Bridges Conference, August 2022

A Fish Pattern on a Regular Triply Periodic Polyhedron

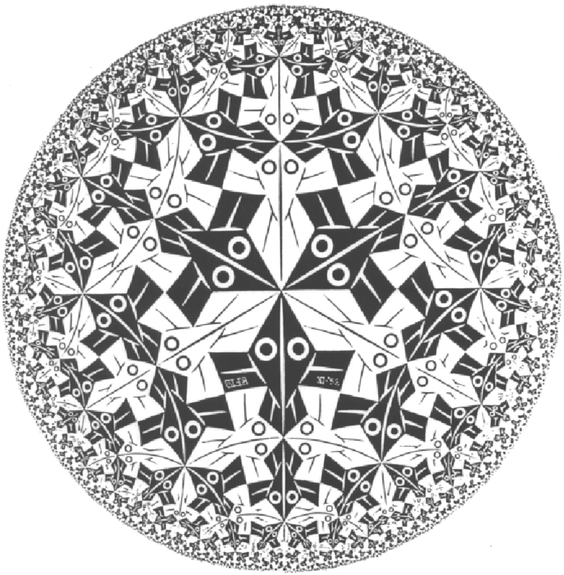
Douglas Dunham
Dept. of Computer Science
Univ. of Minnesota, Duluth
Duluth, MN 55812, USA

Lisa Shier
University of Maryland
Global Campus
New Market, AL 35761, USA

Outline

- Background and motivation
  - M.C. Escher's *Circle Limit I* and *Circle Limit III*
  - Regular $\{p, q \mid r\}$ triply periodic polyhedra
  - Previous polyhedra and their aesthetic problems
- The papercrafted part of a $\{4, 6 \mid 4\}$ polyhedron
- A part of the $\{6, 6 \mid 3\}$ polyhedron that solves all the problems
- Future work
- Contact information

Escher's Woodcut Circle Limit I



Aesthetic Problems with Circle Limit I per Escher

1. The fish were not consistently colored along backbone lines — they alternated from black to white and back every two fish lengths.
2. The fish also changed direction every two fish lengths — thus there was no "traffic flow" (Escher's words) in a single direction along the backbone lines.
3. The fish are very angular and not "fish-like"

Escher's Woodcut Circle Limit III

— solved the problems

Regular Triply Repeating Polyhedra

In 1926 H.S.M. Coxeter defined *regular skew polyhedra* (apeirohedra) to be infinite polyhedra repeating in three independent directions in Euclidean 3-space, with the symmetry group of isometries being transitive on flags.

Coxeter denoted them by the extended Schläfli symbol $\{p, q \,|\, r\}$ which denotes the polyhedron composed of p-gons meeting q at each vertex, with regular r-sided polygonal holes.

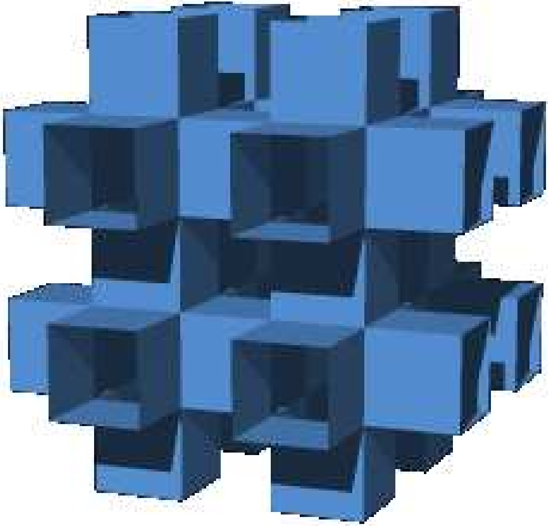
Coxeter and John Flinders Petrie proved that there are exactly three of them: $\{4, 6 \,|\, 4\}$, $\{6, 4 \,|\, 4\}$, and $\{6, 6 \,|\, 3\}$.

Since the sum of the vertex angles is greater than 2π, they are considered to be the hyperbolic analogs of the Platonic solids and the regular Euclidean tessellations $\{3, 6\}$, $\{4, 4\}$, and $\{6, 3\}$

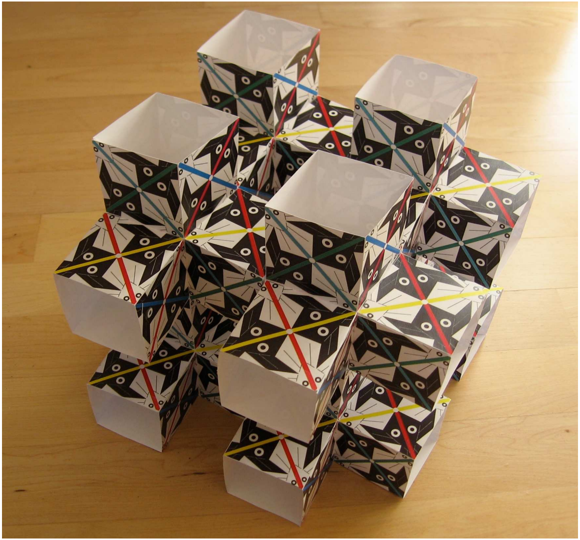
In 2012 Dunham was the first person to decorate those solids with Escher-inspired patterns.

The simplest regular skew polyhedron: $\{4, 6 \,|\, 4\}$

Also called the *Mucube* (for Multi-cube). It consists of invisible "hub" cubes connected by "strut" cubes, hollow cubical cylinders with their open ends connecting neighboring hubs.



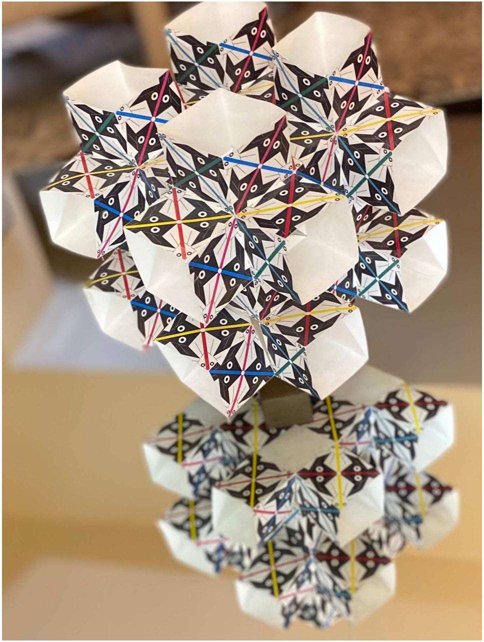
An old patterned $\{4, 6 \,|\, 4\}$ with fish



Problems with the old fish polyhedron

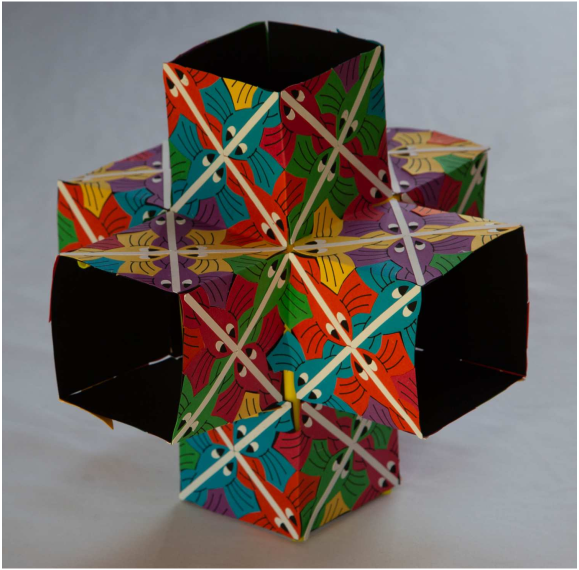
1. The same three problems Escher saw in *Circle Limit I*.
2. A fourth problem: the backbone lines of a particular color are not parallel — which can be seen in a mirror.

The old fish polyhedron on a mirror



A new papercrafted fish pattern on the $\{4, 6 \,|\, 4\}$ polyhedron

Fixes the first and third problems.



The papercrafted $\{4, 6 \,|\, 4\}$ polyhedron on a mirror

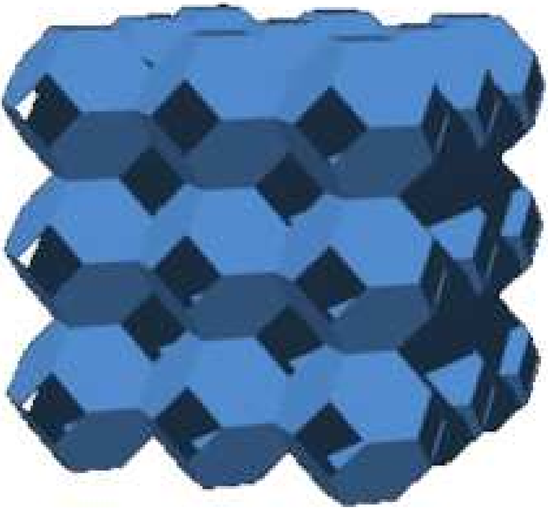
Fixes the fourth problem too, but not the second one.

Colors of fish on the $\{4, 6 \,|\, 4\}$ polyhedron

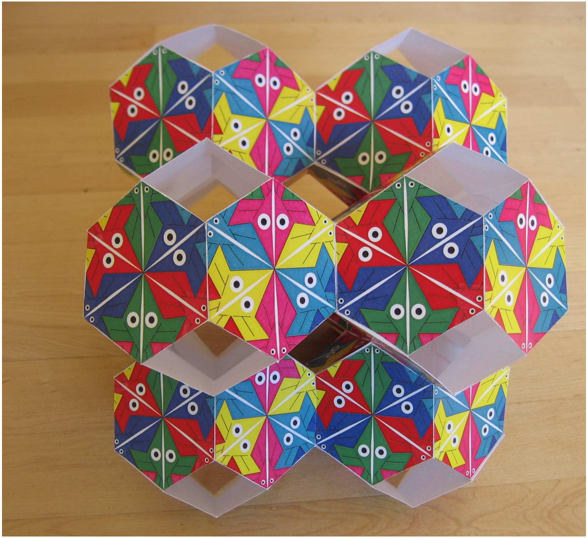
1. There are six families of fish backbone lines that are parallel to the face diagonals of a cube.
2. All the fish in one family are the same color.

The dual of the Mucube is the $\{6, 4 \,|\, 4\}$ polyhedron

Also called the *Muoctahedron* (for Multi-octahedron). It consists of truncated octahedra in a cubic lattice arrangement, connected on their invisible square faces (which are also the square holes between the truncated octahedra).

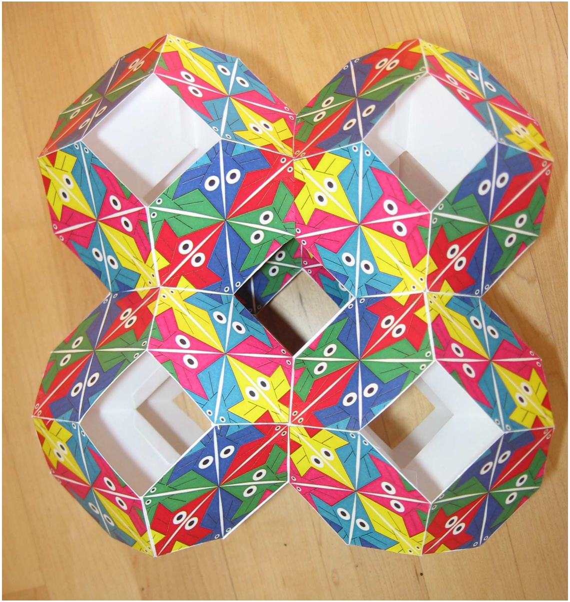


An angular fish pattern on the $\{6, 4 \mid 4\}$ polyhedron



A top view of the fish pattern on the $\{6, 4 \,|\, 4\}$ polyhedron

It solves Escher's first problem, but still has problems two and three.



The $\{6, 6 \mid 3\}$ polyhedron is self-dual

Also called the *Mutetrahedron* (for Multi-tetrahedron). It consists of truncated tetrahedra in a diamond lattice arrangement, connected by their missing triangular faces to faces of invisible regular tetrahedra between them.

The new $\{6,6\,|\,3\}$ patterned polyhedron

Also fixes the second, "traffic flow", problem.

Colors of fish on the $\{6, 6 \mid 3\}$ polyhedron

1. Again, there are six families of fish backbone lines that go through the centers of the hexagon faces of the $\{6, 6 \mid 3\}$ polyhedron.
2. And again, the fish in one family are the same color.
3. Each of the families is parallel to one of the sides of a tetrahedron — which can be one of the truncated tetrahedra, since all the (patterned) truncated tetrahedra in the $\{6, 6 \mid 3\}$ polyhedron are translates of one another.
4. In each family half the lines of fish go one direction, and the other half go the opposite direction — so that fish of one color on one truncated tetrahedron go in opposite directions on adjacent faces.

Future Work

- We would like to make a papercrafted version of the new $\{6, 6 \mid 3\}$ patterned polyhedron.
- We would like to explore putting other patterns on the $\{p, q \mid r\}$ polyhedra, and on less regular triply periodic $\{p, q\}$ polyhedra.

Acknowledgements and Contact

We would sincerely like to thank all the Bridges 2022 organizers!

Contact Information:

Doug Dunham
Email: ddunham@d.umn.edu
Web: http://www.d.umn.edu/~ddunham

Lisa Shier
Email: kwajshier@yahoo.com
Blog: "Fun with a Sewing Machine"
http://funwithasewingmachine.blogspot.com/