# The Symmetry of “Circle Limit IV” and Related Patterns

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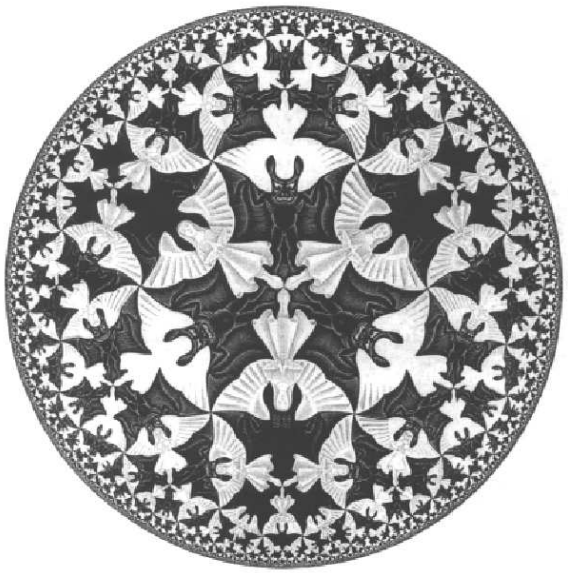


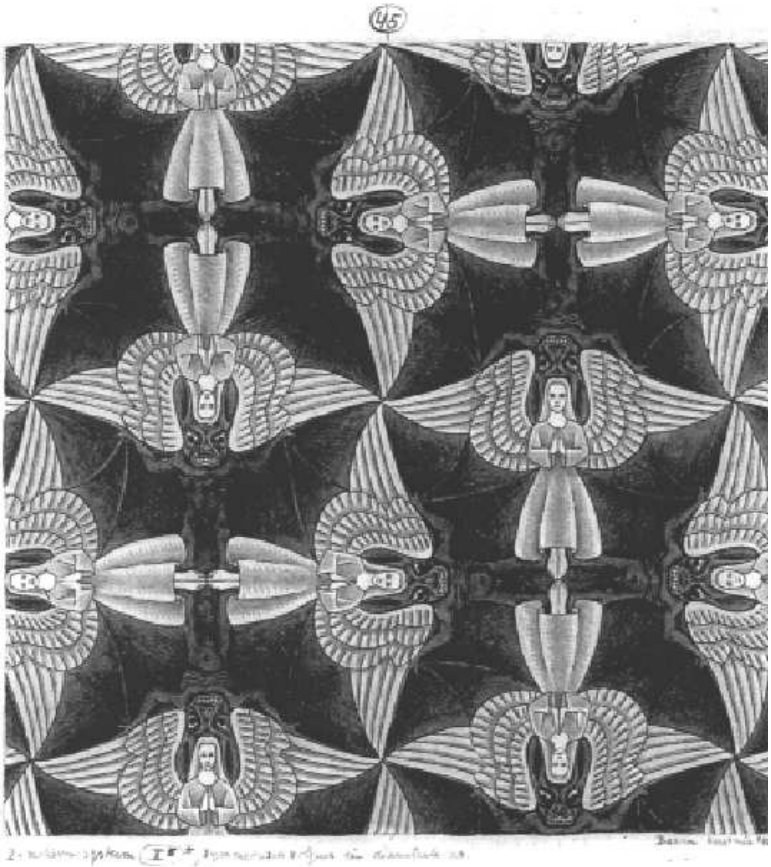
# Outline

- **History**
- **The correct orientation for *Circle Limit IV***
- **The symmetry group of the *Circle Limit IV* pattern**
- **Related patterns**
- **Future Work**

# History

- **1941 - Escher creates Notebook Drawing # 45.**
- **1942 - Escher carves the “Heaven and Hell” maple sphere.**
- **1957–1960 - Escher corresponds with H.S.M. Coxeter, gaining inspiration to create his four “Circle Limit” woodcuts, culminating in *Circle Limit IV*, thus mapping the angels and devils pattern onto each of the three “classical” geometries: the Euclidean plane, the sphere, and the hyperbolic plane.**
- **Since then, at times, *Circle Limit IV* has appeared incorrectly oriented.**
- **2008 - Two books appear with *Circle Limit IV* incorrectly oriented.**
- **2009 - At the Joint Mathematics Meeting two more books appear with *Circle Limit IV* incorrectly oriented.**

# Escher's Notebook Drawing # 45 (1941) (*M.C. Escher: Visions of Symmetry* page 150)



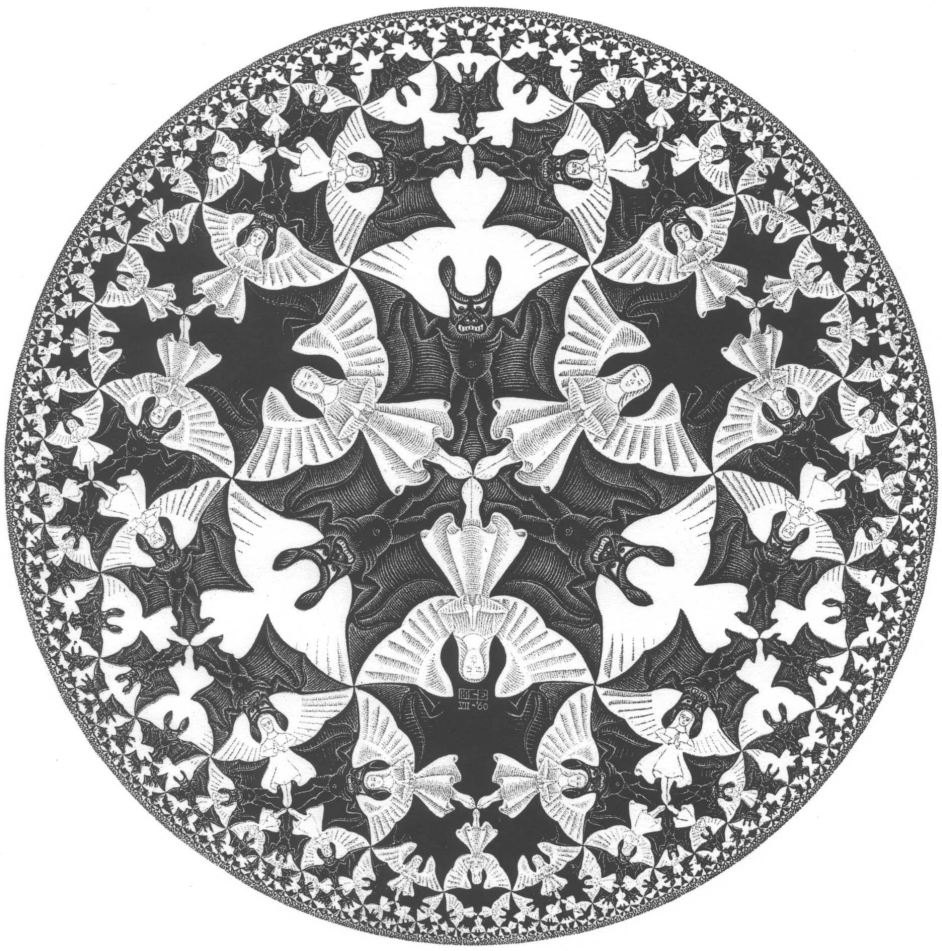
# “Heaven and Hell” Maple Sphere (1942)

"Heaven and Hell" carved sphere, 1942. Maple, stained in two colors, diameter 235 mm.

# Escher's Description (in a letter to C.V.S. Roosevelt, *M.C. Escher: Visions of Symmetry* Harry N. Abrams, 2004 ISBN 0-8109-4308-5, page 245):

**It has two poles and an equator. One pole represents "heaven," with only white angels on a black background, which I carved much deeper than the angel figures. The other pole shows "hell," with only black devils on a deeply carved white background. At the equator both angels and devils are visible and equivalent, carved at the same sphere-level.**

# *Circle Limit IV* (1960)



# What is the correct orientation? Answer:

# A Sub-question

## Why aren’t there signatures in all three of the blank devils’ faces next to the central angels?

# Escher’s Signature (graphic)

# Escher’s Signature (text)

MCE
VII-’60

# Why Examine Orientation Now?

- ***The Symmetries of Things***
  **Conway, Burgiel, Goodman-Strass**
  **A.K. Peters, 2008**

- ***Euclidean and Non-Euclidean Geometries: Development and History***
  **Marvin Greenberg**
  **W.H. Freeman, 2008**

- ***Analysis, Geometry, and Modeling in Finance: Advanced Methods in Option Pricing***
  **Pierre Henry-Labordère**
  **Chapman & Hall/CRC Press, 2008**

- ***Math and Art: an Introduction to Visual Mathematics***
  **Sasho Kalajdzievski**
  **Chapman & Hall/CRC Press, 2008**

# *The Symmetries of Things* page 224 ($90^\circ$)

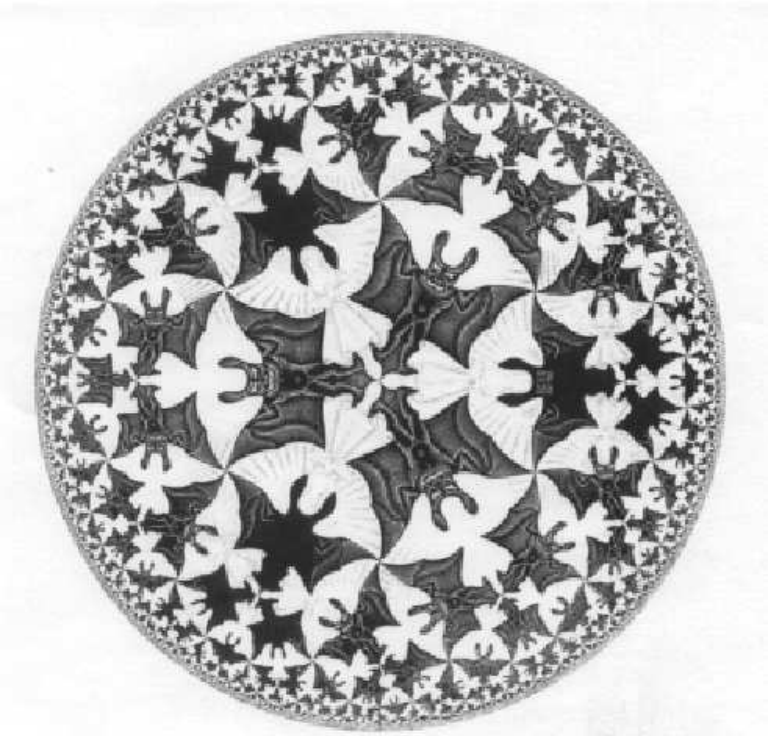
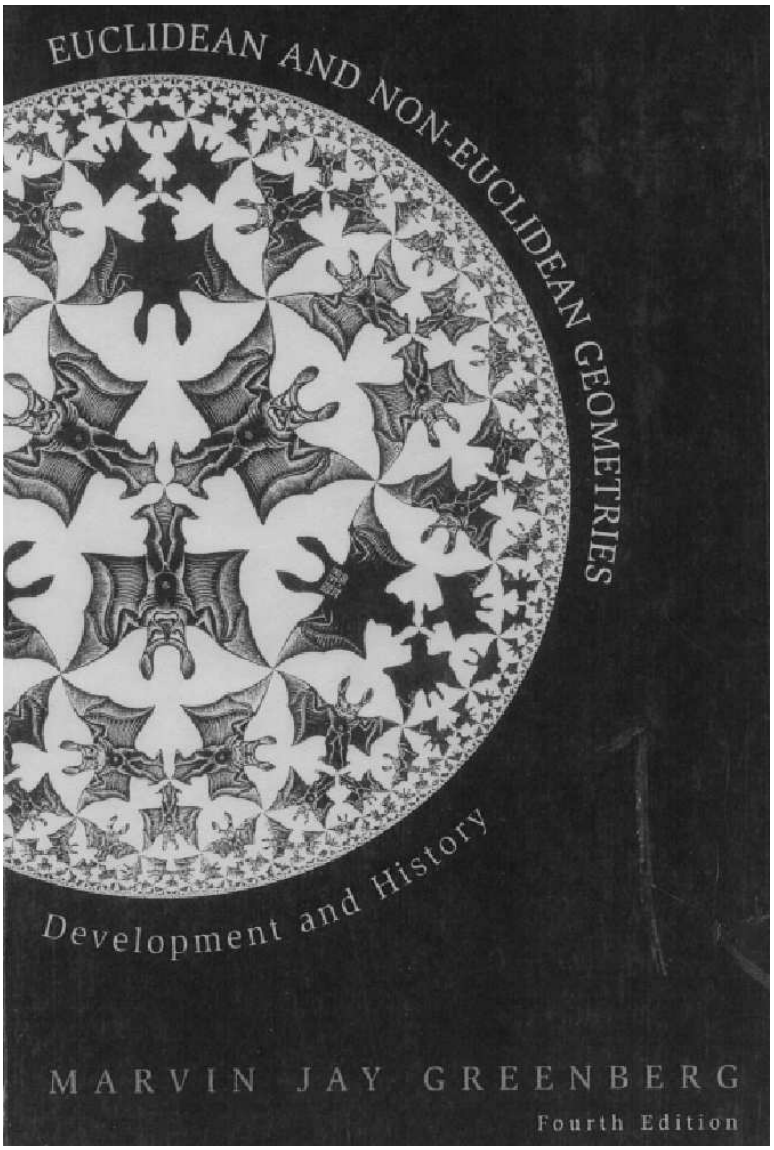
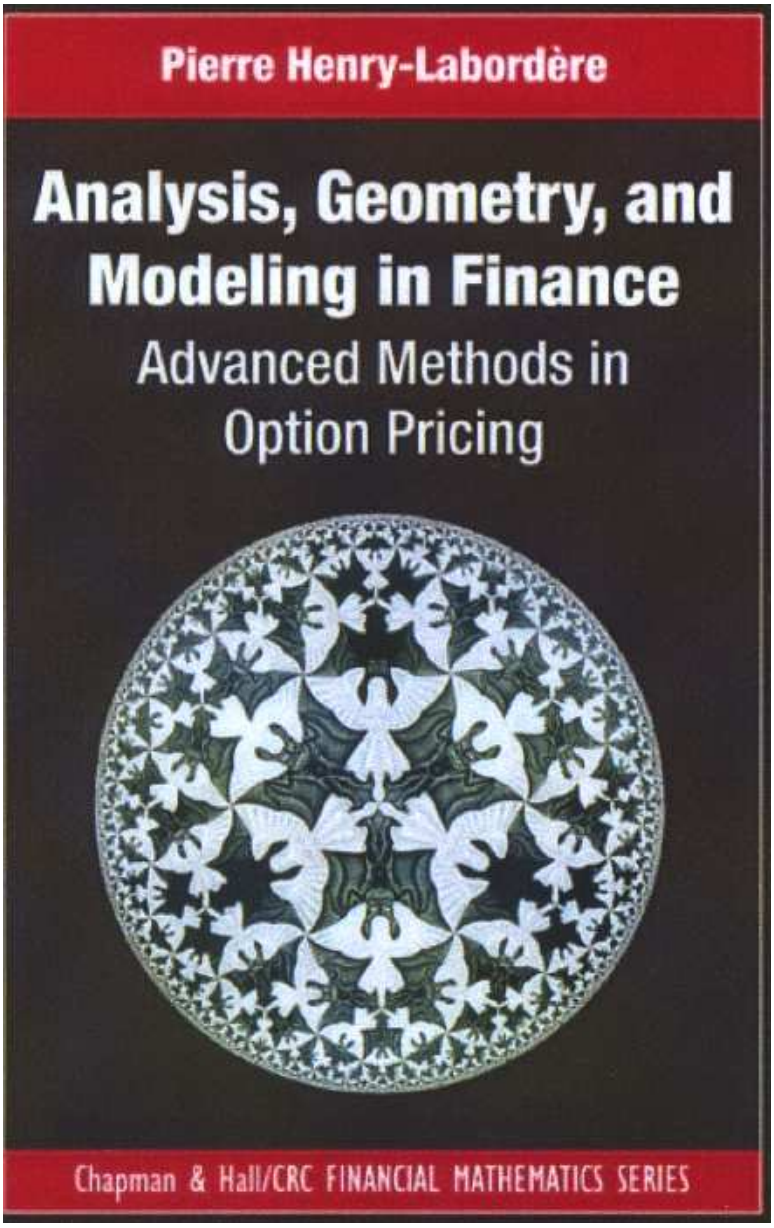


Figure 17.4. At first glance, M. C. Escher's *Circle Limit IV*, often called "Angles and Devils," seems to have signature **4∗3**, but in fact it has signature **+3333**.

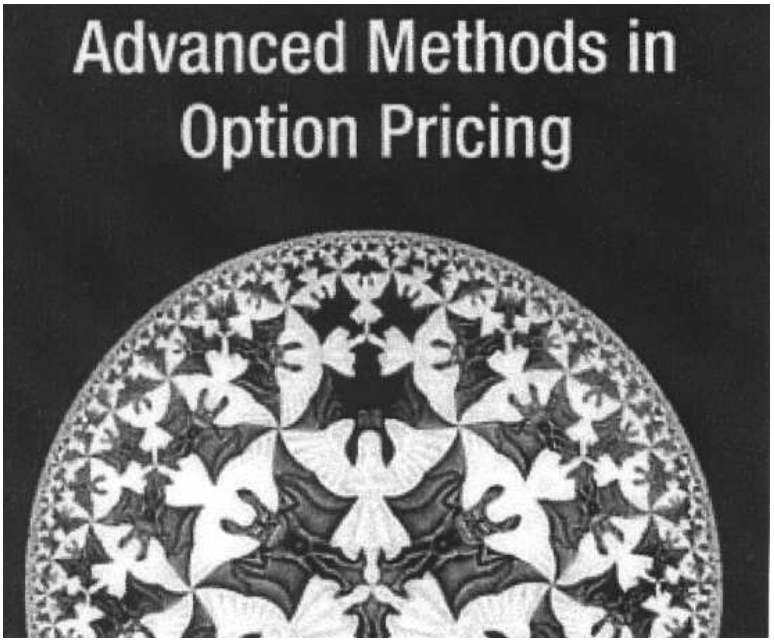
# *Euclidean and Non-Euclidean Geometries*
# (60°)



# ***Analysis, Geometry, and Modeling in Finance*** **(**180°**)**



# *Analysis, Geometry, and Modeling in Finance* close-up



# *Math and Art* pages 166, 167 (120°)

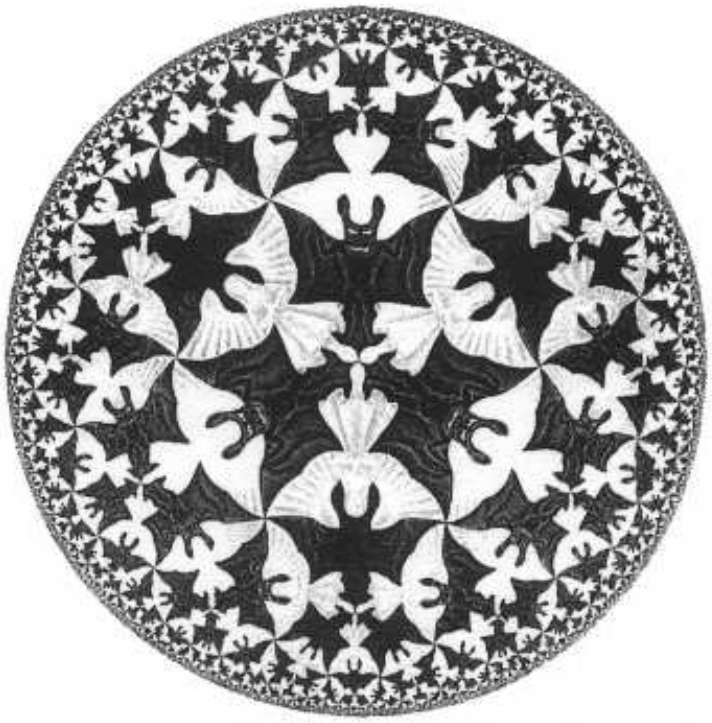
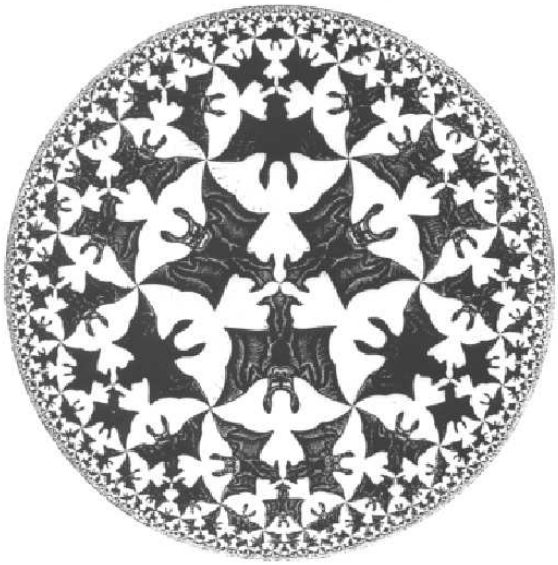


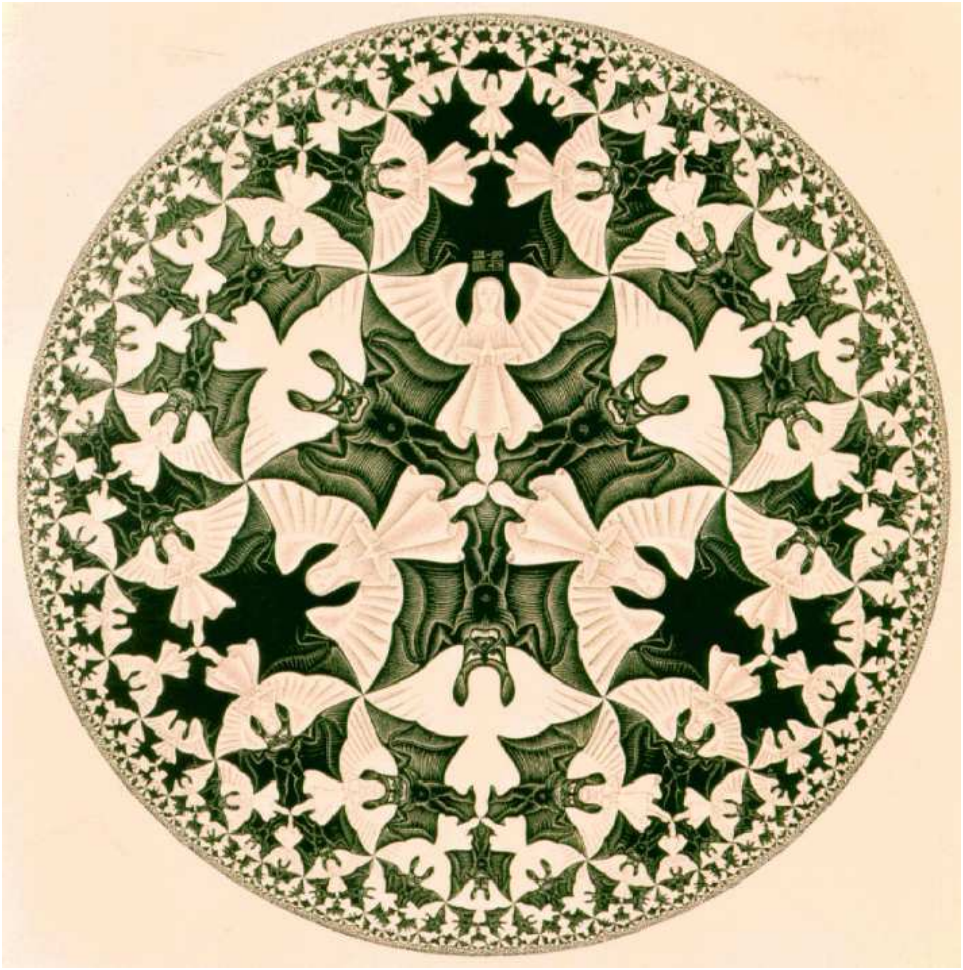
FIGURE 4.5.9 M. C. Escher. *Circle limit 4*, woodcut, 1960. (This woodcut is known as *Heaven and Hell*, and also known as *Angels and Devils*.)

# The World of M.C. Escher His Life and Complete Graphic Work (Harry N. Abrams, 1981)

- **Page 98 (large image): correctly oriented**
- **Page 322 (small image — Catalogue number 436): (**60°**)**

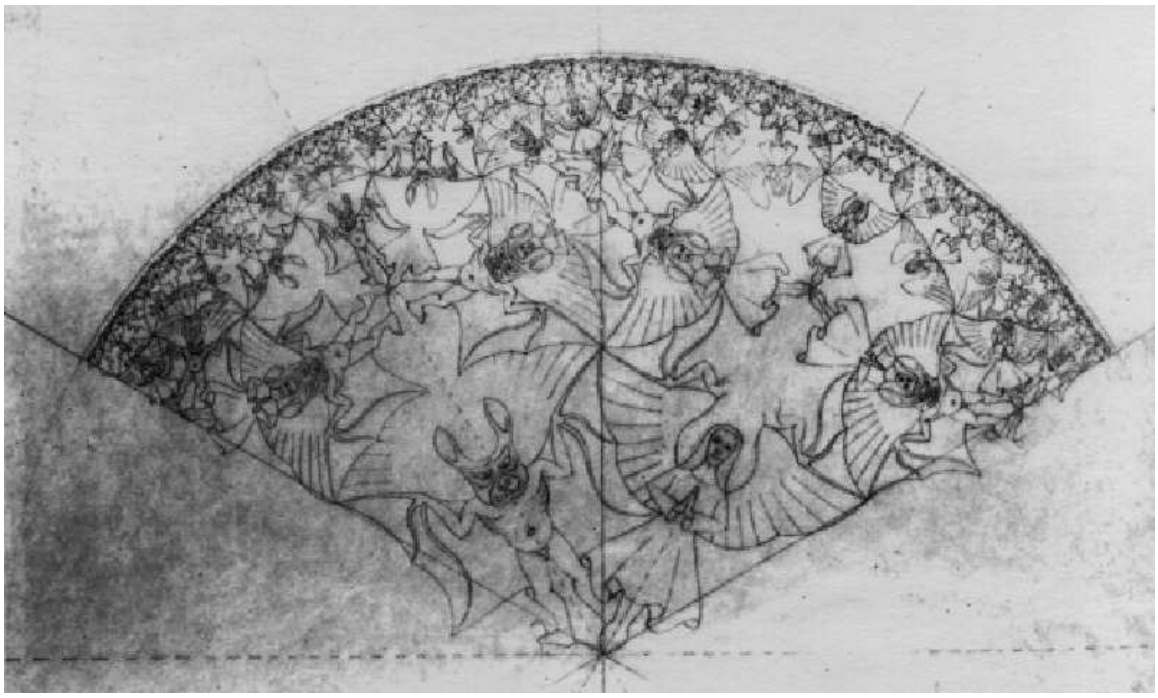


# M.C. Escher web site



# The Signature

# What is the Symmetry group of *Circle Limit IV* ? $C_3$?



# It seems to be $D_3$

**Escher (in *M.C. Escher The Graphic Work* Barnes & Noble/TASCHEN 2007 ISBN 0-7607-9669-6, page 10):**

**Here too, we have the components diminishing in size as they move outwards. The six largest (three white angels and three black devils) are arranged about the centre and radiate from it. The disc is divided into six sections in which, turn and turn about, the angels on a black background and then the devils on a white one gain the upper hand. In this way, heaven and hell change place six times. In the intermediate, "earthly" stages, they are equivalent.**

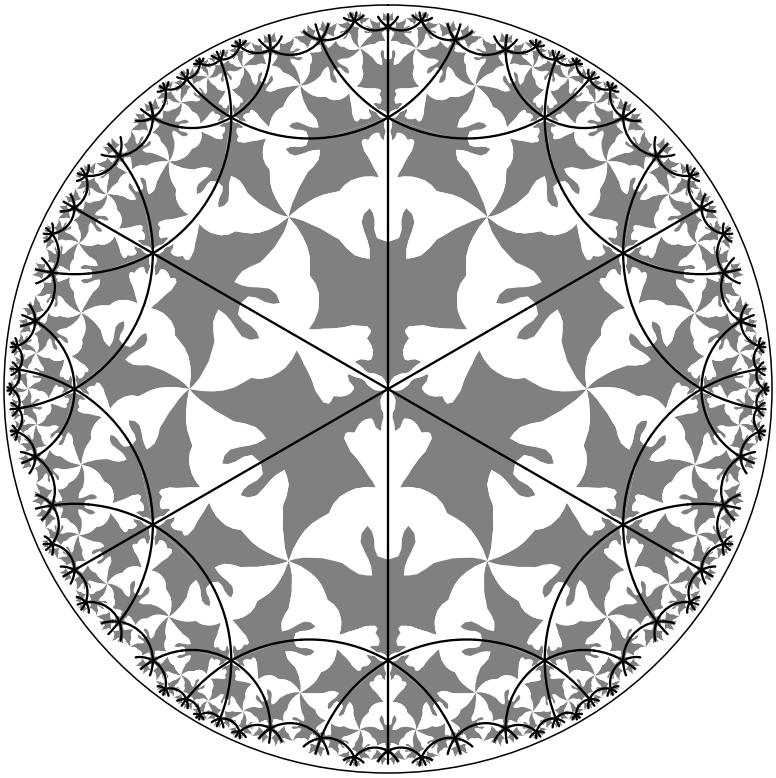
# Symmetry Groups of Angels and Devils Patterns

**The symmetry group of Notebook Drawing 45 is denoted [$4^+$, $4$] in H.S.M. Coxeter's notation, 4*2 in orbifold notation, and p4g in crystallographic notation.**

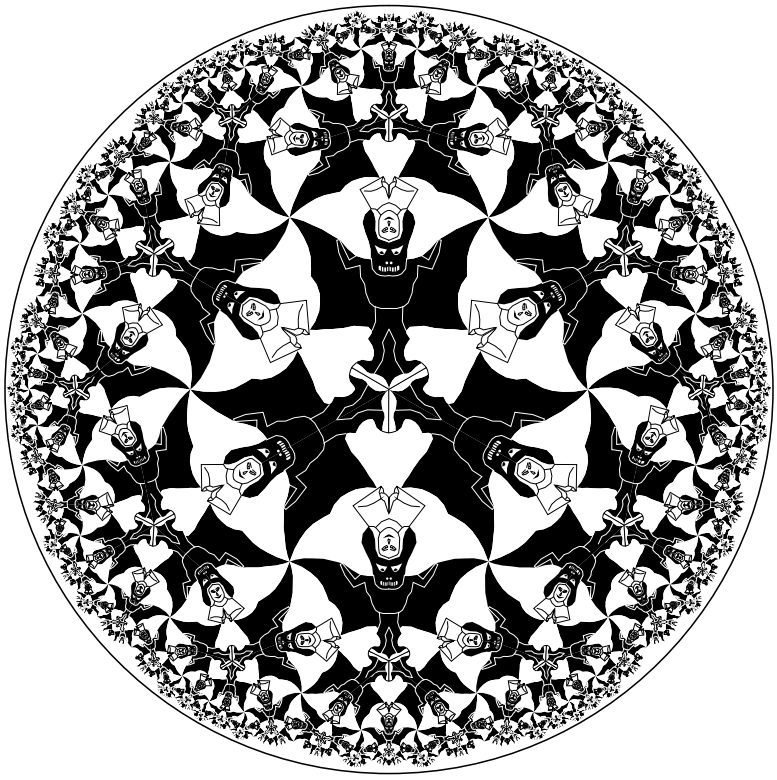
**If all the angels and devils were on the same sphere-level on the maple ball, the symmetry group would be [$3^+$, $4$] (Coxeter's notation) or 3*2 (orbifold notation).**

**If the details of all or none of the angels and devils were filled in for the *Circle Limit IV* pattern, the resulting pattern would have symmetry group [$4^+$, $6$] or 4*3.**

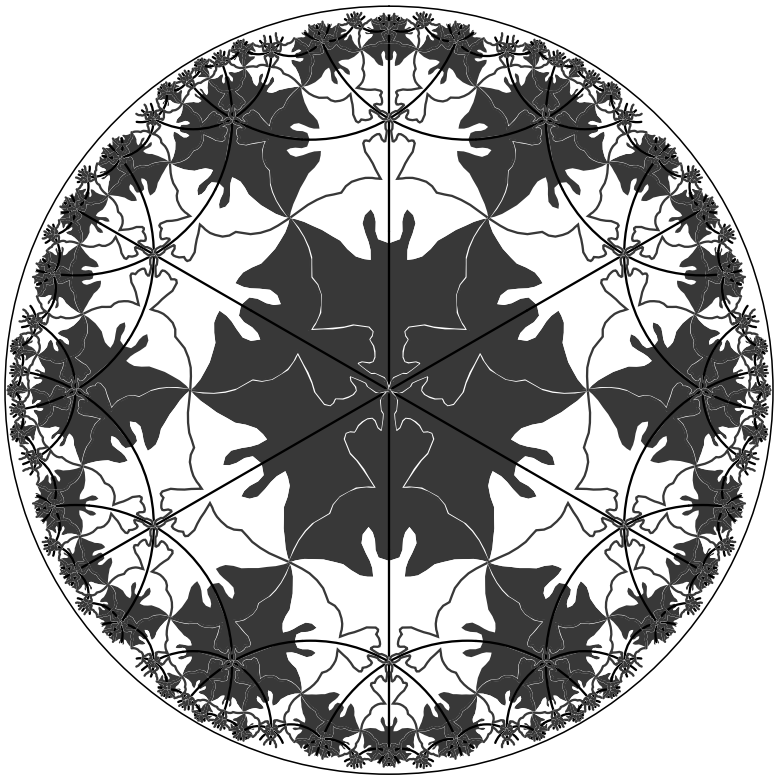
# A Pattern with Symmetry Group $[4^+, 6]$ or 4*3



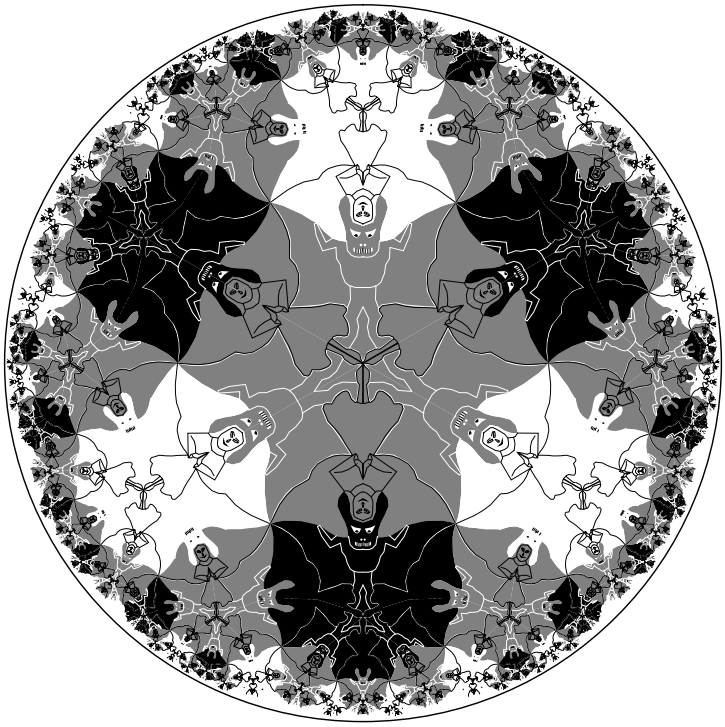
# Another Pattern with Symmetry Group $[4^+, 6]$ or 4*3



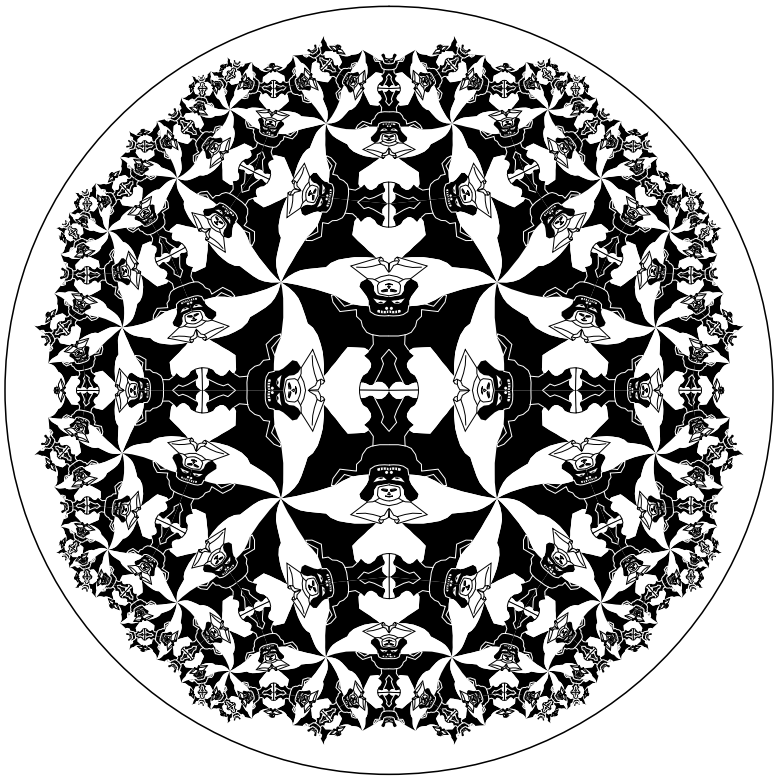
# A Pattern with Symmetry Group 2*33



# A Pattern with Symmetry Group *3333



# A Pattern with Symmetry Group [$5^+$, 4] or 5*2



# Future Work

- **Develop a general theory of “Circle Limit IV” patterns.**
- **Enhance the program to draw these general “Circle Limit IV” patterns.**
- **Use 3D printing technology to create a “Circle Limit IV” disk pattern with areas where angels are raised and devils are recessed, areas where the opposite is true, and intermediate areas where neither is dominant.**