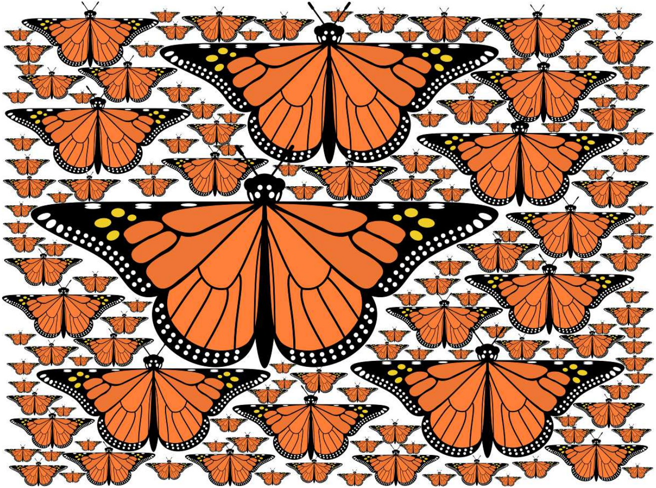
New Kinds of Fractal Patterns

Douglas Dunham
Dept. of Computer Science
Univ. of Minnesota, Duluth
Duluth, MN 55812, USA

John Shier
6935 133rd Court
Apple Valley, MN 55124 USA



Outline

- Background and the “Area Rule”
- The basic algorithm
- Wallpaper patterns
- Patterns with p6 symmetry
- Patterns with varying orientations
- A pattern with a complicated motif
- Conclusions and future work
- Contact information

Background

Our goal is to create patterns by randomly filling a **region** R with successively smaller copies of a **motif**, thus creating a fractal pattern.

This goal can be achieved if the motifs follow an "area rule" which we describe in the next slide.

The resulting algorithm is quite robust in that it has been found to work for hundreds of patterns in (combinations of) the following situations:

- The region R is connected or not.
- The region R has holes — i.e. is not simply connected.
- The motif is not connected or simply connected.
- The motifs have different (even random) orientations.
- The pattern has multiple (even all different) motifs.
- If R is the fundamental region for one of the 17 plane crystallographic (or "wallpaper") groups, that region can be replicated using isometries from the group to tile the plane. The code is different and more complicated in this case.

The Area Rule

If we wish to fill a region R of area A with successively smaller copies of a motif (or motifs), it has been found experimentally that this can be done for $i = 0, 1, 2, \ldots$, with the area A_i of the i-th motif obeying an inverse power law:

$$A_i = \frac{A}{\zeta(c, N)(N + i)^c}$$

where where $c > 1$ and $N > 0$ are parameters, and $\zeta(c, N)$ is the Hurwitz zeta function: $\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(q+k)^s}$ (and thus $\sum_{k=0}^{\infty} A_i = A$).

We call this the **Area Rule**

From this Area Rule, one can compute the **fractal dimension** of the pattern to be $2/c$

The Basic Algorithm

The algorithm works by successively placing copies m_i of the motif at locations inside the bounding region R.

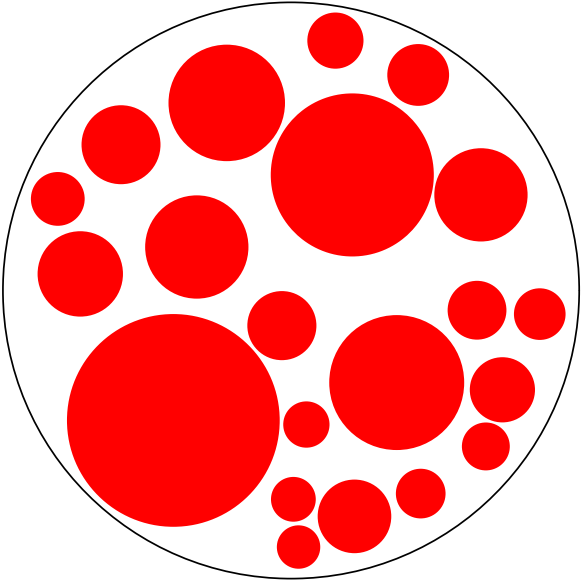
This is done by repeatedly picking a random **trial** location (x, y) inside R until the motif m_i placed at that location doesn't intersect any previously placed motifs.

We call such a successful location a **placement**. We store that location in an array so that we can find successful locations for subsequent motifs.

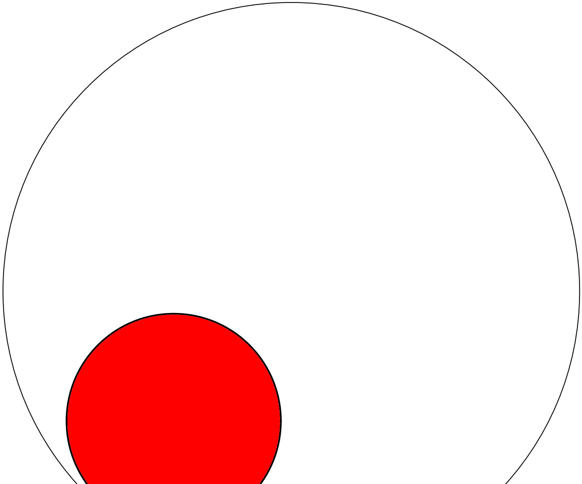
We show an example of how this works in the following slides.

A pattern of 21 circles partly filling a circle

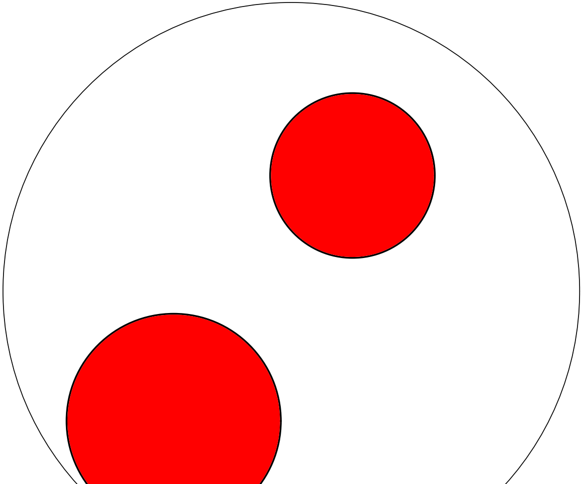
(Note: $c = 1.30$ and $N = 2$ in this example)



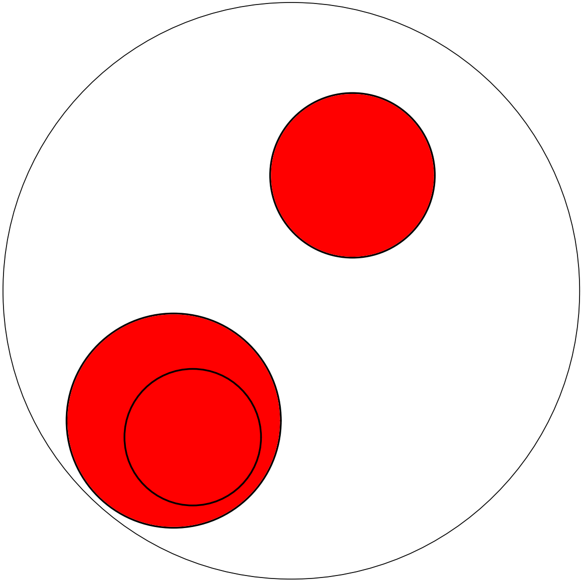
Placement of the first motif



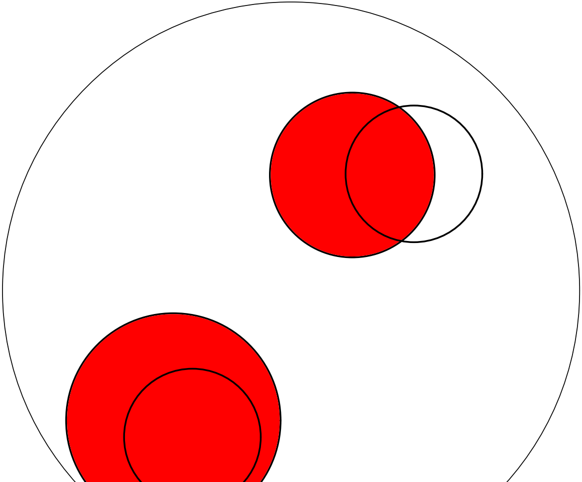
Placement of the second motif



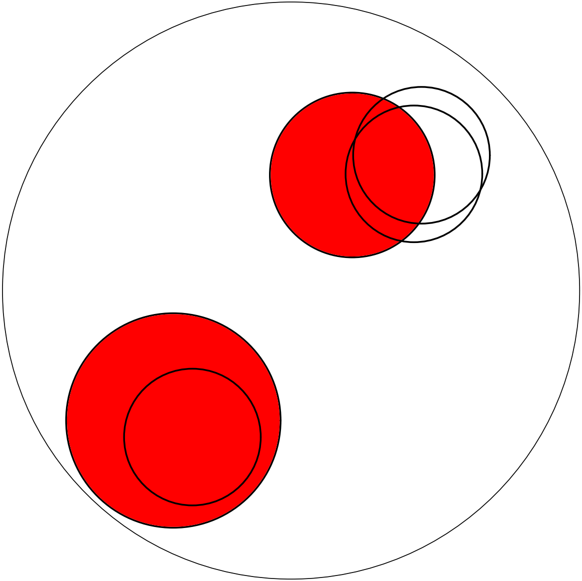
First trial for the third motif



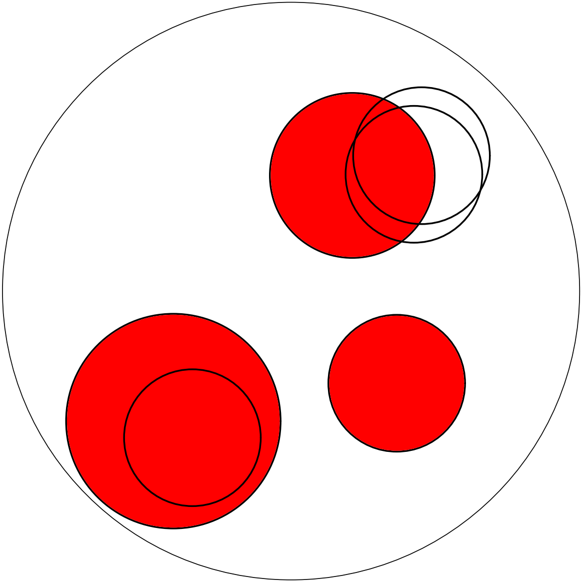
Second trial for the third motif



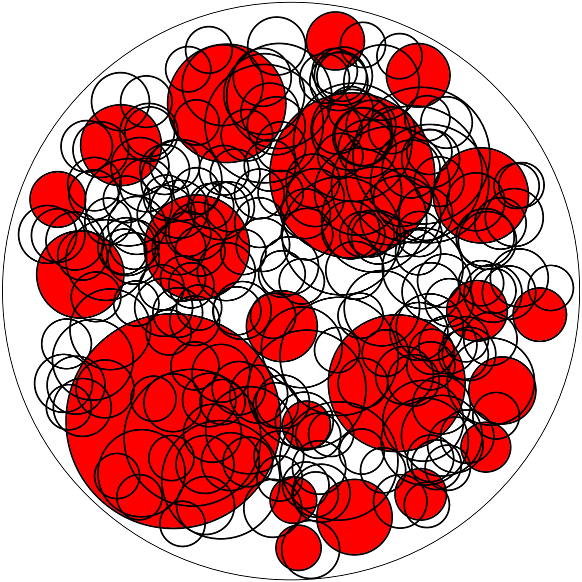
Third trial for the third motif



Successful placement of the third motif



All 245 trials for placement of the 21 circles



The Basic Algorithm

For each $i = 0, 1, 2, \ldots$

Repeat:

Randomly choose a point within R to place the i-th motif m_i.

Until (m_i doesn't intersect any of $m_0, m_1, ..., m_{i-1}$)
Add m_i to the list of successful placements

Until some stopping condition is met, such as a maximum value of *i* or a minimum value of A_i.

A Fractal Area-filling Pattern of Peppers

Wallpaper Patterns with p6 Symmetry

It has been know for more than 100 years that there are 17 different plane *crystallographic groups* — symmetry groups for patterns in the Euclidean plane that repeat in two independent directions.

These groups are also called *wallpaper groups* and the corresponding patterns are called *wallpaper patterns*.

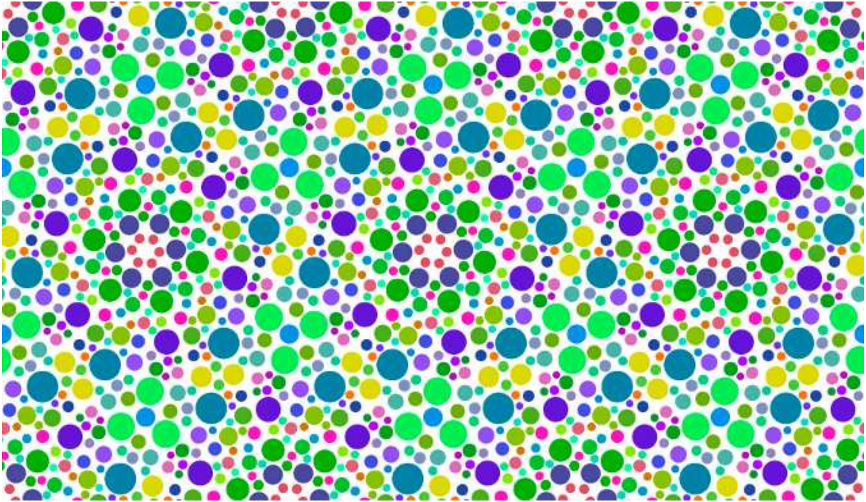
In 1952 the International Union of Crystallography (ICU) established a notation for these groups. A commonly used shorthand followed.

Later, John Conway popularized the more general *orbifold* notation.

In the past we have created patterns with symmetry groups $p1$, $p2mm, p4mm, p6mm$, and $p4$.

In this talk we treat patterns with $p6$ symmetry.

A p6 Pattern of Circles with some on 3-fold Centers



Rotation Patterns

There are four wallpaper groups generated entirely by rotations: $p2, p3, p4$, and $p6$. As mentioned above, we only show samples of patterns with $p6$ symmetry. The previous slide shows such a pattern.

An issue that arises here is what to do if a motif overlaps a center of rotation in the fundamental region. We could just discard that trial.

Alternatively, in the case of a k-fold rotation center, if the motif also has (at least) k-fold rotational symmetry we align the motif with that rotation center.

Also since only part of the motif is within the fundamental region, we need to make an adjustment to the area rule.

In the next slide we show the modified algorithm.

The Modified Algorithm

For each $i = 0, 1, 2, \ldots$

Repeat:

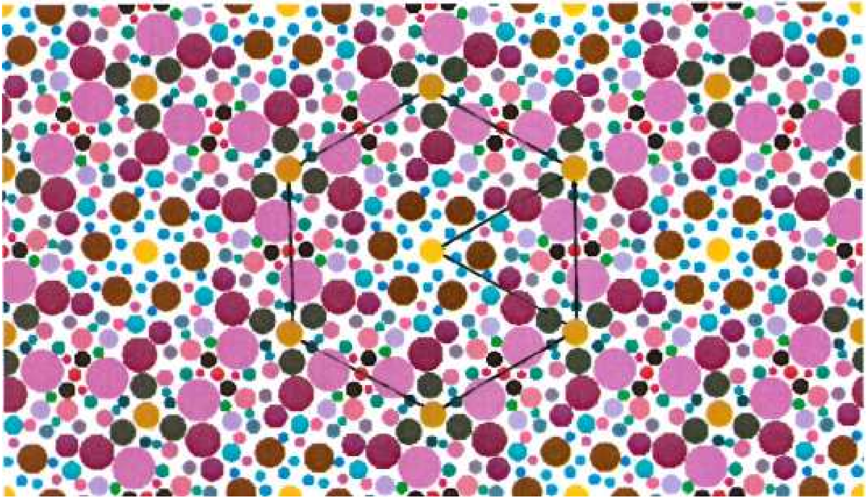
Randomly choose a point within R to place the i-th motif m_i.
If $\mathbf{m_i}$ has k-fold symmetry and overlaps a k-fold rotation point

Move $\mathbf{m_i}$ to be centered on that k-fold rotation point

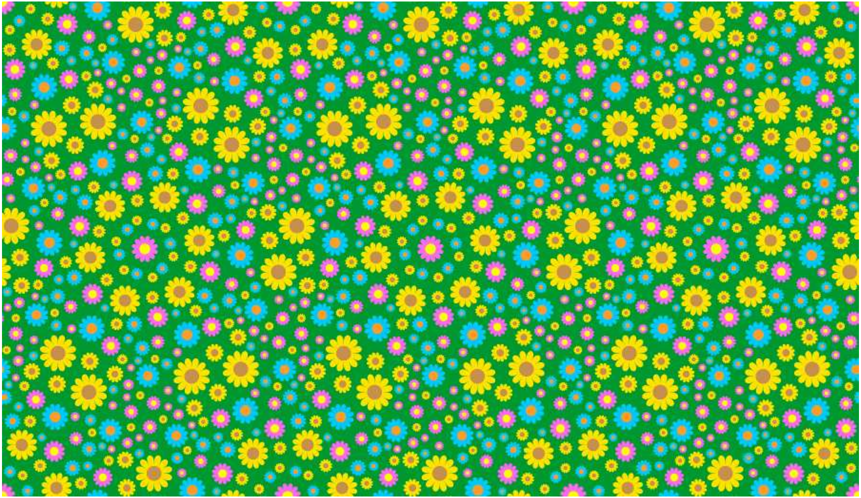
Until (m_i doesn't intersect any of $m_0, m_1, ..., m_{i-1}$)
Add m_i to the list of successful placements

Until some stopping condition is met, such as a maximum value of i or a minimum value of A_i.

A p6 Pattern of Circles with some on each Rotation Center



A p6 Pattern of Flowers

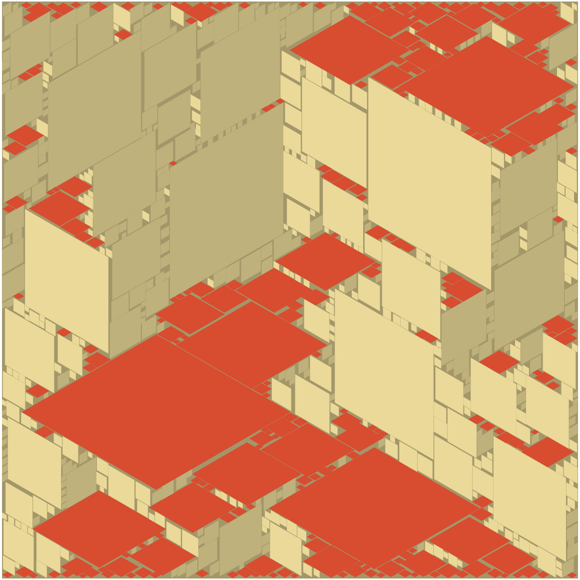


Patterns Restricted by Motif Orientation

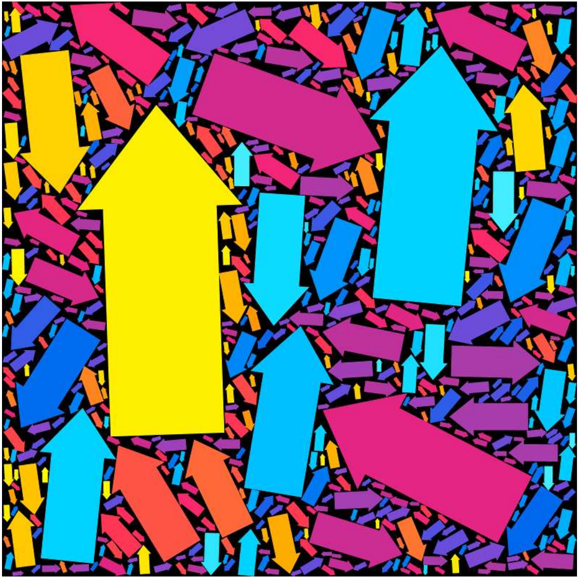
There are several possible choices for orienting orientable motifs:

- We could give them all the same orientation as with the butterflies of the title slide.
- We could cycle among a finite number of fixed orientations as in the next slide.
- We could use random orientations as in the second slide below.
- We could orient the motifs toward a fixed point as is done in the third and fourth slides below.
- We could orient the motifs according to an “orientation field” that depends on position, as shown in the fifth and following slides below.

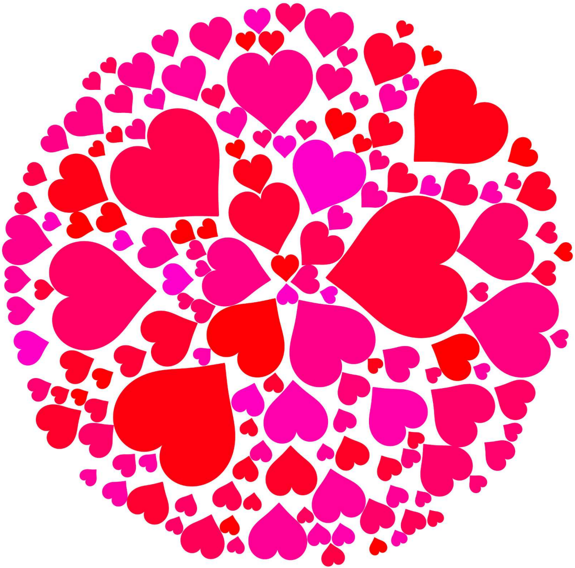
Rhombi in three orientations



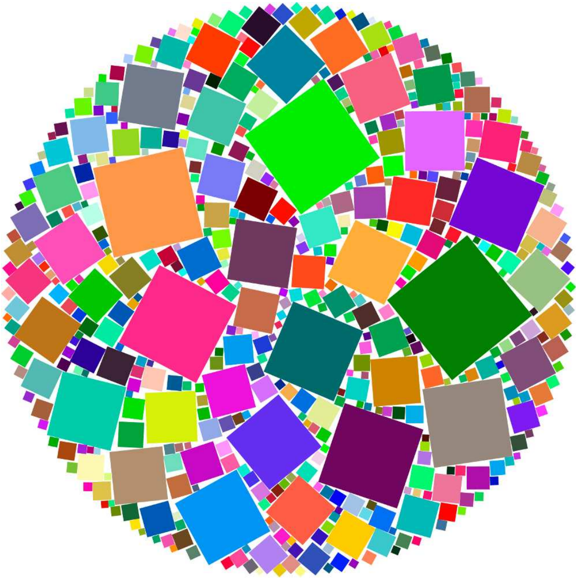
Randomly oriented arrows



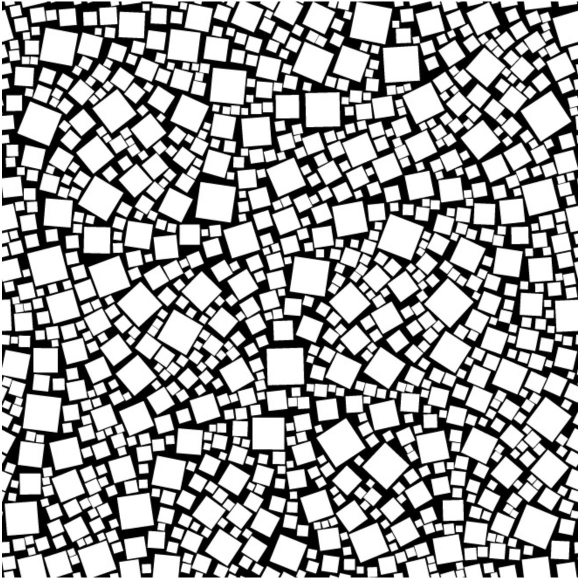
Centrally oriented hearts



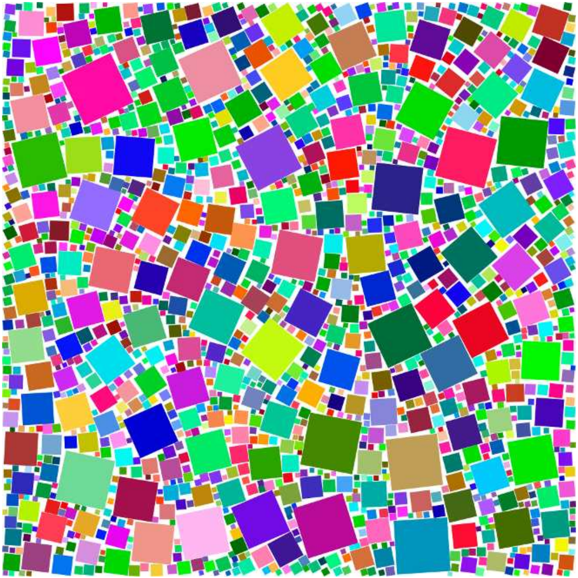
Centrally oriented squares



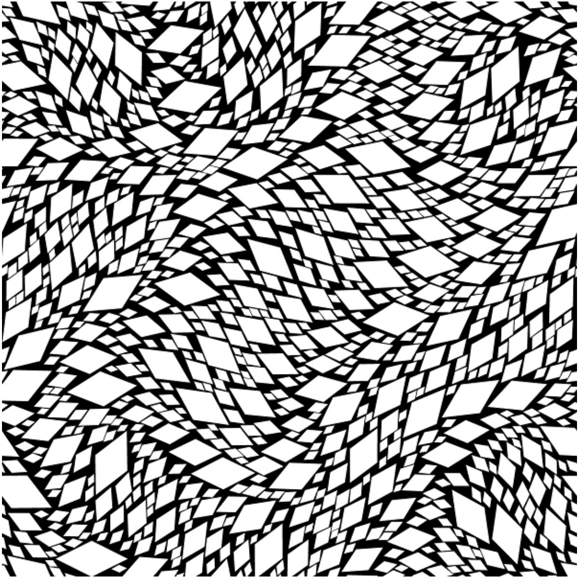
A flowing pattern of squares



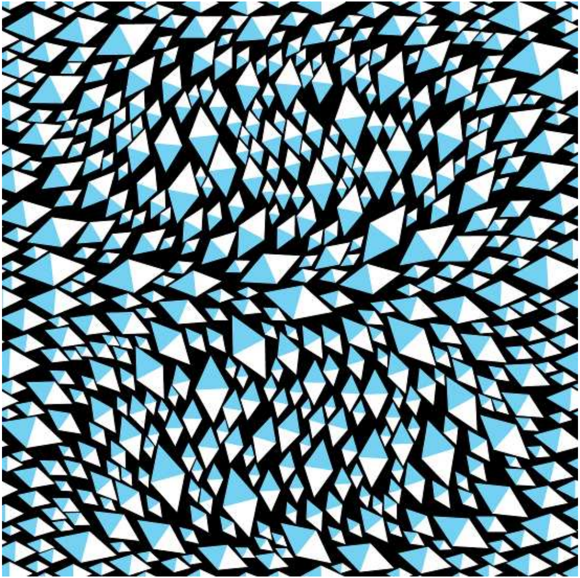
Another flowing pattern of squares



A flowing pattern of rhombi



Another flowing pattern of rhombi



Patterns with Complicated Motifs

We have used several techniques to create more complicated motifs.

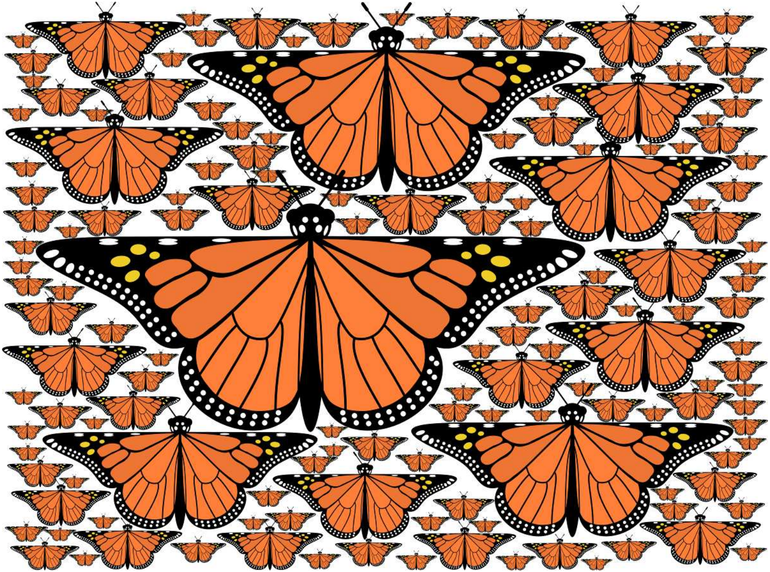
- We have used finite Fourier polynomials to outline a motif.
- We use differential scaling and rotation to obtain new features — applying that to circles gives ellipses in any desired orientation.
- We use Bezier curves for smoothly flowing lines.

A pattern of buses

A male monarch butterfly

A female monarch butterfly

The monarch butterfly pattern



Future Work

- Here and in the past we have shown fractal patterns with $p1, p2mm, p4, p4mm, p6$, and $p6mm$ as their symmetry groups, but we haven't implemented algorithms for all the wallpaper groups. It would seem possible to create locally fractal patterns having the global symmetries of the other plane symmetry groups using our techniques.
- It would also seem possible to generate patterns of the sphere and hyperbolic plane that are locally fractal, but are globally symmetric.

Acknowledgements and Contact

We would like to thank *all* the organizers of Bridges 2017.

And of course we also owe Reza Sarhangi a *great* debt for his wonderful inspiration over the years.

Contact Information:
Doug Dunham
Email: ddunham@d.umn.edu
Web: http://www.d.umn.edu/~ddunham

John Shier
Email: johnpf99@frontiernet.net
Web: http://www.john-art.com/stat_geom_linkpage.html