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Fractal Wallpaper Patterns

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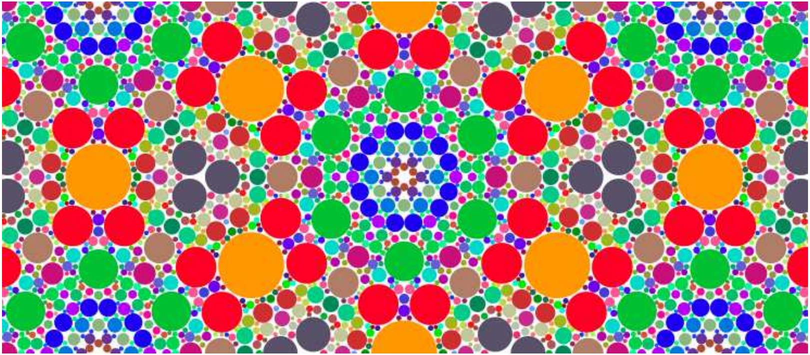


Outline

- Background and the "Area Rule"
- The algorithm
- A conjecture
- Patterns with symmetry $p1$ ($= o$ in orbifold notation)
- Patterns with symmetry $p2mm$ ($= *2222$ orbifold notation)
- Patterns with symmetry $p4mm$ ($= *442$ orbifold notation)
- Patterns with symmetry $p6mm$ ($= *632$ orbifold notation)
- A $p3m1$ ($= *333$) on a triply periodic polyhedron
- Future Work
- Contact information

Background

Our original goal was to create patterns by randomly filling a region R with successively smaller copies of a motif, creating a fractal pattern.

This goal can be achieved if the motifs follow an "area rule" which we describe in the next slide.

The resulting algorithm is quite robust in that it has been found to work for hundreds of patterns in (combinations of) the following situations:

- The region R is connected or not.
- The region R has holes — i.e. is not simply connected.
- The motif is not connected or simply connected.
- The motifs have multiple (even random) orientations.
- The pattern has multiple (even all different) motifs.
- If R is a rectangle, the pattern can be **periodic** — it can repeat horizontally and vertically, and thus tile the plane. The code is different and more complicated in this case.

The Area Rule

If we wish to fill a region R of area A with successively smaller copies of a motif (or motifs), it has been found experimentally that this can be done for $i = 0, 1, 2, \ldots$, with the area A_i of the i-th motif obeying an inverse power law:

$$A_i = \frac{A}{\zeta(c, N)(N + i)^c}$$

where where $c > 1$ and $N > 0$ are parameters, and $\zeta(c, N)$ is the Hurwitz zeta function: $\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(q+k)^s}$ (and thus $\sum_{k=0}^{\infty} A_i = A$).

We call this the **Area Rule**

The Algorithm

The algorithm works by successively placing copies m_i of the motif at locations inside the bounding region R.

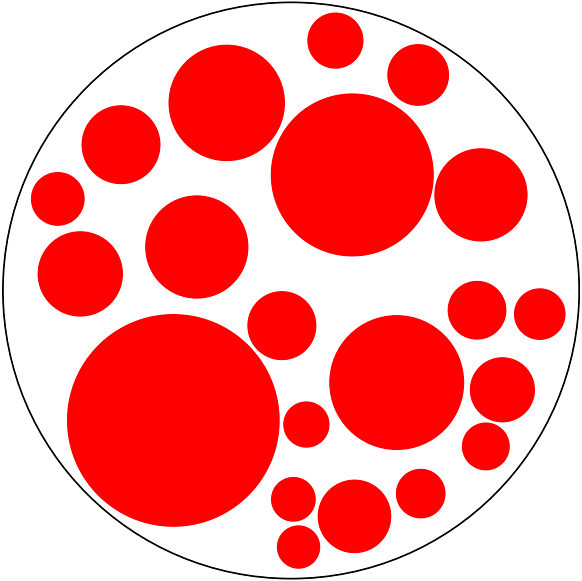
This is done by repeatedly picking a random **trial** location (x, y) inside R until the motif m_i placed at that location doesn't intersect any previously placed motifs.

We call such a successful location a **placement**. We store that location in an array so that we can find successful locations for subsequent motifs.

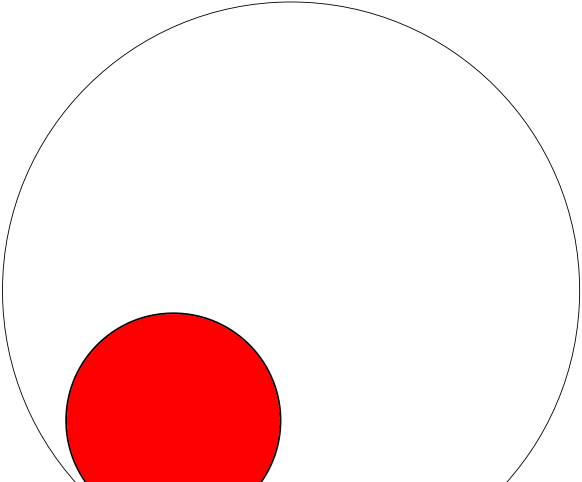
We show an example of how this works in the following slides.

A pattern of 21 circles partly filling a circle

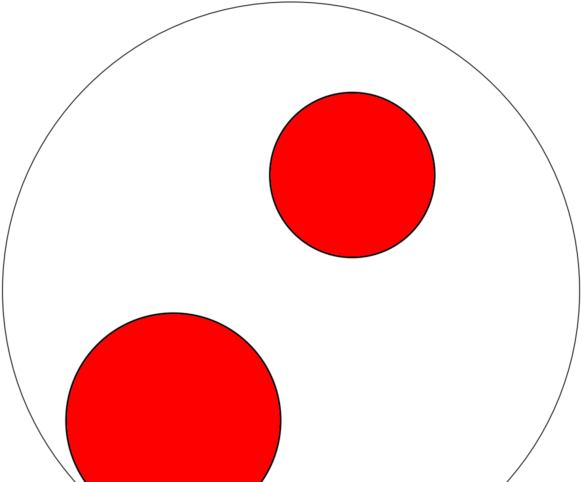
(Note: $c = 1.30$ and $N = 2$ in this example)



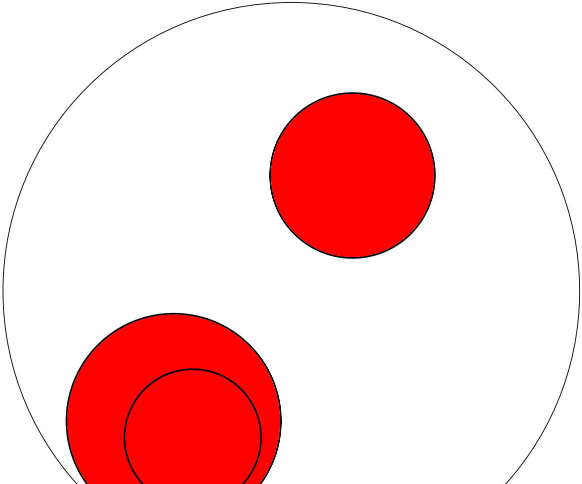
Placement of the first motif



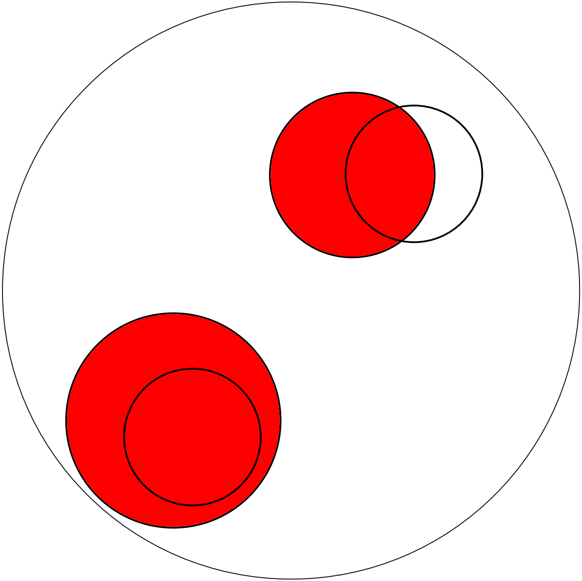
Placement of the second motif



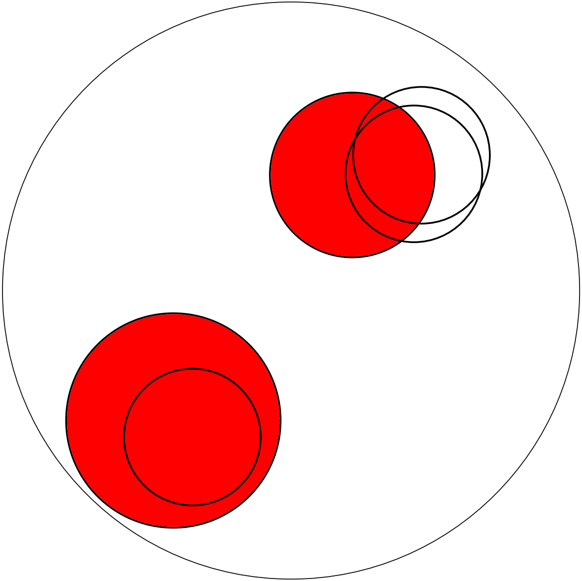
First trial for the third motif



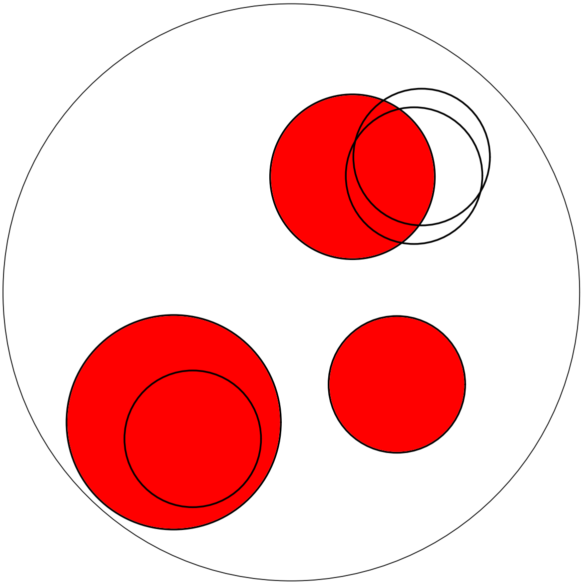
Second trial for the third motif



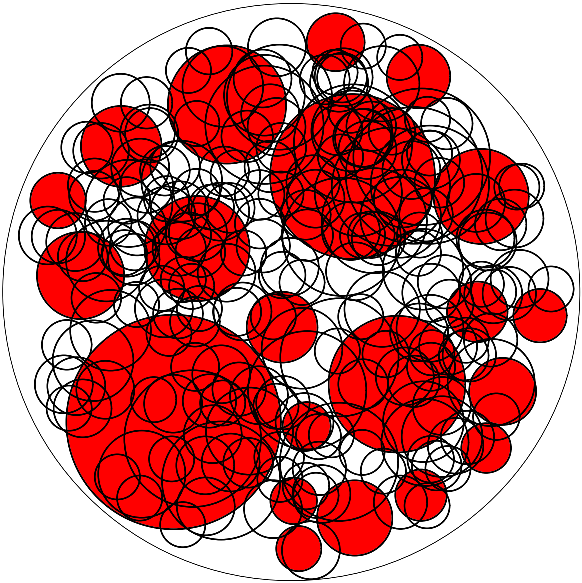
Third trial for the third motif



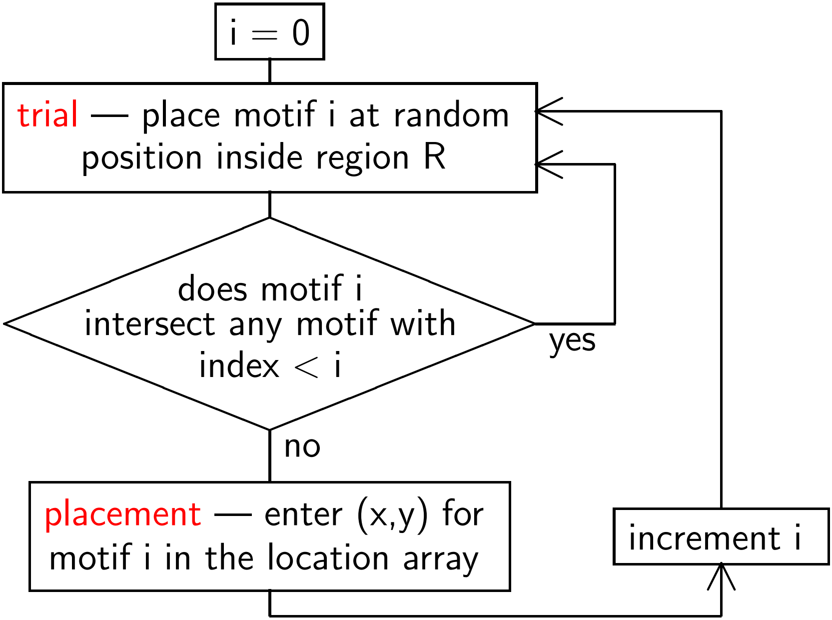
Successful placement of the third motif



All 245 trials for placement of the 21 circles



A Flowchart for the Algorithm



A Conjecture

Conjecture: The algorithm will randomly fill *any* reasonably defined region R with *any* reasonably defined motif(s), and it will not halt for $1 < c < c\_max$ and $N > N\_min > 0$, for appropriate values of c_max and N_min (which depend on the shapes of R and the motifs).

Typically values of c_max seem to be somewhat less than 1.5; often the values of N that were used were 2 or greater (not necessarily integer).

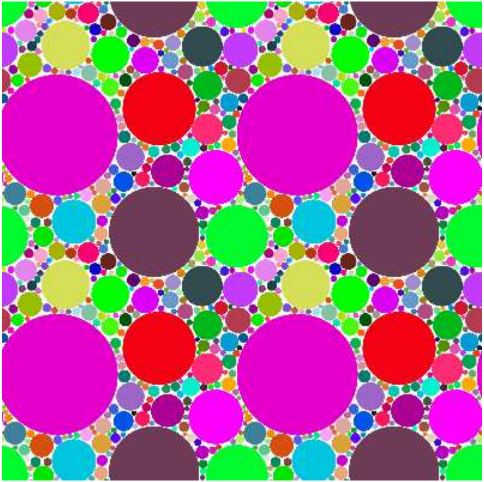
This algorithm has been implemented in dimensions 1, 2, 3, and 4, though we note that 1D patterns are not very interesting, and the "front" motifs in 3D and 4D obscure the motifs behind them.

In 1D, in which the motifs are line segments, it has been proved that the algorithm never halts for any c with $1 < c < 2$.

Also, the fractal dimensions of the patterns (not the unused portion of R) can be calculated to be $1/c, 2/c$, and $3/c$ in the 1D, 2D, and 3D cases respectively, which leads to the conjecture that the fractal dimension is d/c in d-dimensional space.

Repeating Wallpaper Patterns with $p1$ Symmetry

For patterns with p1 symmetry, we relax the rule that the motif is within the rectangular region R.

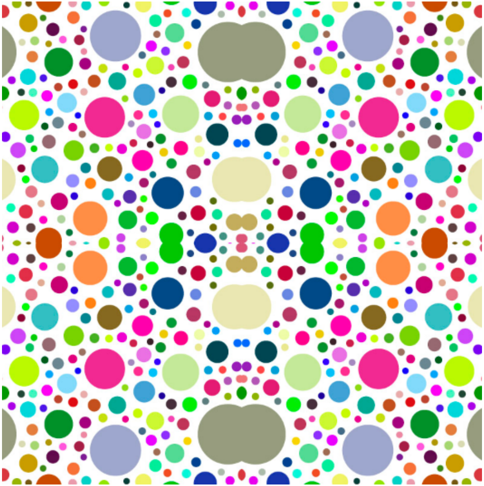


A pattern of peppers with p1 symmetry.

Patterns with $p2mm$ ($= *2222$) Symmetry

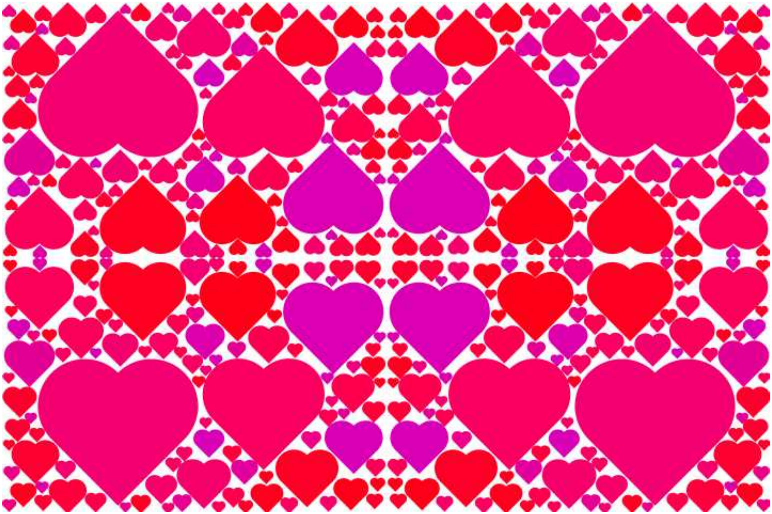
Problem: What if a motif falls on a mirror boundary of R?

Cure 1: Leave it there — produces "fused" motifs.



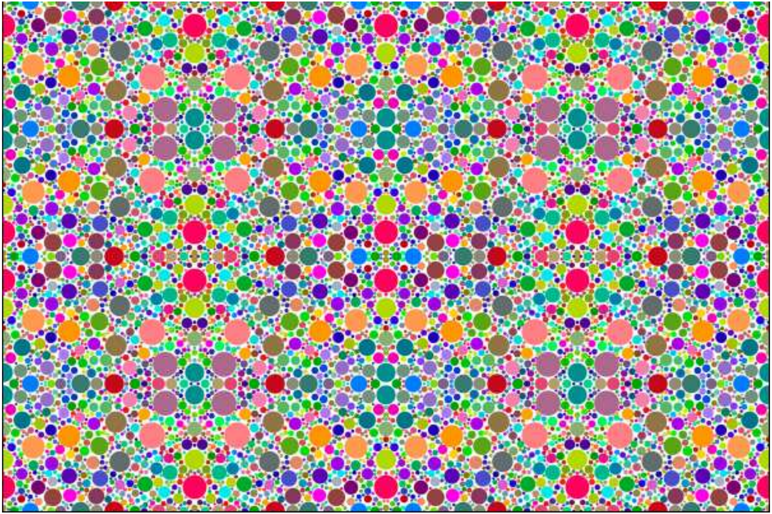
A pattern of hearts with p2mm symmetry.

Cure 2: Avoid mirror boundaries.



A pattern of circles with p2mm symmetry.

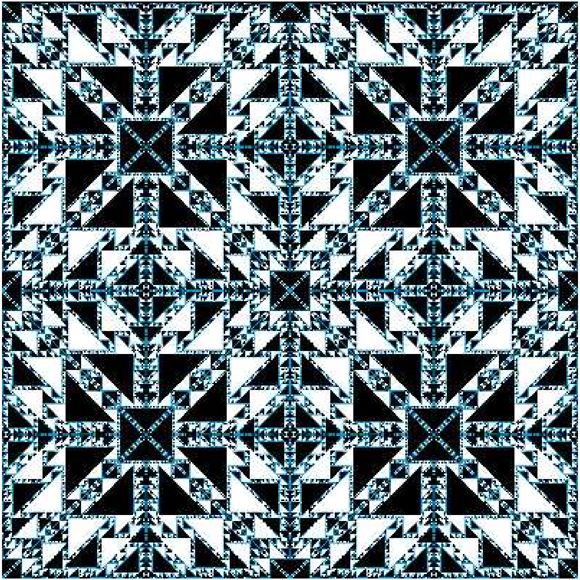
Cure 3: Center motifs with mirror symmetry on the boundary.



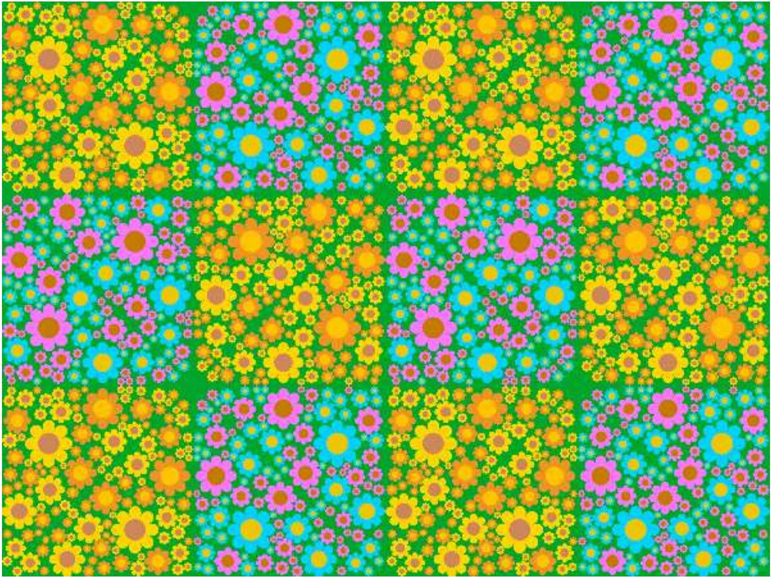
Patterns with $p4mm$ ($= *442$) Symmetry

A "Rorschach" pattern with $p4mm$ symmetry.

A pattern of black & white triangles with p4mm symmetry.

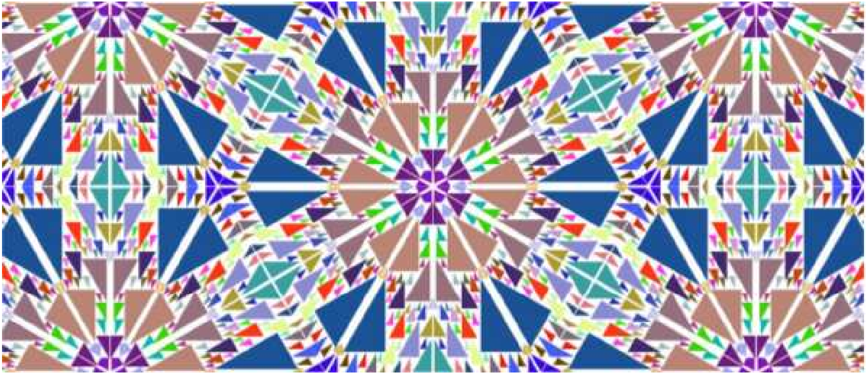


A flower pattern with p4mm color symmetry.

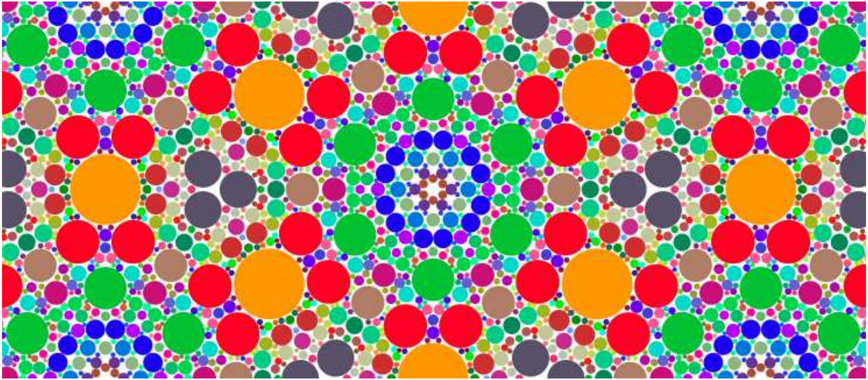


Patterns with $p6mm$ ($= *632$) Symmetry

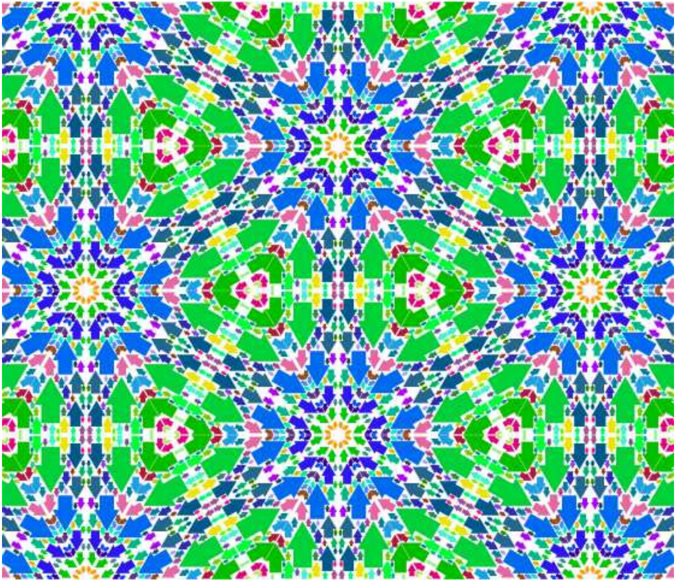
A triangle pattern with $p6mm$ symmetry avoiding mirror lines.



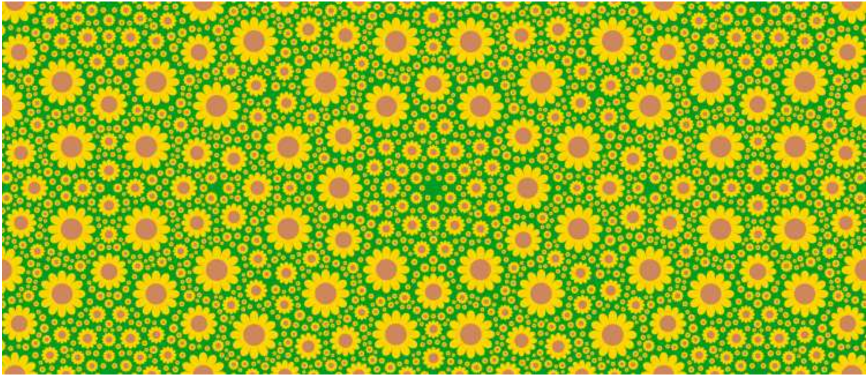
The title slide with circles centered on mirrors & p6mm symmetry.



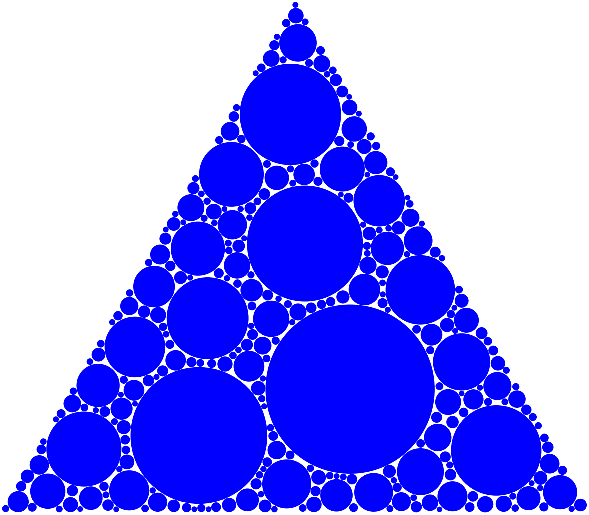
A p6mm arrow pattern that avoids the mirror lines.



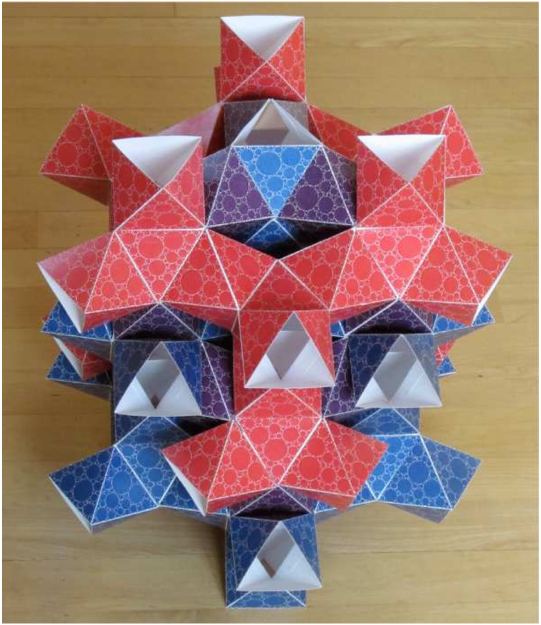
A p6mm flower pattern with some flowers on mirror lines.



A Motif for a $p3m1$ ($= *333$) Pattern



A triply peridic polyhedron using triangles of the previous slide.



Future Work

- Our work so far has only produced four of the 17 kinds of wallpaper patterns. In the future we hope to be able to create patterns with all such symmetries.
- The patterns shown here can tile the Euclidean plane, but it seems possible that similar techniques could be used to tile the sphere or the hyperbolic plane.
- A small number of our patterns have color symmetry, but it would seem that there is much more to explore in this area.
- Finally, there are a few things that can be proved mathematically about these patterns, but there are a number of conjectures that have yet to be proved — such as the non-halting of the 2D algorithm for different motifs and reasonable values of c and N.

Acknowledgements and Contact

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