More “Circle Limit III” Patterns

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Abstract

M.C. Escher used the Poincaré model of hyperbolic geometry when he created his four “Circle Limit” patterns. The third one of this series, *Circle Limit III*, is usually considered to be the most attractive of the four. In *Circle Limit III*, four fish meet at right fin tips, three fish meet at left fin tips, and three fish meet at their noses. In this paper, we show patterns with other numbers of fish that meet at those points, and describe some of the theory of such patterns.

1. Introduction

Figures 1 and 2 below show computer renditions of the Dutch artist M.C. Escher’s hyperbolic patterns *Circle Limit I* and *Circle Limit III* respectively. Escher made criticisms of his first attempt at creating hyperbolic art,

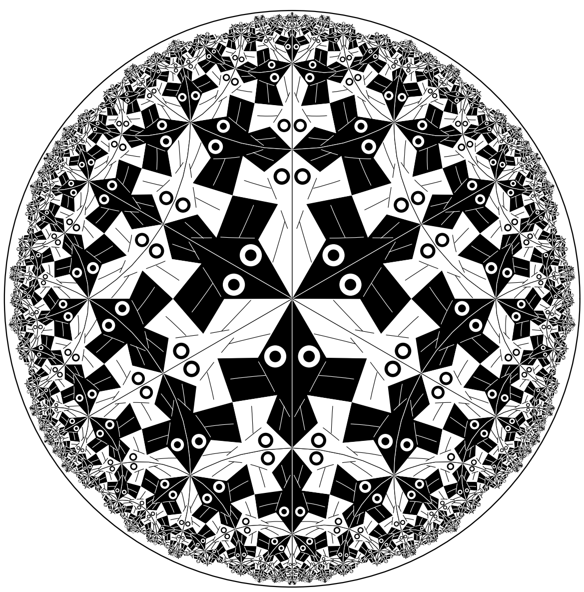

Figure 1: A rendition of Escher’s *Circle Limit I*.

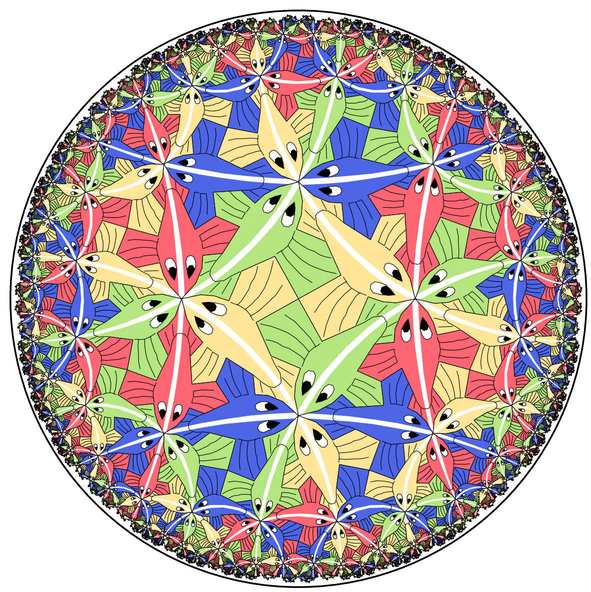

Figure 2: A rendition of Escher’s *Circle Limit III*.

Circle Limit I. However, he later redressed those deficiencies in *Circle Limit III*. In a letter to the Canadian mathematician H.S.M. Coxeter, Escher wrote:

> *Circle Limit I*, being a first attempt, displays all sorts of shortcomings... There is no continuity, no “traffic flow,” nor unity of colour in each row... In the coloured woodcut *Circle Limit III*, the shortcomings of *Circle Limit I* are largely eliminated. We now have none but “through traffic”

> series, and all the fish belonging to one series have the same colour and swim after each other head to tail along a circular route from edge to edge... Four colours are needed so that each row can be in complete contrast to its surroundings. ([6], pp. 108-109, reprinted in [4])

Escher had been inspired to create his "Circle Limit" patterns by a figure in one of Coxeter's papers [2]. That figure "gave me quite a shock" according to Escher in a letter to Coxeter, since the figure showed Escher how to make "circle-limit" patterns. Some of this important Coxeter-Escher interaction is recounted in [3].

In turn, Coxeter, was later inspired to write two papers explaining the mathematics behind Escher's *Circle Limit III* [3, 4]. In the same issue of *The Mathematical Intelligencer* containing Coxeter's second paper, an anonymous editor wrote the following caption for the cover of that issue, which showed Escher's *Circle Limit III*:

> Coxeter's enthusiasm for the gift M.C. Escher gave him, a print of Circle Limit III, is understandable. So is his continuing curiosity. See the articles on pp. 35–46. He has not, however said of what general theory this pattern is a special case. Not as yet. [1]

It seems that Coxeter did not describe such a general theory, or at least did not publish it. The goals of this paper are to provide the beginnings of a general theory and to show some sample patterns. Before developing our theory, we start with a short review of some hyperbolic geometry. With this background, we can then present a general theory of "Circle Limit III" fish patterns. Next, we examine special cases that are more amenable to calculation, showing sample patterns along the way. Finally, we indicate directions of further research.

2. Hyperbolic Geometry

Escher used the *Poincaré disk model* of hyperbolic geometry for his "Circle Limit" patterns. In this model, Euclidean objects are used to represent objects in hyperbolic geometry. The points of hyperbolic geometry in this model are just the (Euclidean) points within a Euclidean bounding circle. The hyperbolic lines are represented by circular arcs orthogonal to the bounding circle (including diameters). For example, the backbone lines and other features of the fish lie along hyperbolic lines in Figure 1. The hyperbolic measure of an angle is the same as its Euclidean measure in the disk model — we say such a model is *conformal* — so all fish in a "Circle Limit III" pattern have roughly the same Euclidean shape. We note that equal hyperbolic distances correspond to ever smaller Euclidean distances toward the edge of the disk. For example, all the black fish in Figure 1 are hyperbolically the same size, as are the white fish; all the fish in *Circle Limit III* are the same (hyperbolic) size. The Poincaré disk model was appealing to Escher since an infinitely repeating pattern could be shown in a bounded area and shapes remained recognizable even for small copies of the motif, as a consequence of conformality. Escher was more interested in the Euclidean properties of the disk model than the fact that it could be interpreted as hyperbolic geometry.

One might guess that the backbone arcs of the fish in *Circle Limit III* are also hyperbolic lines, but this is not the case. Even Escher believed the backbone arcs were orthogonal to the bounding circle, but he accurately drew them as non-orthogonal circular arcs. They are *equidistant curves* in hyperbolic geometry: curves at a constant hyperbolic distance from the hyperbolic line with the same endpoints on the bounding circle. For each hyperbolic line and a given distance, there are two equidistant curves, called *branches*, at that distance from the line, one each side of the line. In the Poincaré disk model, those two branches are represented by circular arcs making the same (non-right) angle with the bounding circle and having the same endpoints as the corresponding hyperbolic line. Equidistant curves are the hyperbolic analog of small circles in spherical geometry: a small circle of latitude in the northern hemisphere is equidistant from the equator (a great circle or "line" in spherical geometry), and has a corresponding small circle of latitude in the southern hemisphere the same distance from the equator. We usually do not use the term "equidistant curve" in Euclidean geometry since parallel lines have that property (and are not curved).

There is a *regular tessellation*, $\{m, n\}$ of the hyperbolic plane by regular m-sided polygons meeting n at a vertex provided $(m-2)(n-2) > 4$. Escher used the regular tessellations $\{6, 4\}$ and $\{8, 3\}$ as the basis of his Circle Limit patterns ($\{6, 4\}$ for *Circle Limit I* and *IV*, and $\{8, 3\}$ for *Circle Limit II* and *III*). Figure 3 shows the tessellation $\{8, 3\}$ (heavy lines) superimposed on the *Circle Limit III* pattern. As one traverses edges of this tessellation, alternately going left, then right at each vertex, one obtains a zig-zagging path called a *Petrie polygon*. The midpoints of the edges of a Petrie polygon lie on a hyperbolic line by symmetry. The vertices of the Petrie polygon lie alternately on each of the two equidistant curve branches associated to that line — this is shown in Figure 4 with the Petrie polygon drawn with thick lines, the "midpoint" line and the equidistant curves drawn in a medium line, all superimposed on the $\{8, 3\}$ tessellation (lightest lines). Escher only used one branch for fish backbones from each pair of equidistant curves in *Circle Limit III*. If he had consistently used the other branch, the pattern would have been rotated about the center by 45 degrees.

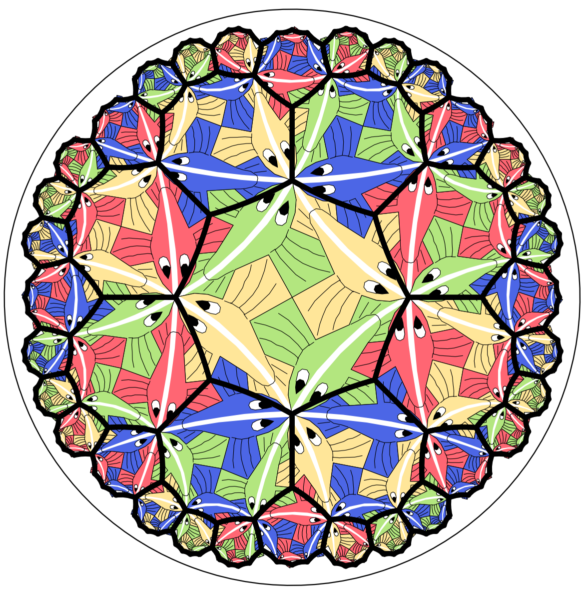

Figure 3: The tessellation $\{8, 3\}$ underlying the *Circle Limit III* pattern.

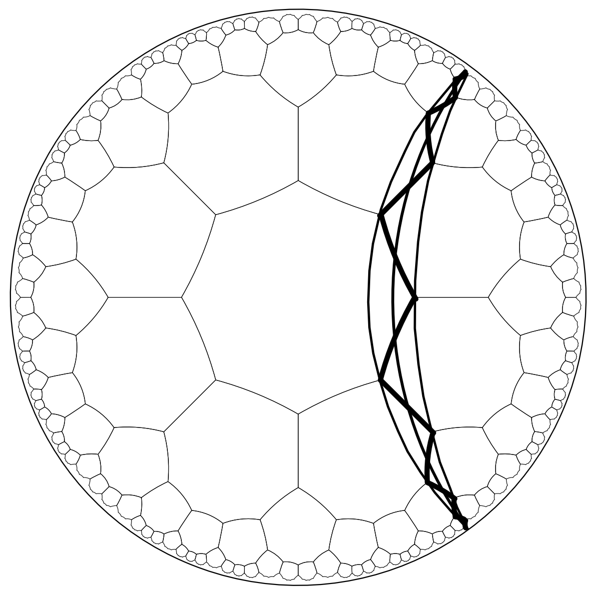

Figure 4: A Petrie polygon (heavy), a hyperbolic line and two equidistant curves (medium) associated to the $\{8, 3\}$ tessellation (lightest lines).

3. The General Theory of "Circle Limit III" Patterns

If we examine *Circle Limit III*, we see that four fish meet at right fin tips, three meet at left fin tips, and three meet at their noses (and tails). We generalize these numbers to patterns of fish with p fish meeting at right fins, q fish meeting at left fins, and r fish meeting at their noses. We will label such a pattern (p, q, r). So *Circle Limit III* would be called $(4, 3, 3)$ in this notation. One could conceptually "reflect" all the fish of a (p, q, r) pattern across their backbone lines to obtain a (q, p, r) pattern, but this is a true hyperbolic reflection only when $p = q$.

One restriction that we make is that r be odd so that the fish swim head-to-tail, in order to achieve "traffic flow." Also, by examining the left fins of *Circle Limit III*, we also require that p and q be at least 3, since two fins could not have tips that meet. And of course r must be at least 3 too (since r is odd and greater than 1). Consequently, the "smallest" example of such a pattern is $(3,3,3)$, realized by Escher in his *Notebook Drawing 123* [8, Page 216]. This $(3,3,3)$ pattern is based on the regular tessellation {3,6} of the Euclidean plane by equilateral triangles, each triangle containing three half-fish. It is interesting that this simpler drawing is dated several years after the much more complicated *Circle Limit III*. We also note that we do not consider *Notebook Drawing 122* to be a valid "Circle Limit III" pattern, since, as in *Circle Limit I*, fish meet "head-on", not head-to-tail. This pattern is based on the Euclidean tessellation of squares, each square containing four half-fish, and would be denoted $(4,4,2)$ if we allowed r to be even.

There is another natural tessellation that we can associate with the *Circle Limit III* pattern, obtained by dividing the octagons in Figure 3 into four "kites" — convex quadrilaterals with two pairs of adjacent equal sides. Each kite can serve as a *fundamental region* for the pattern since it contains exactly the right fish pieces to assemble one complete fish. Figure 5 shows this kite tessellation superimposed on the *Circle Limit III* pattern. In general for a (p,q,r) pattern, one can use a kite-shaped fundamental region with vertex angles $\frac{2\pi}{p}$, $\frac{\pi}{r}$, $\frac{2\pi}{q}$, and $\frac{\pi}{r}$. We note that a quadrilateral is hyperbolic precisely when the sum of its interior angles is less than 2π, which translates to the following inequality for our kites: $\frac{2\pi}{p}+\frac{\pi}{r}+\frac{2\pi}{q}+\frac{\pi}{r}<2\pi$ or in other words: $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$. Figure 6 shows the kite tessellation corresponding to the (4,3,5) pattern and containing fish backbones along equidistant curves.

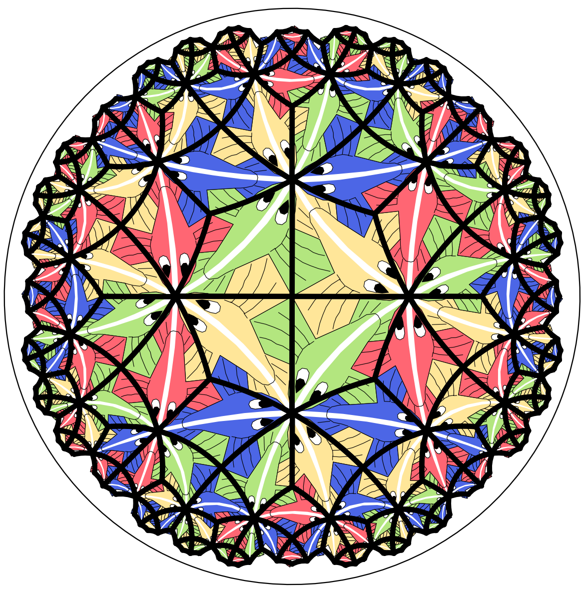

Figure 5: The kite tessellation superimposed on the *Circle Limit III* pattern.

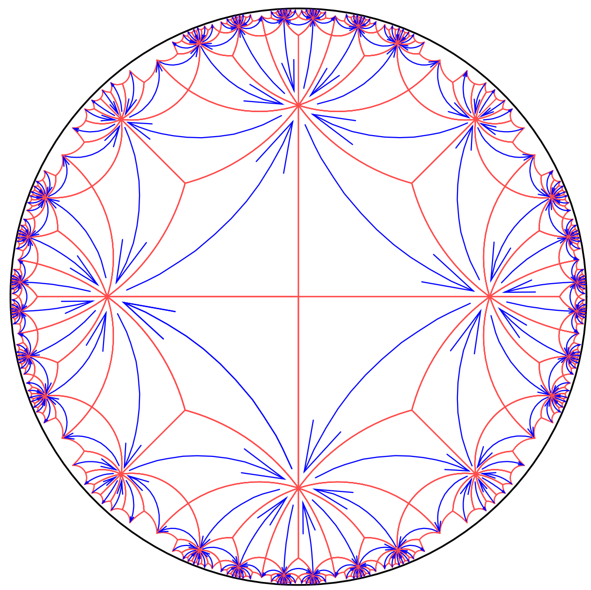

Figure 6: The kite tessellation (lighter lines) of the (4,3,5) pattern, with fish backbones (darker arrows) along equidistant curves.

4. Special Cases

There are two special cases that can be analyzed in more detail. The first case we consider is when $p=q$, so that the fish are symmetric. In this case the backbone curves are (hyperbolic) lines. When $p \neq q$ and the fish are not symmetric, their backbone lines bend away from the side with the larger number of fish meeting at a fin tip. For the second special case, we assume that q and r are both 3.

In the first case, the fish are symmetric by reflection across the hyperbolic backbone lines. Thus we can use half a fish for the motif and an isosceles triangle that is half of a kite (with angles $\frac{2\pi}{p}$, $\frac{\pi}{2r}$, and $\frac{\pi}{2r}$) for the corresponding fundamental region. Figure 7 below shows a $(4, 4, 3)$ pattern, but with angular fish in the style of *Circle Limit I*. Escher's *Notebook Drawing 123*, mentioned above, is the "smallest" example of this special case, since it is a $(3, 3, 3)$ pattern.

Figure 8 is a fin-centered version of Figure 7 that answers most of Escher's criticisms of his *Circle Limit I* pattern. This pattern was obtained by the following sequence of steps. First, two of the noses of the white fish (and tails of the black fish) were made narrower so that three white fish could also meet nose-to-nose. This solved the "traffic flow" problem (and made the white fish congruent to the black fish). The fish would now all swim the same direction along a backbone line, but would be alternately colored black and white. To solve this "unity of colour" problem we use the minimum number of colors, three, to re-color the fish, yeilding the pattern of Figure 7. Finally, Figure 8 is derived from Figure 7 by hyperbolically translating a 4-fold fin meeting point to the center of the bounding circle. Fortunately Escher did not follow this sequence, so that we have the beautiful *Circle Limit III* pattern instead.

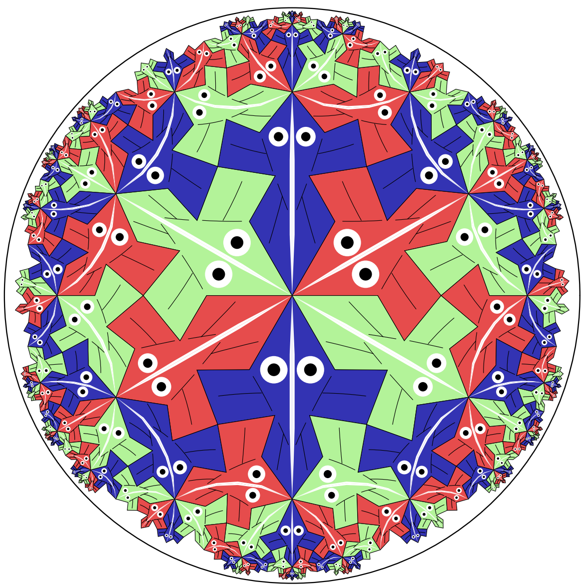

Figure 7: A (4,4,3) pattern derived from the *Circle Limit I* pattern.

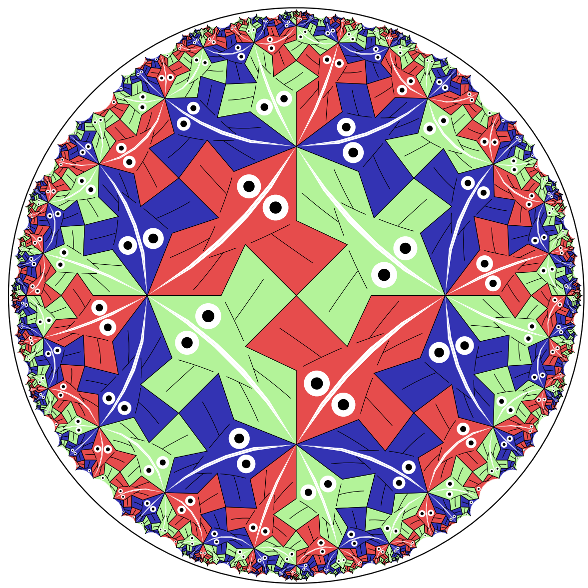

Figure 8: A (4,4,3) pattern in the style of *Circle Limit I* (derived from Figure 7).

In the second special case, when $q = r = 3$, we can calculate the angle, ω, that the "backbone" equidistant curves make with the bounding circle. Again, Escher's *Notebook Drawing 123*, a $(3, 3, 3)$ pattern, is the "smallest" example of this special case too. Coxeter computed ω for *Circle Limit III* ($p = 4$) in two ways: first by using hyperbolic trigonometry [3], and later by using Euclidean techniques [4]. We follow Coxeter's first method to calculate ω in terms of p. First, we note that the regular tessellation $\{2p, 3\}$ can be superimposed on a $(p, 3, 3)$ pattern just as in Figure 3 (with nose and left fin points at alternate vertices of the $2p$-gons). We next make additions to Figure 4 to obtain Figure 9: we add lines ON, OM, and NL, and label points L, M, N, and O. Angles $\angle OMN$ and $\angle MLN$ are right angles.

We wish to calculate the distance $\overline{NL}$ (an overline above a line segment denotes its hyperbolic length) from the left equidistant curve to hyperbolic line passing through L and M. This distance is related to *angle of parallelism*, the angle between NL and the hyperbolic line $N\infty$ (not shown — it is different than the equidistant curve going through N and ∞ which is shown and clearly makes a right angle with NL). If α denotes the angle of parallelism, this important relation in hyperbolic geometry is: $\cos\alpha = \tanh\overline{NL}$ [7, Page 402]. It turns out that angle of parallelism α is the same as the angle of intersection ω of the bounding circle with the equidistant curve at that distance from its hyperbolic line. Thus we have:

$$\cos\omega = \tanh \overline{NL}$$

We can calculate $\overline{NL}$ by solving the two right triangles $\triangle OMN$ and $\triangle MLN$ using standard formulas from hyperbolic trigonometry [7, Page 403, Theorem 10.3]. First, we use $\triangle OMN$ to compute $\overline{MN}$ by:

$$\cosh \overline{MN} = \cos(\frac{\pi}{2p})/\sin(\frac{\pi}{3}) = \frac{2}{\sqrt{3}}\cos(\frac{\pi}{2p})$$

Next, we use $\triangle MLN$ to compute $\overline{LN}$ from $\overline{MN}$ by:

$$\tanh \overline{LN} = \cos(\frac{\pi}{3}) \tanh \overline{MN} = \frac{1}{2}\tanh \overline{MN} = \frac{1}{2}\sqrt{1 - 1/\cosh^2(\overline{MN})}$$

using the relation $\tanh^2 = 1 - 1/\cosh^2$. Finally, can combine these equations to obtain:

$$\cos\omega = \frac{1}{2}\sqrt{1 - 3/4\cos^2(\frac{\pi}{2p})}$$

When $p = 4$, $\cos\frac{\pi}{8} = \sqrt{\frac{2+\sqrt{2}}{4}}$, consequently $\cos\omega = \sqrt{\frac{3\sqrt{2}-4}{8}}$ and ω is approximately $79.97°$. Coxeter obtained a different, but equivalent expression for ω. Similarly, when $p = 5$, $\cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$, $\cos\omega = \sqrt{\frac{3\sqrt{5}-5}{40}}$, and $\omega \approx 78.07°$. Figure 10 shows a $(5, 3, 3)$ pattern which was used as the basis for the 2003 Mathematics Awareness Month poster and whose background is described in [5].

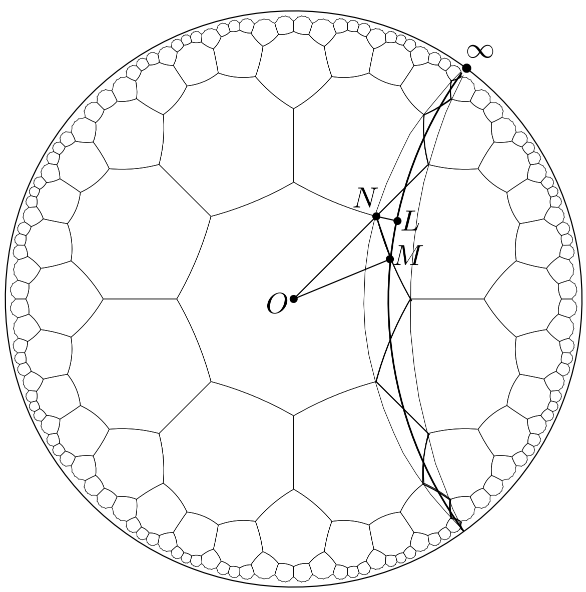

Figure 9: Two right triangles MON and MLN added to Figure 4 used to calculate ω.

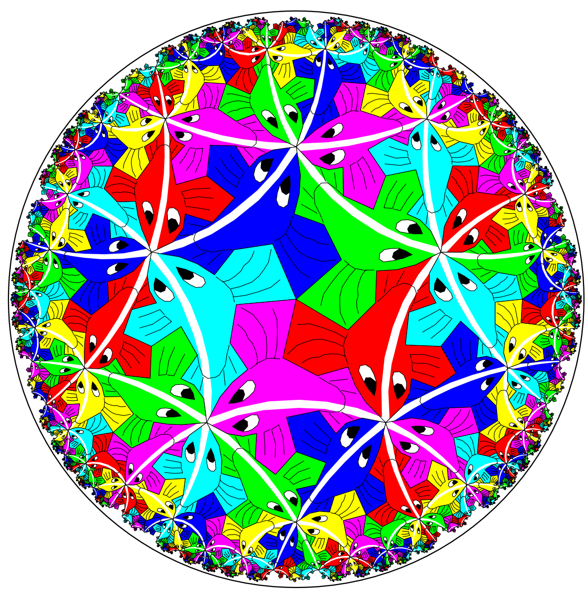

Figure 10: A $(5, 3, 3)$ fish pattern.

We can also take the limit as p goes to infinity and obtain the limiting equation $\cos\omega = 1/4$ or $\omega \approx 75.52°$. Thus ω lies in the interval $(\cos^{-1}(\frac{1}{4}), \frac{\pi}{2}]$ for finite p, q, and r. Of course we cannot actually draw a pattern with an infinite number of fish meeting in the center of the disk (but an infinite number of fish fins *can* meet on the bounding circle, as indicated below).

The concept of (p, q, r) is symmetric in p and q, of course. Figures 11 and 12 below show examples in which $p = r = 3$. We have put the right fin tips at the center of the disk in agreement with Escher's *Circle Limit III*. Figure 11 shows a $(3, 4, 3)$ pattern related to *Circle Limit III* except that the numbers of fish meeting at left and right fin tips have been switched. Figure 12 shows a $(3, 5, 3)$ pattern that bears the same relationship to our Figure 10 above. Note that these figures are not just translations (or reflections) of *Circle Limit III* and Figure 10, since the number of fish meeting at right (and left) fins is different.

There are two interesting aspects of $(3, q, 3)$ patterns. First, the three backbone lines closest to the center have "straightened out" into chords of the bounding circle, resulting in a Euclidean equilateral triangle of backbones in the center. When I first created such a pattern (by translating to the origin a left fin meeting point of the *Circle Limit III* pattern) about 20 years ago, I told Coxeter that I was astonished by this phenomenon. He replied (words to the effect):

> Well, you shouldn't have been. Any triangle made up of three congruent circular arcs meeting at 60-degree angles must obviously be a Euclidean equilateral triangle.

It may be possible to exploit the simple geometry of these $(3, q, 3)$ patterns to compute ω more easily.

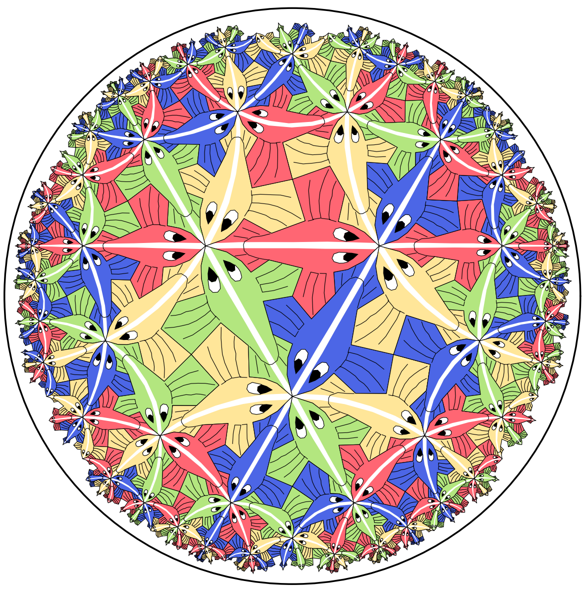

Figure 11: A (3,4,3) pattern related to Escher's *Circle Limit III*.

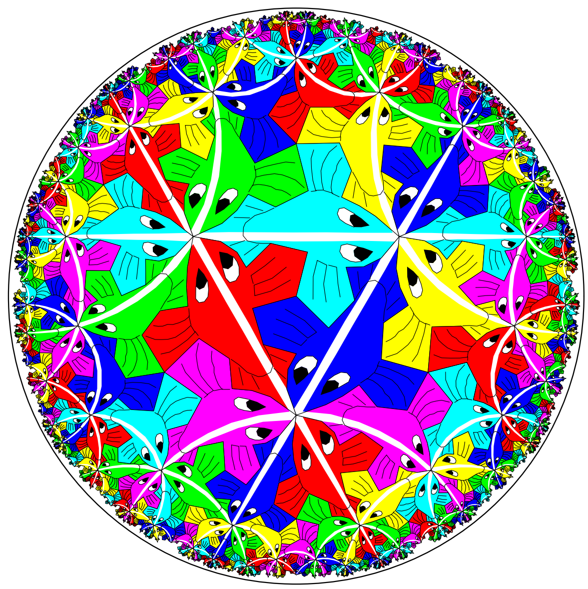

Figure 12: A (3,5,3) pattern related to our pattern of Figure 10.

A second aspect of $(3, q, 3)$ patterns is that we can conceive of taking the limit of them as q tends to infinity. As q goes to infinity, the left fin tips would get farther and farther from the center of the disk, until in the limit, they would be on the bounding circle. It would theoretically be possible to draw such a pattern, but our current software can only handle finite values of p, q and r.

5. Conclusions and Future Work

We have described a general theory of (p, q, r) "Circle Limit III" patterns and shown some examples. It would seem to be a worthy goal to find a general formula for ω in terms of p, q, and r. With current software, (p, q, r) patterns can only be created one at a time. It would certainly be useful to have a program that could automatically create a new (p, q, r) pattern with different values of p, q, and r from an existing pattern. Another interesting direction would be to investigate (and draw) (p, q, r) patterns with one of q, or r being infinity (patterns may also exist with both q and r being infinity). A seemingly difficult problem is to automate the process of determining a coloring for a (p, q, r) pattern that has the same color along any line of fish and adheres to the map-coloring principle that adjacent fish have different colors.

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