

A Fish Pattern on a Triply Periodic Polyhedron

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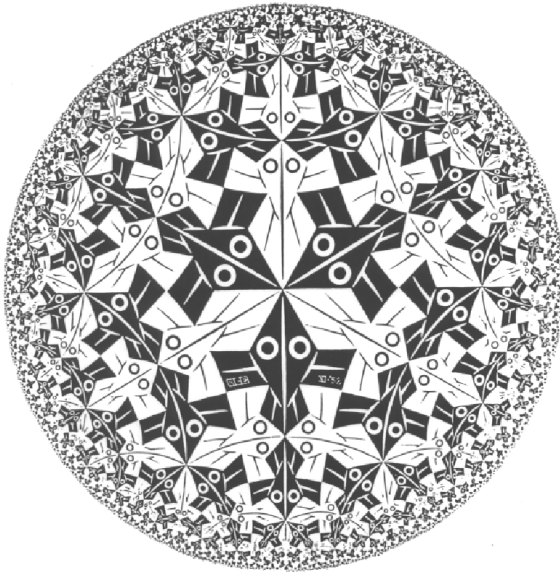
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Outline

- ▶ Background and motivation
 - ▶ M.C. Escher's *Circle Limit I* and *Circle Limit III*
 - ▶ Regular $\{p, q \mid r\}$ triply periodic polyhedra
 - ▶ Previous polyhedra and their aesthetic problems
- ▶ The papercrafted part of a $\{4, 6 \mid 4\}$ polyhedron
- ▶ A part of the $\{6, 6 \mid 3\}$ polyhedron that solves all the problems
- ▶ Future work
- ▶ Contact information

Escher's Woodcut Circle Limit I



Aesthetic Problems with Circle Limit I per Escher

1. The fish were not consistently colored along backbone lines — they alternated from black to white and back every two fish lengths.
2. The fish also changed direction every two fish lengths — thus there was no “traffic flow” (Escher’s words) in a single direction along the backbone lines.
3. The fish are very angular and not “fish-like”

Escher's Woodcut Circle Limit III

— solved the problems



Regular Triply Repeating Polyhedra

In 1926 H.S.M. Coxeter defined *regular skew polyhedra* (apeirohedra) to be infinite polyhedra repeating in three independent directions in Euclidean 3-space, with the symmetry group of isometries being transitive on flags.

Coxeter denoted them by the extended Schläfli symbol $\{p, q | r\}$ which denotes the polyhedron composed of p -gons meeting q at each vertex, with regular r -sided polygonal holes.

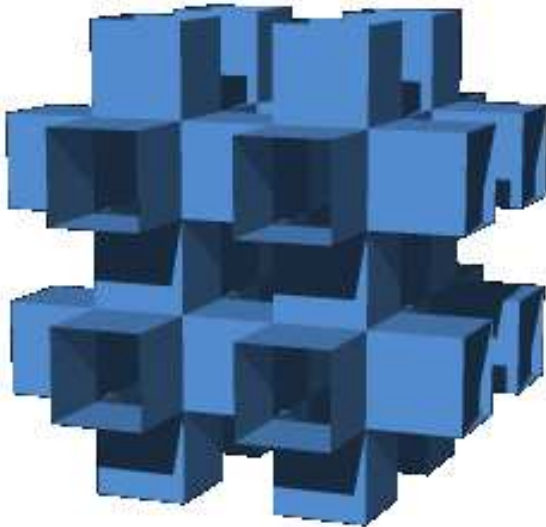
Coxeter and John Flinders Petrie proved that there are exactly three of them: $\{4, 6 | 4\}$, $\{6, 4 | 4\}$, and $\{6, 6 | 3\}$.

Since the sum of the vertex angles is greater than 2π , they are considered to be the hyperbolic analogs of the Platonic solids and the regular Euclidean tessellations $\{3, 6\}$, $\{4, 4\}$, and $\{6, 3\}$

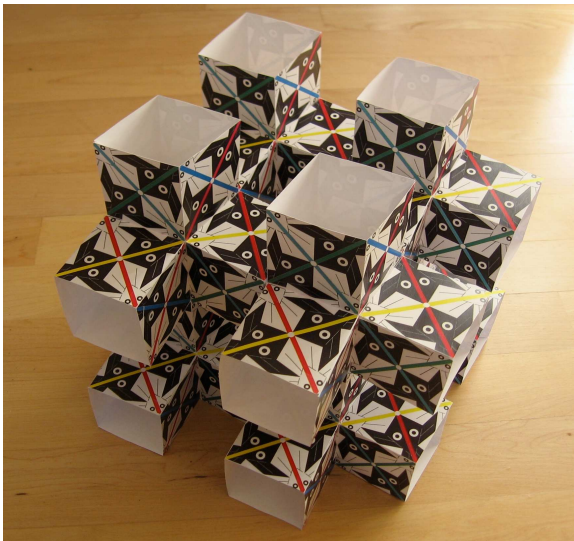
In 2012 Dunham was the first person to decorate those solids with Escher-inspired patterns.

The simplest regular skew polyhedron: $\{4, 6 | 4\}$

Also called the *Mucube* (for Multi-cube). It consists of invisible “hub” cubes connected by “strut” cubes, hollow cubical cylinders with their open ends connecting neighboring hubs.



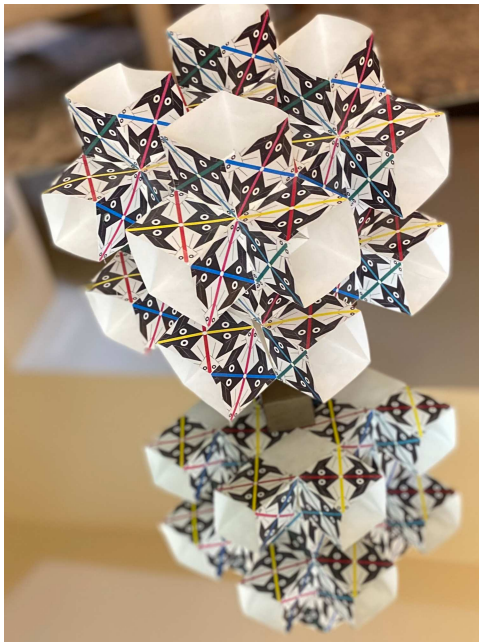
An old patterned $\{4, 6 | 4\}$ with fish



Problems with the old fish polyhedron

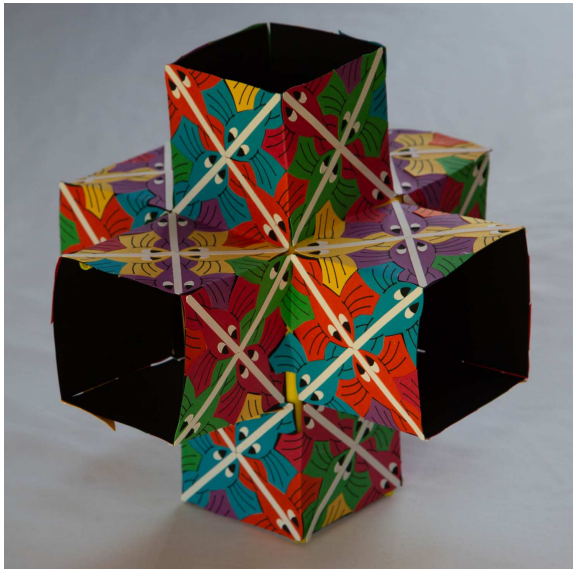
1. The same three problems Escher saw in *Circle Limit I*.
2. A fourth problem: the backbone lines of a particular color are not parallel — which can be seen in a mirror.

The old fish polyhedron on a mirror



A new papercrafted fish pattern on the $\{4, 6 | 4\}$ polyhedron

Fixes the first and third problems.



The papercrafted $\{4, 6 | 4\}$ polyhedron on a mirror
Fixes the fourth problem too, but not the second one.

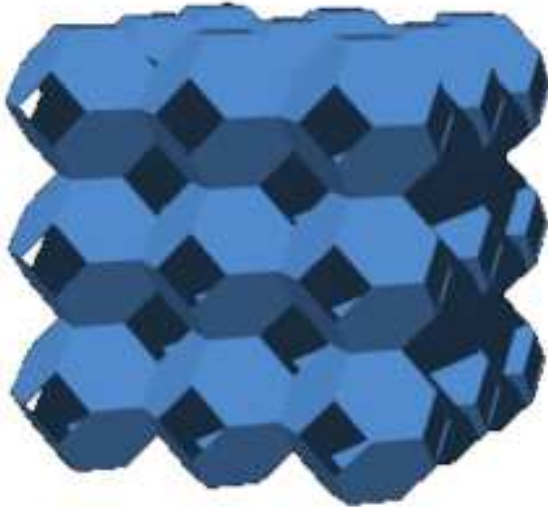


Colors of fish on the $\{4, 6 | 4\}$ polyhedron

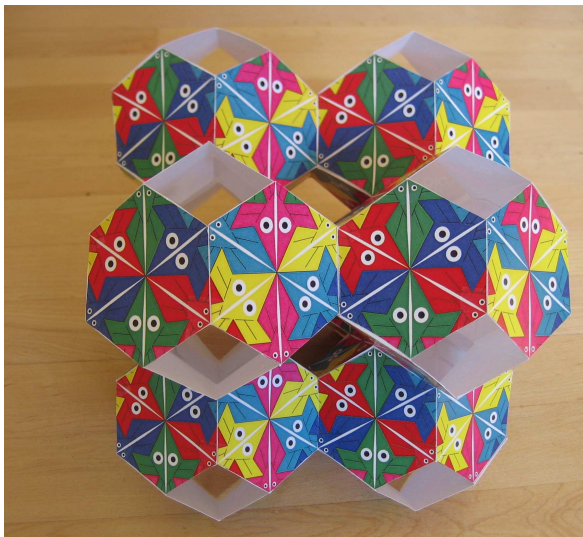
1. There are six families of fish backbone lines that are parallel to the face diagonals of a cube.
2. All the fish in one family are the same color.

The dual of the Mucube is the $\{6, 4 | 4\}$ polyhedron

Also called the *Muoctahedron* (for Multi-octahedron). It consists of truncated octahedra in a cubic lattice arrangement, connected on their invisible square faces (which are also the square holes between the truncated octahedra).



An angular fish pattern on the $\{6, 4 | 4\}$ polyhedron



A top view of the fish pattern on the $\{6, 4 | 4\}$ polyhedron

It solves Escher's first problem, but still has problems two and three.



The $\{6, 6 | 3\}$ polyhedron is self-dual

Also called the *Mutetrahedron* (for Multi-tetrahedron). It consists of truncated tetrahedra in a diamond lattice arrangement, connected by their missing triangular faces to faces of invisible regular tetrahedra between them.



The new $\{6, 6 | 3\}$ patterned polyhedron
Also fixes the second, "traffic flow", problem.



Colors of fish on the $\{6, 6 | 3\}$ polyhedron

1. Again, there are six families of fish backbone lines that go through the centers of the hexagon faces of the $\{6, 6 | 3\}$ polyhedron.
2. And again, the fish in one family are the same color.
3. Each of the families is parallel to one of the sides of a tetrahedron — which can be one of the truncated tetrahedra, since all the (patterned) truncated tetrahedra in the $\{6, 6 | 3\}$ polyhedron are translates of one another.
4. In each family half the lines of fish go one direction, and the other half go the opposite direction — so that fish of one color on one truncated tetrahedron go in opposite directions on adjacent faces.

Future Work

- ▶ We would like to make a papercrafted version of the new $\{6, 6 | 3\}$ patterned polyhedron.
- ▶ We would like to explore putting other patterns on the $\{p, q | r\}$ polyhedra, and on less regular triply periodic $\{p, q\}$ polyhedra.

Acknowledgements and Contact

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