Repeating Fractal Patterns with 4-Fold Symmetry

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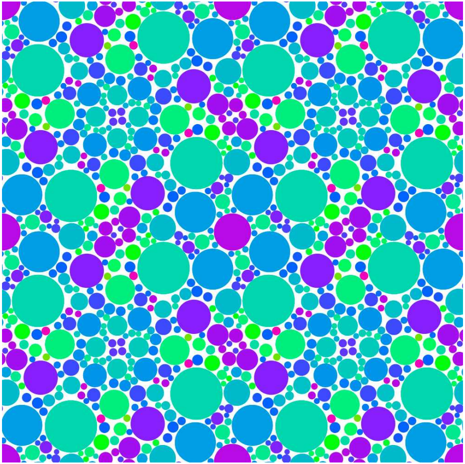


Outline

- Background and the “Area Rule”
- The basic algorithm
- Wallpaper patterns
- Another p4 pattern
- Patterns with p6 symmetry
- Conclusions and future work
- Contact information

Background

Our original goal was to create patterns by randomly filling a region R with successively smaller copies of a motif, creating a fractal pattern.

This goal can be achieved if the motifs follow an "area rule" which we describe in the next slide.

The resulting algorithm is quite robust in that it has been found to work for hundreds of patterns in (combinations of) the following situations:

- The region R is connected or not.
- The region R has holes — i.e. is not simply connected.
- The motif is not connected or simply connected.
- The motifs have multiple (even random) orientations.
- The pattern has multiple (even all different) motifs.
- If R is the fundamental region for one of the 17 plane crystallographic (or "wallpaper") groups, that region can be replicated using isometries from the group to tile the plane. The code is different and more complicated in this case.

The Area Rule

If we wish to fill a region R of area A with successively smaller copies of a motif (or motifs), it has been found experimentally that this can be done for $i = 0, 1, 2, \ldots$, with the area A_i of the i-th motif obeying an inverse power law:

$$A_i = \frac{A}{\zeta(c, N)(N + i)^c}$$

where where $c > 1$ and $N > 0$ are parameters, and $\zeta(c, N)$ is the Hurwitz zeta function: $\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(q+k)^s}$ (and thus $\sum_{k=0}^{\infty} A_i = A$).

We call this the **Area Rule**

The Basic Algorithm

The algorithm works by successively placing copies m_i of the motif at locations inside the bounding region R.

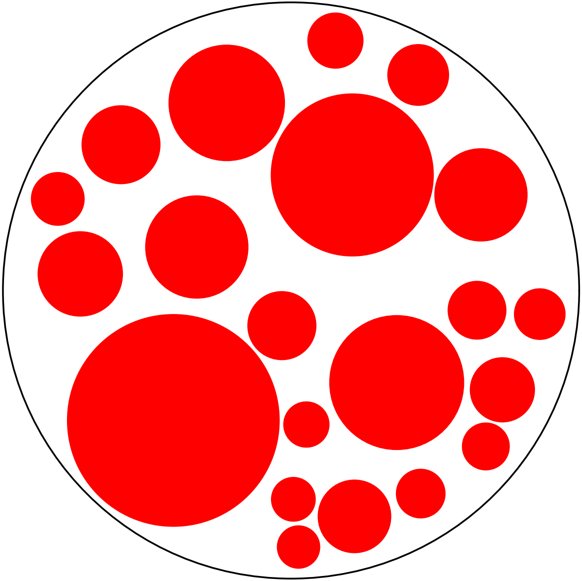
This is done by repeatedly picking a random **trial** location (x, y) inside R until the motif m_i placed at that location doesn't intersect any previously placed motifs.

We call such a successful location a **placement**. We store that location in an array so that we can find successful locations for subsequent motifs.

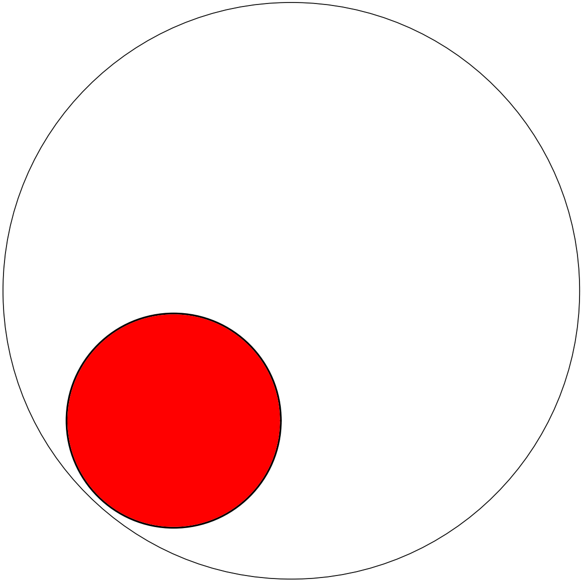
We show an example of how this works in the following slides.

A pattern of 21 circles partly filling a circle

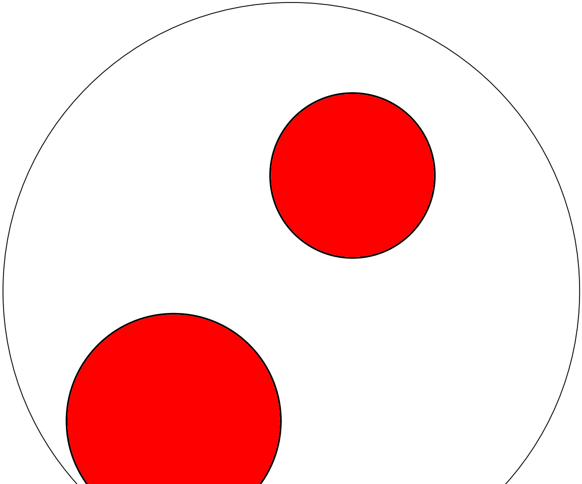
(Note: $c = 1.30$ and $N = 2$ in this example)



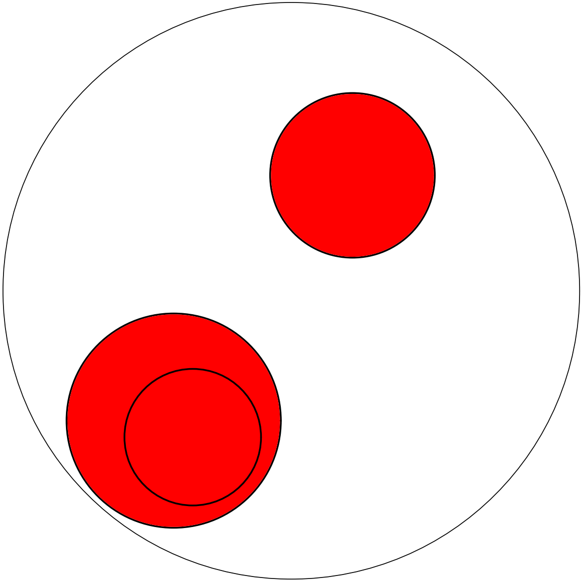
Placement of the first motif



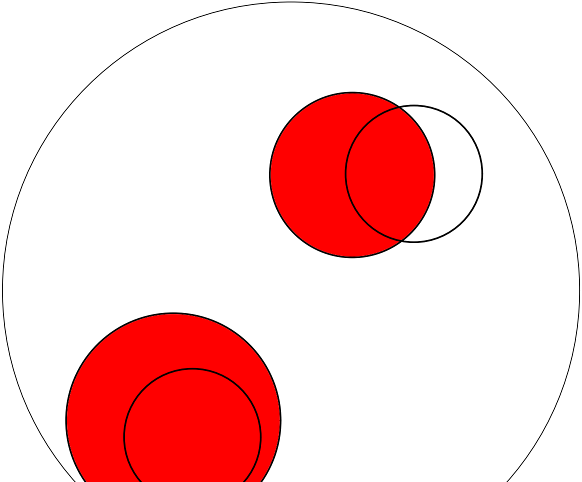
Placement of the second motif



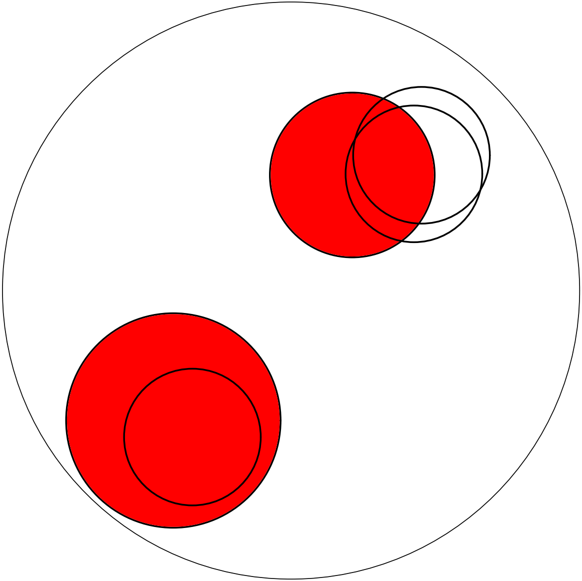
First trial for the third motif



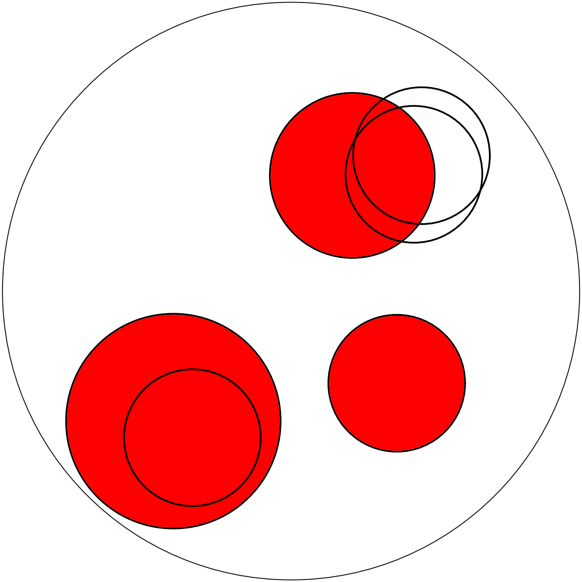
Second trial for the third motif



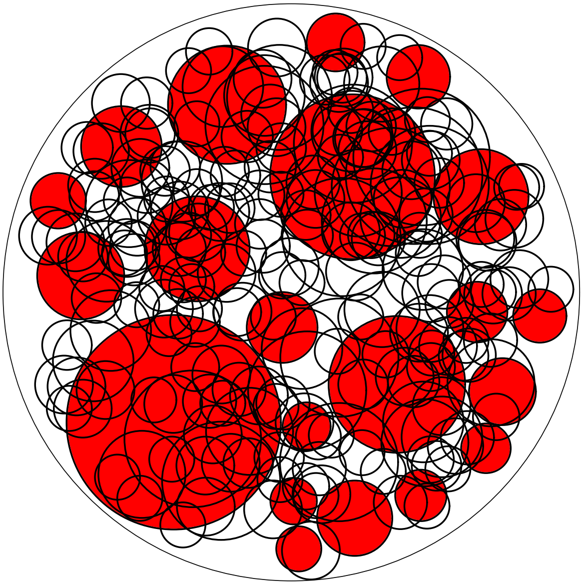
Third trial for the third motif



Successful placement of the third motif



All 245 trials for placement of the 21 circles



The Basic Algorithm

For each $i = 0, 1, 2, \ldots$

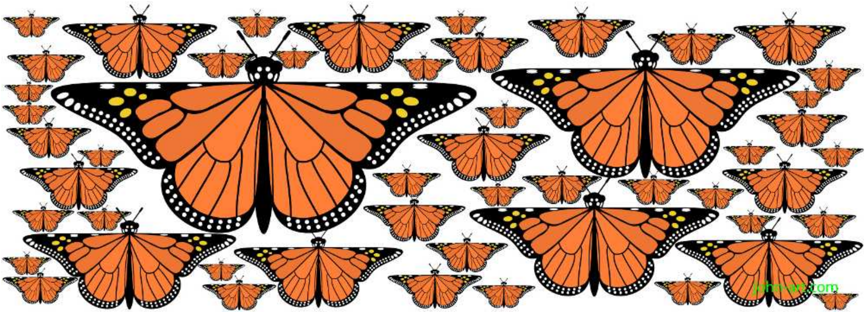
Repeat:

Randomly choose a point within R to place the i-th motif m_i.

Until (m_i doesn't intersect any of $m_0, m_1, ..., m_{i-1}$)
Add m_i to the list of successful placements

Until some stopping condition is met, such as a maximum value of *i* or a minimum value of A_i.

A Fractal Area-filling Pattern of Monarch Butterflies



Wallpaper Patterns

It has been know for more than 100 years that there are 17 different plane *crystallographic groups* — symmetry groups for patterns in the Euclidean plane that repeat in two independent directions.

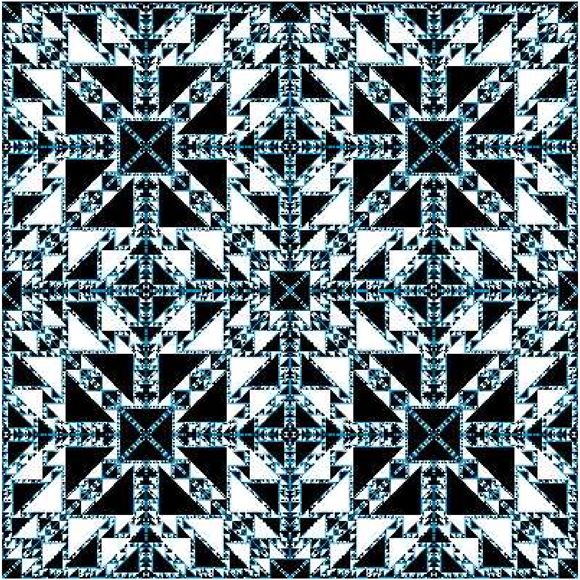
These groups are also called *wallpaper groups* and the corresponding patterns are called *wallpaper patterns*.

In 1952 the International Union of Crystallography (ICU) established a notation for these groups. A commonly used shorthand followed.

Later, John Conway popularized the more general *orbifold* notation.

In this paper we focus on the groups $p4$ and $p6$

A *p4mm* Pattern of Black and White Triangles



Rotation Patterns

There are four wallpaper groups generated entirely by rotations: $p2, p3, p4$, and $p6$. We just show examples of patterns with $p4$ and $p6$ symmetry. The title slide shows a $p4$ pattern.

The issue that arises here is what to do if a motif overlaps a center of rotation in the fundamental region.

In the case of a p-fold rotation center, if the motif also has (at least) p-fold rotational symmetry we align the motif with that rotation center.

Also since only part of the motif is within the fundamental region, we need to make an adjustment to the area rule.

In the next slide we show the modified algorithm for $p4$. A similar modification applies to the $p6$ algorithm.

The Modified Algorithm for p4

For each $i = 0, 1, 2, \ldots$

Repeat:

Randomly choose a point within R to place the i-th motif m_i.
If $\mathbf{m_i}$ has 4-fold symmetry and overlaps a 4-fold rotation point

Move $\mathbf{m_i}$ to be centered on that 4-fold rotation point

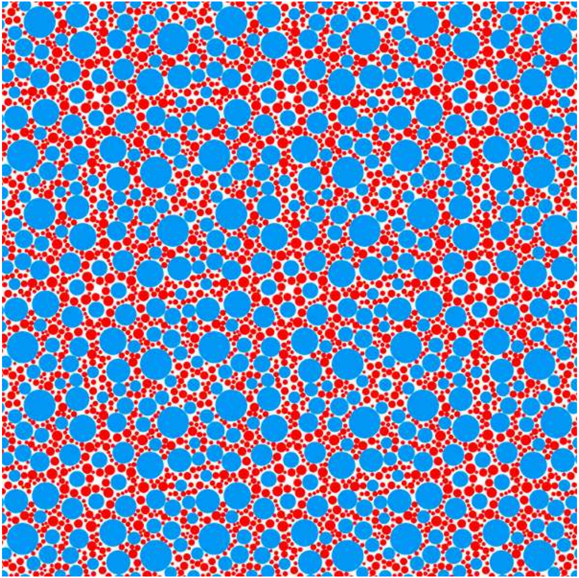
If $\mathbf{m_i}$ has at least 2-fold symmetry and overlaps a 2-fold rotation point

Move $\mathbf{m_i}$ to be centered on that 2-fold rotation point

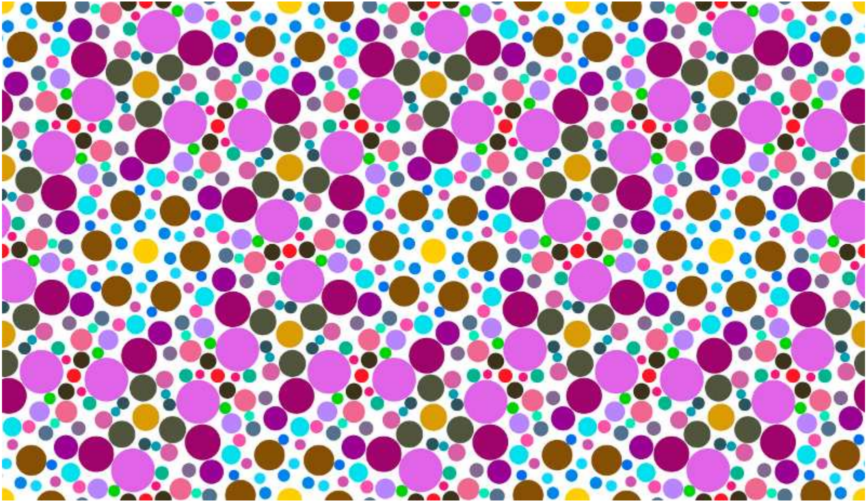
Until (m_i doesn't intersect any of $m_0, m_1, ..., m_{i-1}$)
Add m_i to the list of successful placements

Until some stopping condition is met, such as a maximum value of i or a minimum value of A_i.

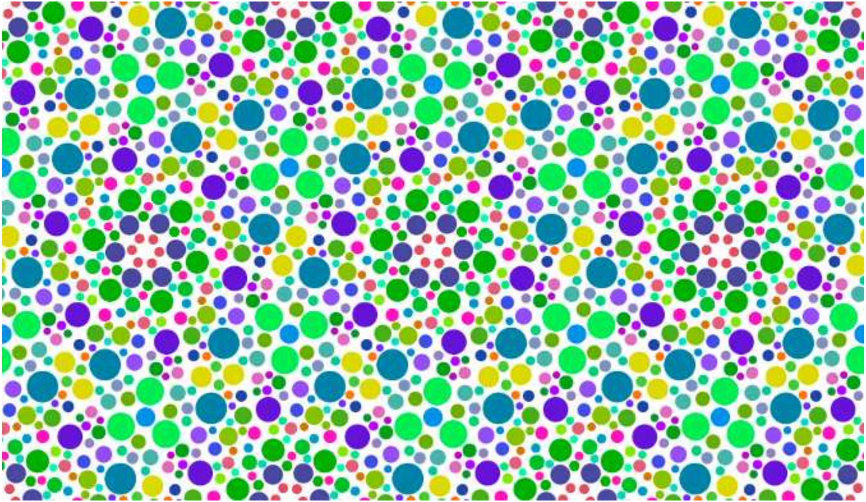
Another p4 Pattern of Circles with some on 4-fold Rotation Centers



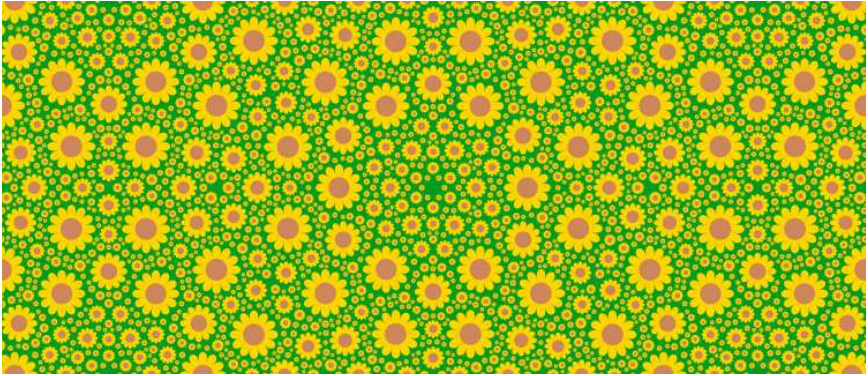
A p6 Pattern of Circles with some on each Rotation Center



Another p6 Pattern of Circles with some on 3-fold Centers



A flower pattern with $p6mm$ symmetrey



Future Work

- We have shown fractal patterns with $p4$ and $p6$ as their symmetry groups, but we haven't implemented algorithms for all the wallpaper groups. It would seem possible to create locally fractal patterns having the global symmetries of the other plane symmetry groups using our techniques.
- It would also seem possible to generate patterns of the sphere and hyperbolic plane that are locally fractal, but are globally symmetric.

Acknowledgements and Contact

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