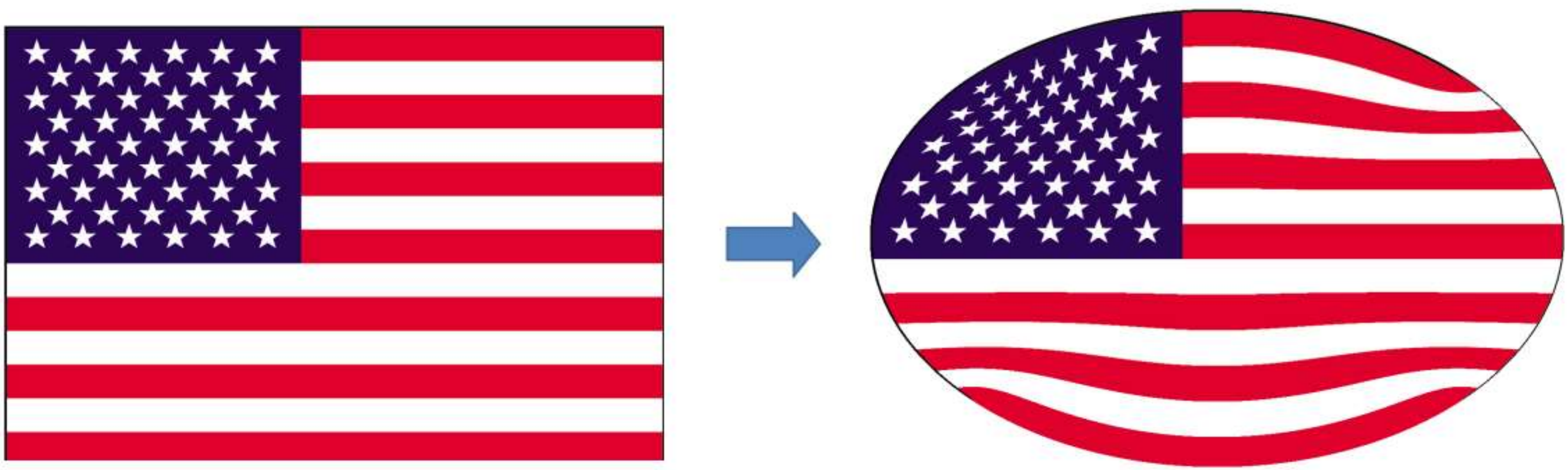
Schwarz Christoffel

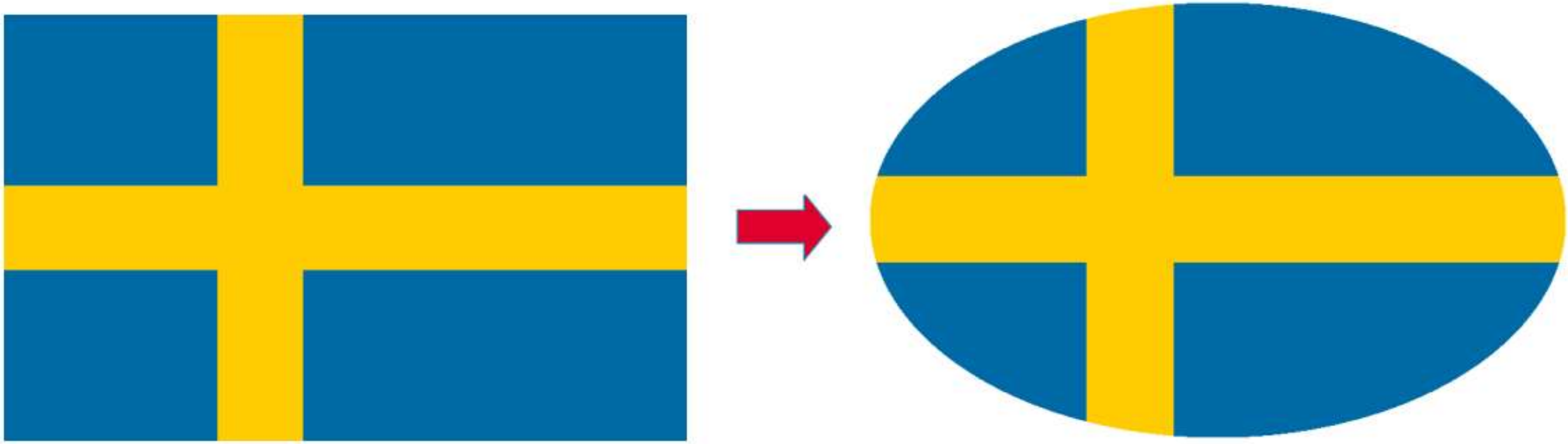
poor man's (squircular)



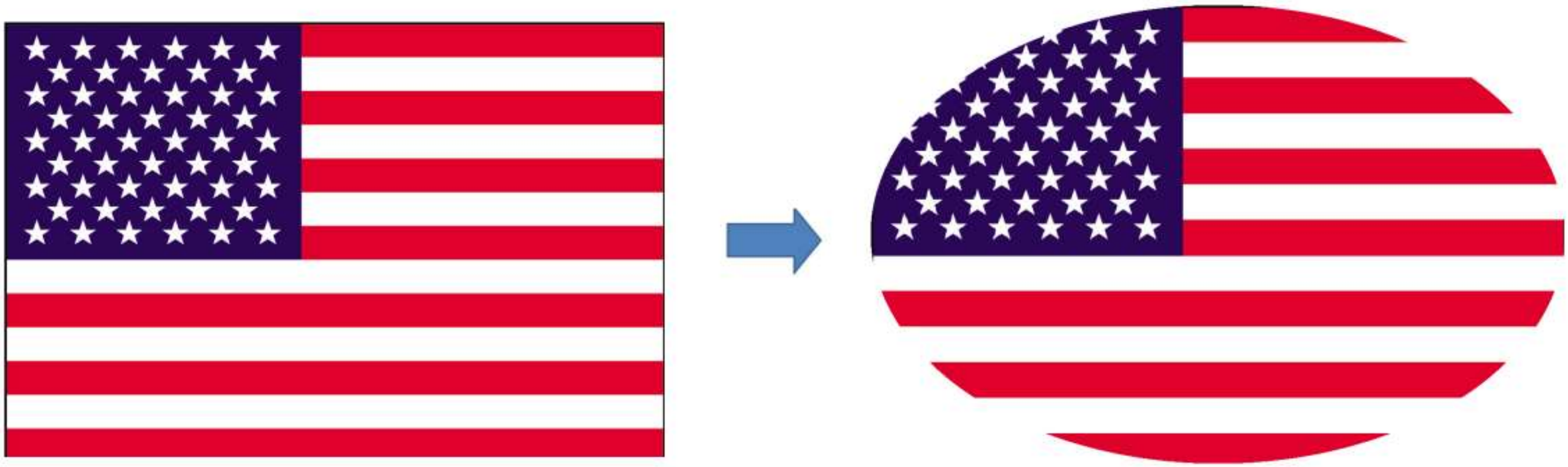


Elliptification



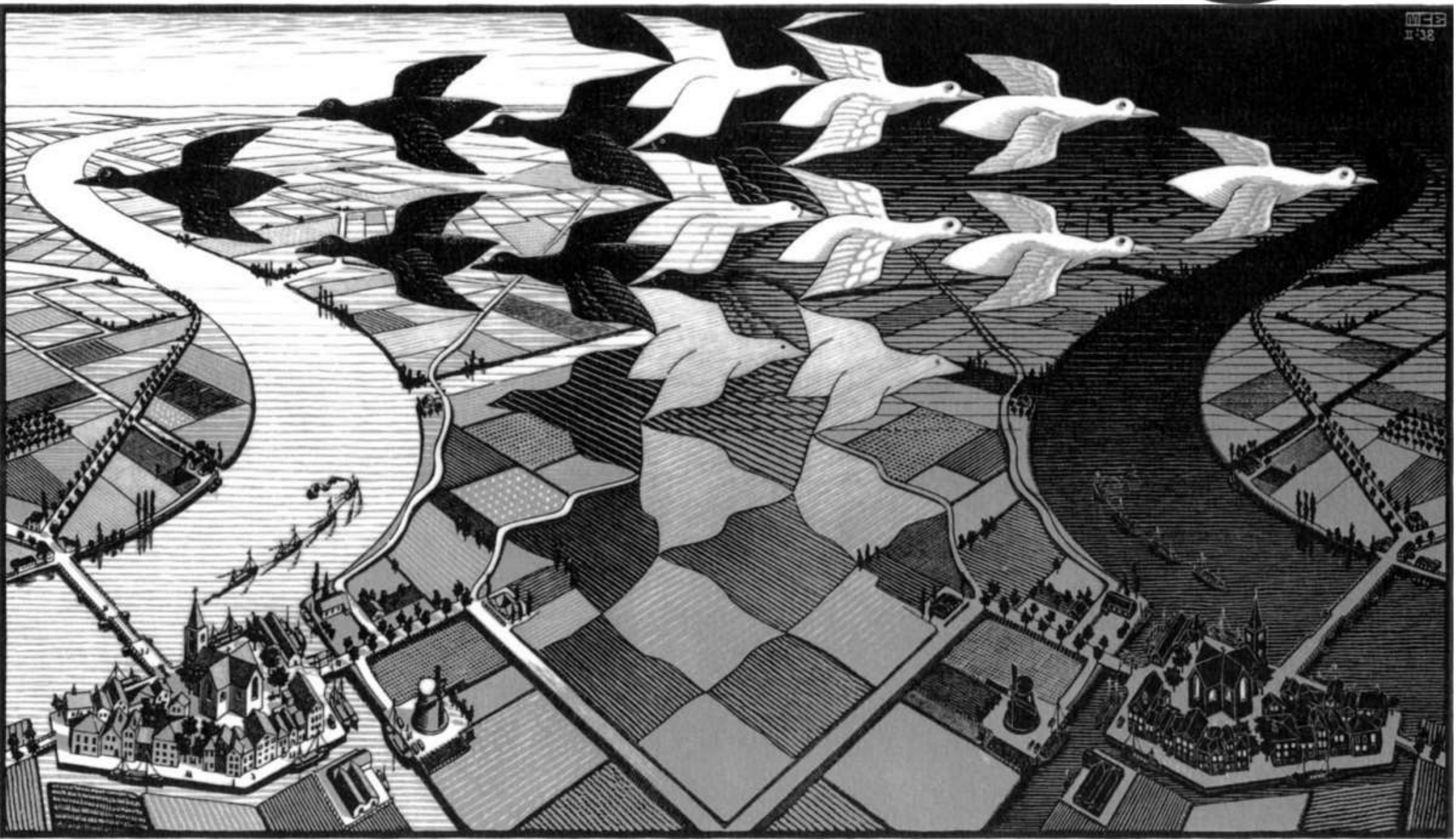


plain cropping

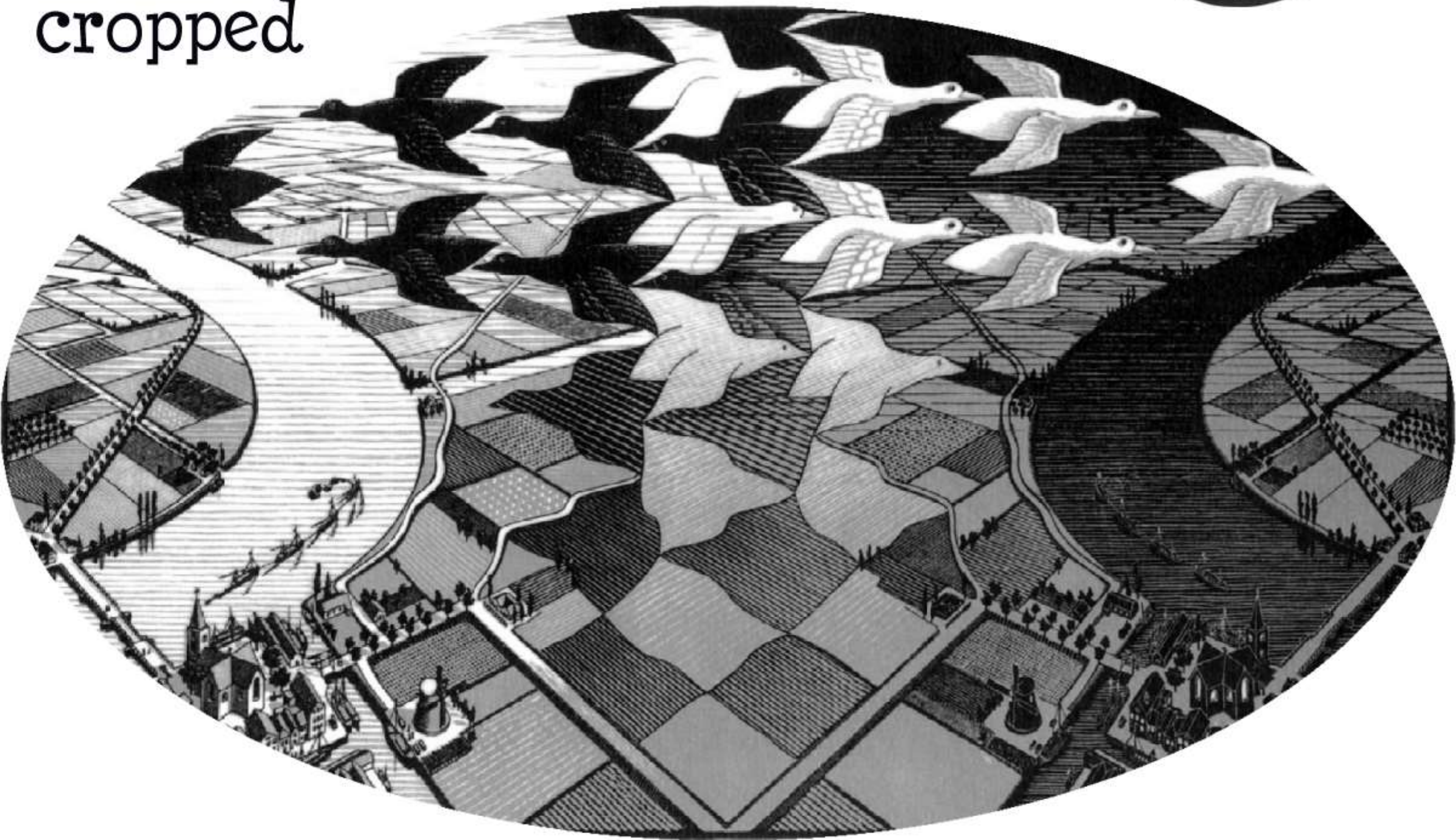


M.C. Escher

"Day and Night" (1938)



cropped



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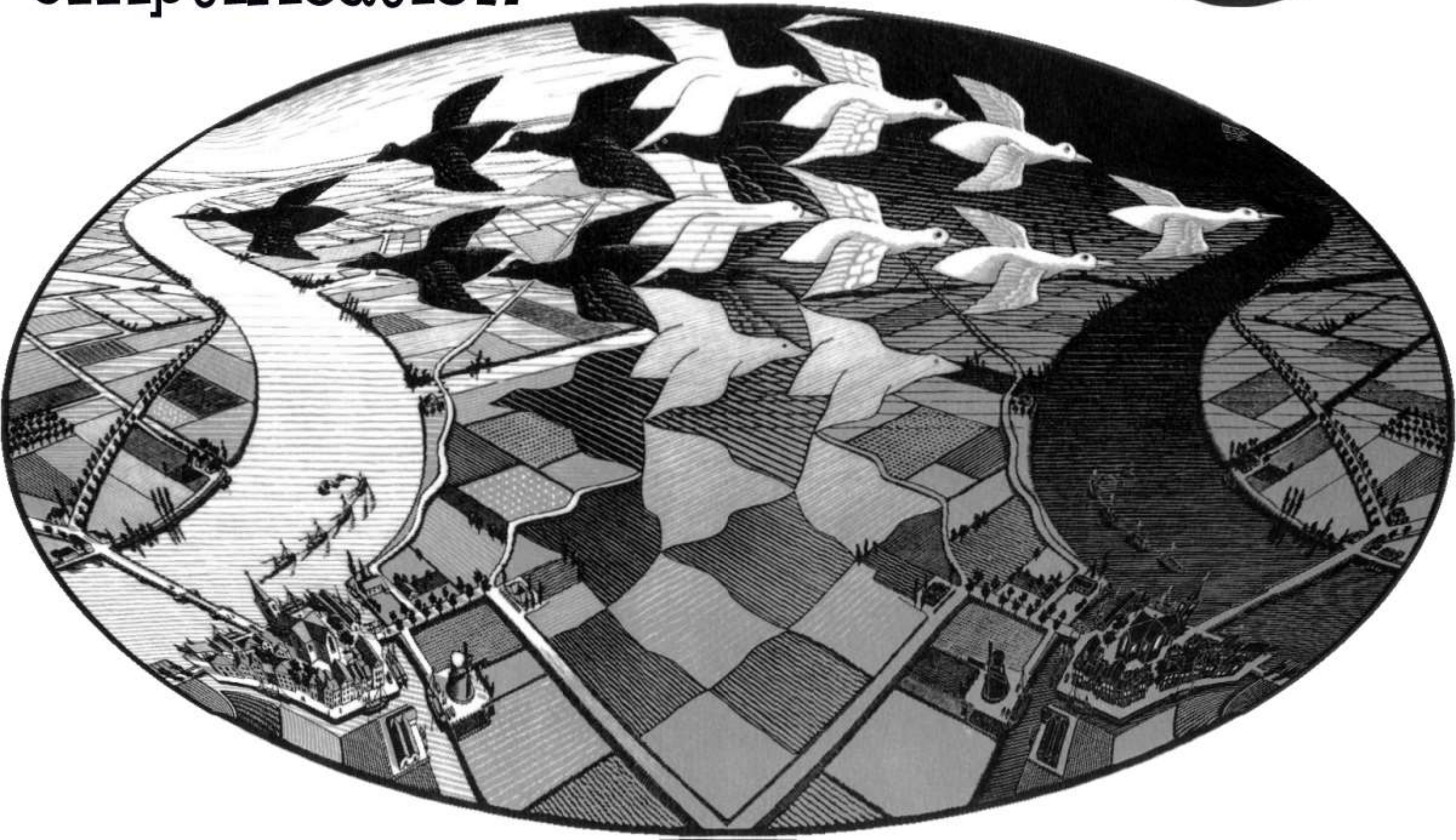

questions

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thanks to

Bruce Torrence
Eve Torrence
& the
anonymous
reviewers

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