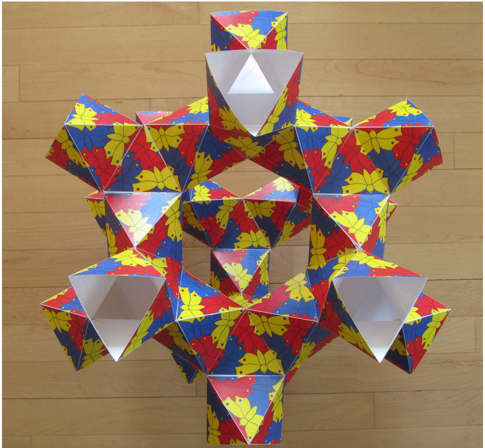
Bridges 2013
Enschede, Netherlands

Escher Patterns on Triply Periodic Polyhedra

Douglas Dunham
University of Minnesota Duluth
Duluth, Minnesota USA



Outline

- Some previously designed patterned (closed) polyhedra
- Triply periodic polyhedra
- Inspiration for this work
- Hyperbolic geometry and regular tessellations
- Relation between periodic polyhedra and regular tessellations
- A Pattern of angels and devils on a $\{4, 5\}$ polyhedron
- A Pattern of butterflies on a $\{3, 8\}$ polyhedron
- A Pattern of butterflies on another $\{3, 8\}$ polyhedron
- Future research

Previously Designed Patterned Polyhedra

- M.C. Escher (1898–1972) created at least 3 such polyhedra.
- In 1977 Doris Schattschneider and Wallace Walker placed Escher patterns on each of the Platonic solids and the cuboctahedron.
- Schattschneider and Walker also put Escher patterns on rotating rings of tetrahedra, which they called "kaleidocycles".
- In 1985 H.S.M. Coxeter showed how to place 18 Escher butterflies on a torus.

Triply Periodic Polyhedra

- A *triply periodic polyhedron* is a (non-closed) polyhedron that repeats in three different directions in Euclidean 3-space.
- We will consider the special case of *uniform* triply periodic polyhedra which have the same vertex figure at each vertex — i.e. there is a symmetry of the polyhedron that takes any vertex to any other vertex..
- We will further specialize to what we call *semiregular triply periodic polyhedra* — polyhedra composed of copies of a single p-sided regular polygon. We denote such polyhedra $\{p, q\}$ if q polygons meet at each vertex.
- We note that p and q do not uniquely specify such polyhedra — unlike that situation for regular tessellations.

Inspirations for this Work

- Two papers by Steve Luecking at ISAMA 2011:
  - Building a Sherk Surface from Paper Tiles
  - Sculpture From a Space Filling Saddle Pentahedron
- Bead sculptures that approximate three triply periodic minimal surfaces (TPMS) by Chern Chuang, Bih-Yaw Jin, and Wei-Chi Wei at the 2012 Joint Mathematics Meeting Art Exhibit.

  As we will see, some TPMS's are related to triply periodic polyhedra.

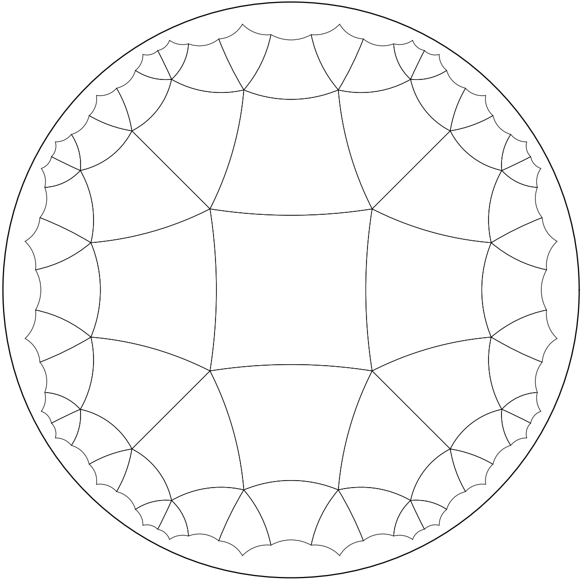
Hyperbolic Geometry and Regular Tessellations

- In 1901, David Hilbert proved that, unlike the sphere, there was no isometric (distance-preserving) embedding of the hyperbolic plane into ordinary Euclidean 3-space.
- Thus we must use *models* of hyperbolic geometry in which Euclidean objects have hyperbolic meaning, and which must distort distance.
- One such model is the *Poincaré disk model*. The hyperbolic points in this model are represented by interior point of a Euclidean circle — the *bounding circle*. The hyperbolic lines are represented by (internal) circular arcs that are perpendicular to the bounding circle (with diameters as special cases).
- This model is appealing to artests since (1) angles have their Euclidean measure (i.e. it is conformal), so that motifs of a repeating pattern retain their approximate shape as they get smaller toward the edge of the bounding circle, and (2) it can display an entire pattern in a finite area.

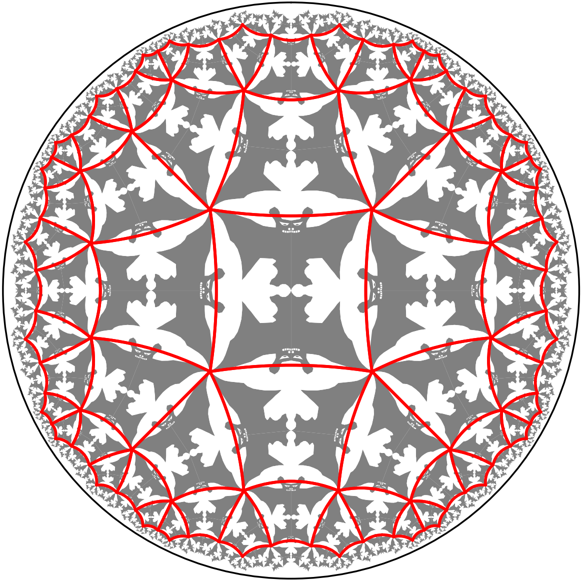
Repeating Patterns and Regular Tessellations

- A *repeating pattern* in any of the 3 "classical geometries" (Euclidean, spherical, and hyperbolic geometry) is composed of congruent copies of a basic subpattern or *motif*.
- The *regular tessellation*, $\{p, q\}$, is an important kind of repeating pattern composed of regular p-sided polygons meeting q at a vertex.
- If $(p-2)(q-2) < 4$, $\{p, q\}$ is a spherical tessellation (assuming $p > 2$ and $q > 2$ to avoid special cases).
- If $(p-2)(q-2) = 4$, $\{p, q\}$ is a Euclidean tessellation.
- If $(p-2)(q-2) > 4$, $\{p, q\}$ is a hyperbolic tessellation. The next slide shows the $\{4, 5\}$ tessellation.
- Escher based his 4 "Circle Limit" patterns, and many of his spherical and Euclidean patterns on regular tessellations.

The Regular Tessellation $\{4, 5\}$



The tessellation $\{4, 5\}$ superimposed on a pattern of angels and devils



Relation between periodic polyhedra and regular tessellations — a 2-Step Process

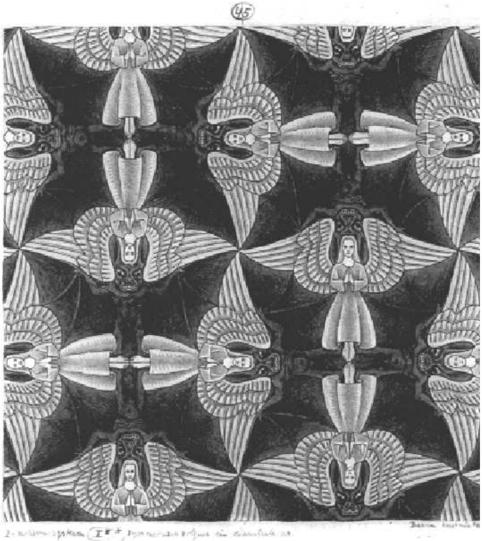
- (1) Some triply periodic polyhedra approximate TPMS's.
- (2) As a minimal surface, a TPMS has negative curvature (except for isolated points of zero curvature), and so its universal covering surface also has negative curvature and thus has the same large-scale geometry as the hyperbolic plane.

  So the polygons of the triply periodic polyhedron can be transferred to the polygons of a corresponding regular tessellation of the hyperbolic plane.
- We show this relationship when we analyze the patterned polyhedra below.
- Just as the regular hyperbolic tessellations are "universal covering tessellations" of the polyhedra, we can consider patterns based on those tessellations as *universal covering patterns* of the patterned polyhedra.

Escher's "Angels and Devils" Patterns

- Escher only realized one pattern, his "Angles and Devils" pattern in each of the three classical geometries:
- His Euclidean Regular Division Drawing # 45 (1941), which is based on the $\{4, 4\}$ tessellation.
- "Heaven and Hell" on a Maple Sphere (1942), based on the $\{4, 3\}$ spherical tessellation.
- His hyperbolic pattern "Circle Limit IV" (1960), based on the $\{6, 4\}$

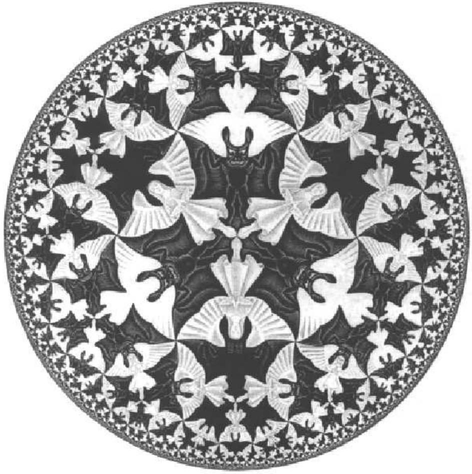
Regular Division Drawing # 45



“Heaven and Hell” on a Maple Sphere

“Heaven and Hell” carved sphere, 1942. Maple, stained in two colors, diameter 235 mm.

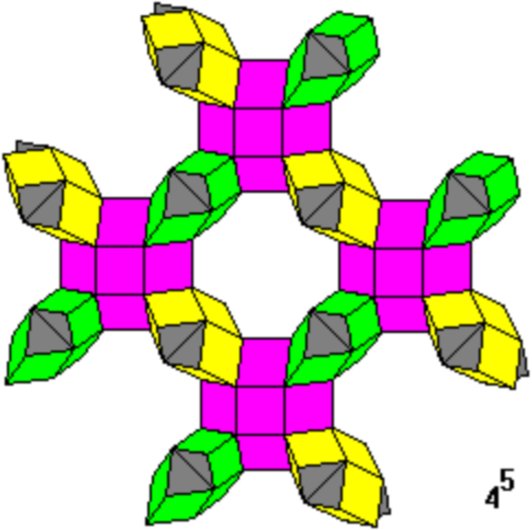
“Circle Limit IV”



Filling a "Gap" — a $\{4,5\}$ Pattern

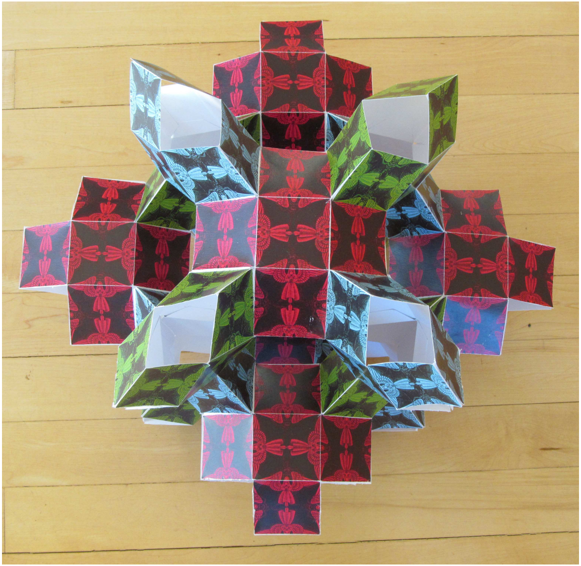
- We have seen the progression from Escher's carved spherical pattern based on the $\{4,3\}$ tessellation, to Notebook Pattern 45, based on the $\{4,4\}$ tessellaion, and *Circle Limit IV* based on the $\{6,4\}$ tessellaion.
- But Escher did not create an "angels and devils" pattern based on the $\{4,5\}$ tessellation.
- We fill that "gap" by displaying and angels and devils pattern on a $\{4,5\}$ polyhedron.

A $\{4,5\}$ Polyhedron

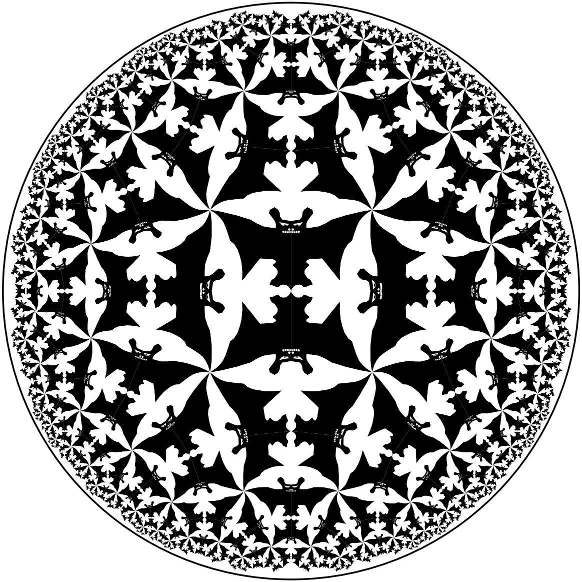


A "Construction Unit" for the $\{4,5\}$ Polyhedron

The $\{4, 5\}$ Polyhedron with Angels and Devils



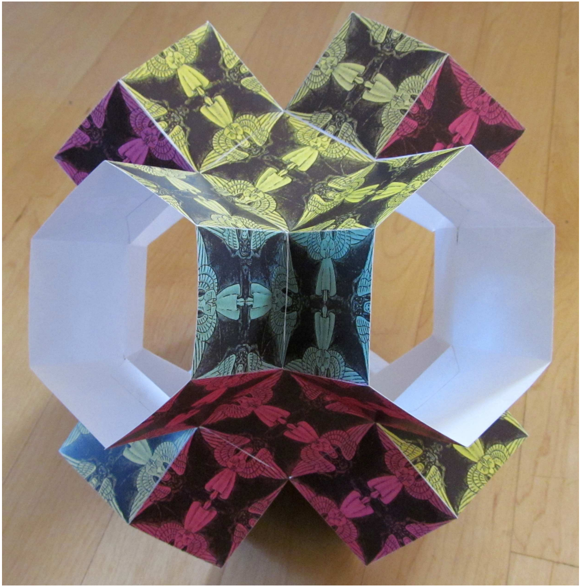
The Hyperbolic "Universal Covering Pattern"



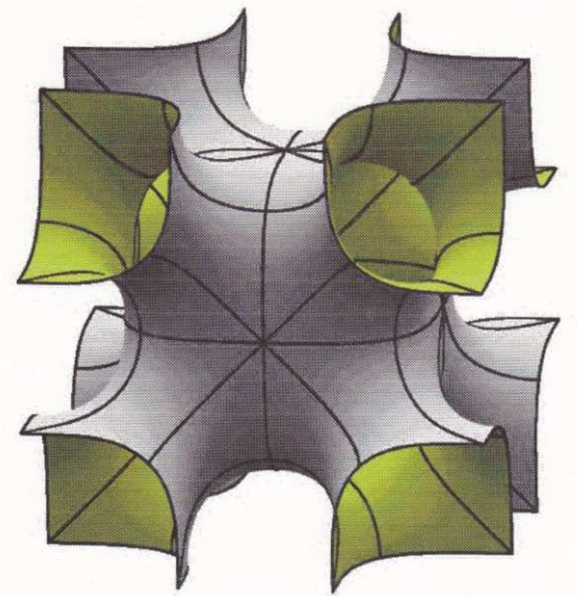
The “Complement” of the $\{4, 5\}$ Polyhedron

- The complement of the $\{4, 5\}$ polyhedron is perhaps easier to understand.
- It is composed of truncated octahedral “hubs” having their hexagonal faces connected by regular hexagonal prisms as “struts”.
- The next slide shows one of the hubs with the struts that are connected to it.

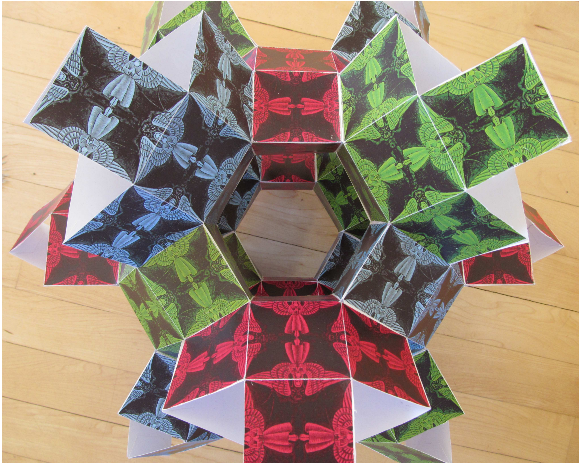
A "Hub" and 8 "struts" of the $\{4, 5\}$ Complement Polyhedron



A Piece of the TPMS IWP Surface Corresponding to the Hub



A View of the Original $\{4, 5\}$ Polyhedron Looking down a Hexagonal Hole

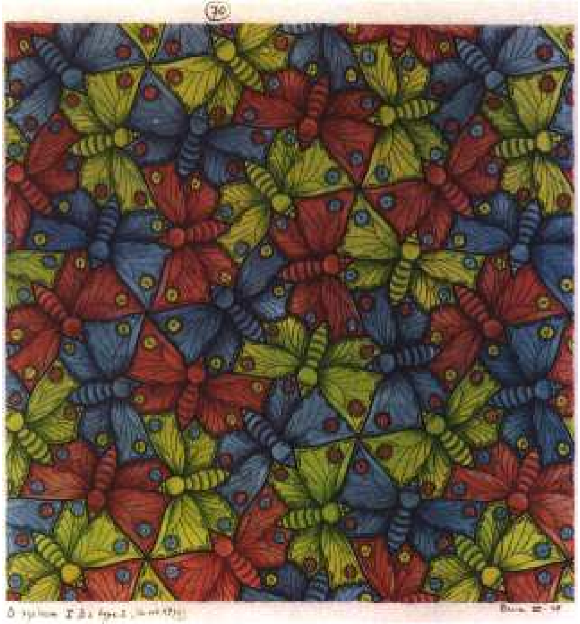


A Pattern of Butterflies on a $\{3, 8\}$ Polyhedron

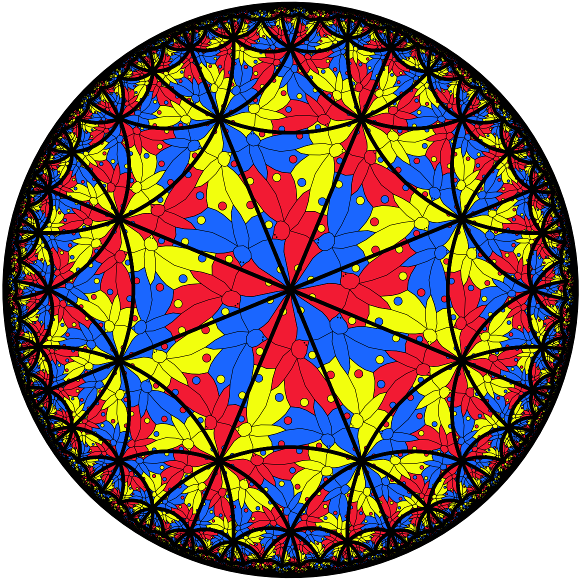
The butterfly pattern on the triply periodic $\{3, 8\}$ polyhedron of the title slide was inspired by Escher's Regular Division Drawing # 70. This polyhedron is related to Schwarz's D-surface, a TPMS with the topology of a thickened diamond lattice. We show:

- Escher's Regular Division Drawing # 70.
- A hyperbolic pattern of butterflies based on the $\{3, 8\}$ tessellation — the "universal covering pattern" of the patterned polyhedron.
- A construction unit of a $\{3, 8\}$ polyhedron consisting of a regular octahedral "hub" and four octahedral "struts" placed on alternate faces of the hubs.
- Part of Schwarz's D-surface corresponding to the construction unit.
- Another view of the patterned $\{3, 8\}$ polyhedron of the title slide down one of its "tunnels".
- A close-up of a vertex of the patterned polyhedron.

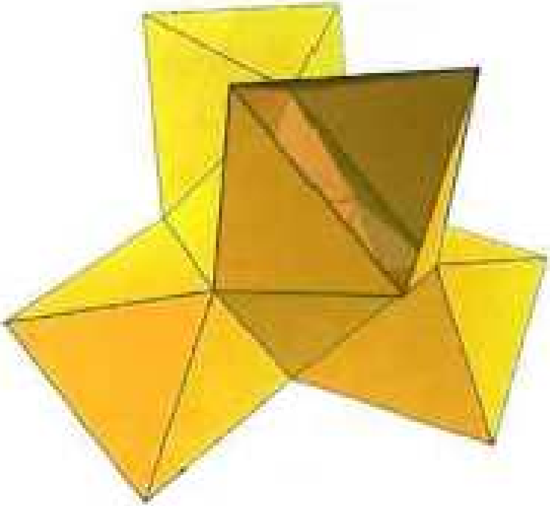
Escher's Regular Division Drawing # 70



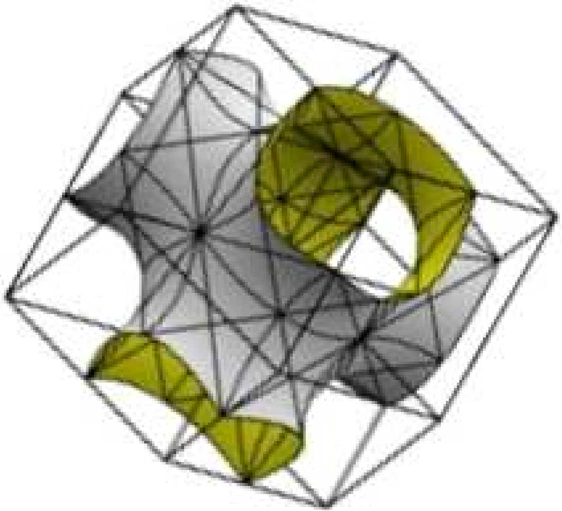
A pattern of butterflies based on the $\{3, 8\}$ tessellation — the "universal covering pattern" for the polyhedron.



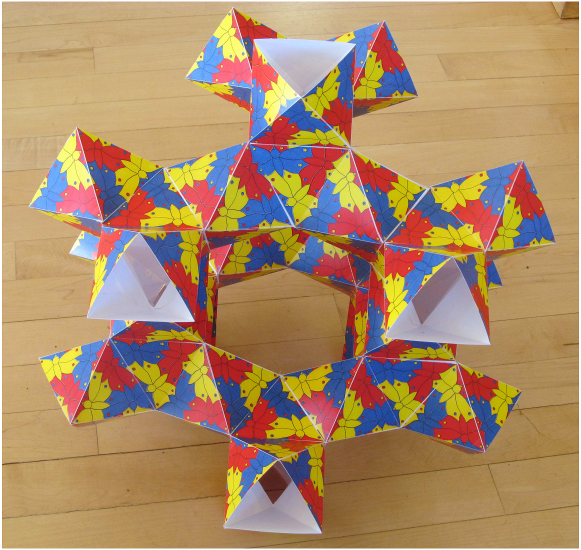
A “construction unit” of the triply periodic polyhedron



A corresponding piece of Schwarz's D-surface



A view down one of the "tunnels" of the $\{3, 8\}$ polyhedron.



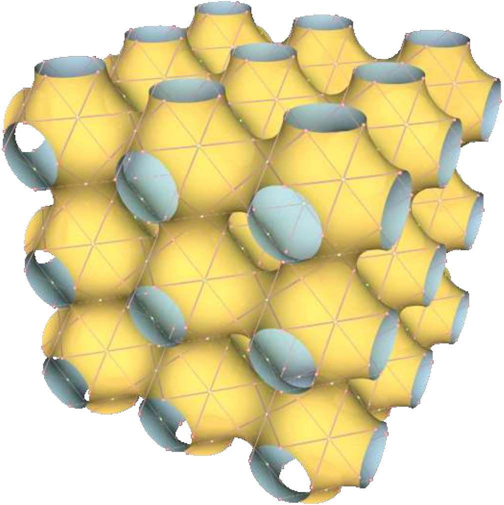
A close-up of a vertex of the patterned polyhedron.

Butterflies on Another $\{3, 8\}$ Polyhedron

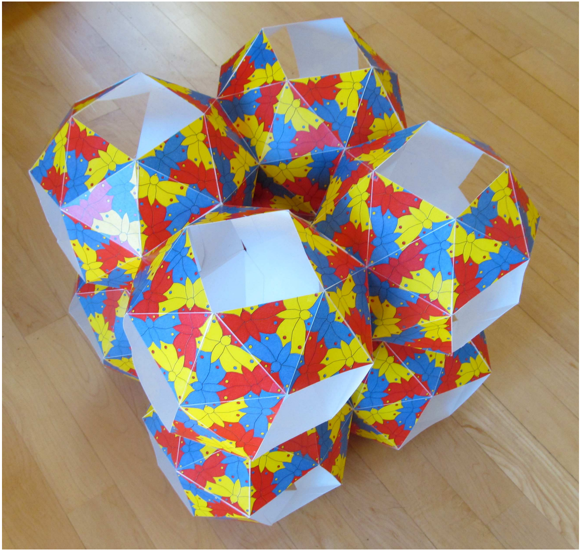
We show the pattern of butterflies on a different the triply periodic $\{3, 8\}$ polyhedron. This butterfly pattern was also inspired by Escher's Regular Division Drawing # 70. Thus the hyperbolic "covering pattern" is the same as for the previous polyhedron. This polyhedron has the same topology as Schwarz's P-surface, a TPMS with the topology of a thickened version of the 3-D coordinate lattice. We show:

- Schwarz's P-surface.
- The $\{3, 8\}$ polyhedron, which is made up of snub cubes arranged in a cubic lattice, attached by their (missing) square faces, and alternating between left-handed and right-handed versions.
- A close-up of the patterned polyhedron.
- A close-up of a vertex of the patterned polyhedron.

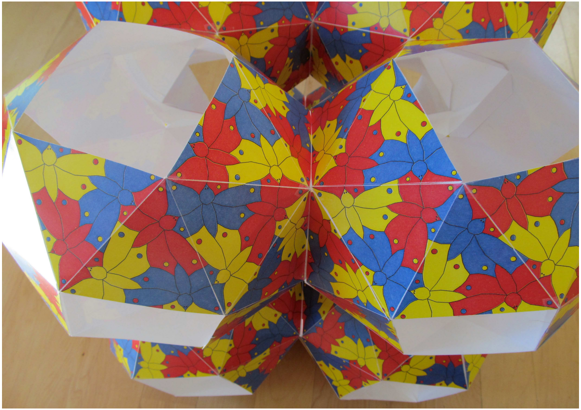
Schwarz's P-surface



Another Patterned $\{3, 8\}$ Polyhedron



A Close-up of the $\{3, 8\}$ Polyhedron



A close-up of a vertex of the patterned polyhedron.

Future Work

- Put other patterns on the triply periodic polyhedra shown above.
- Place patterns on other triply periodic polyhedra, with various regularity conditions.
- Place a butterfly pattern on a $\{3, 7\}$ polyhedron.
- Draw patterns on TPMS's — the gyroid, for example.

Thank You!

Reza and *all* the other Bridges organizers

Contact Information:
Doug Dunham
Email: `ddunham@d.umn.edu`
Web: `http://www.d.umn.edu/~ddunham`