Repeating Hyperbolic Pattern Algorithms — Special Cases

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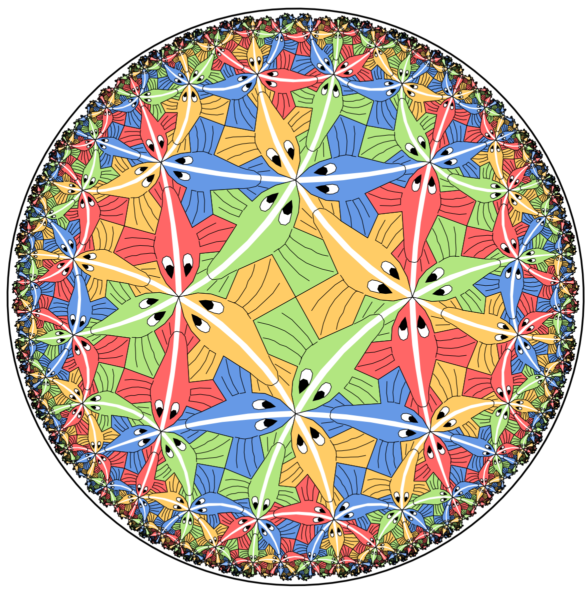


Outline

1. **History**
2. **A Repeating hyperbolic pattern algorithm based on regular tessellations**
3. **A Repeating hyperbolic pattern algorithm based on non-regular tessellations**
4. **Future work**

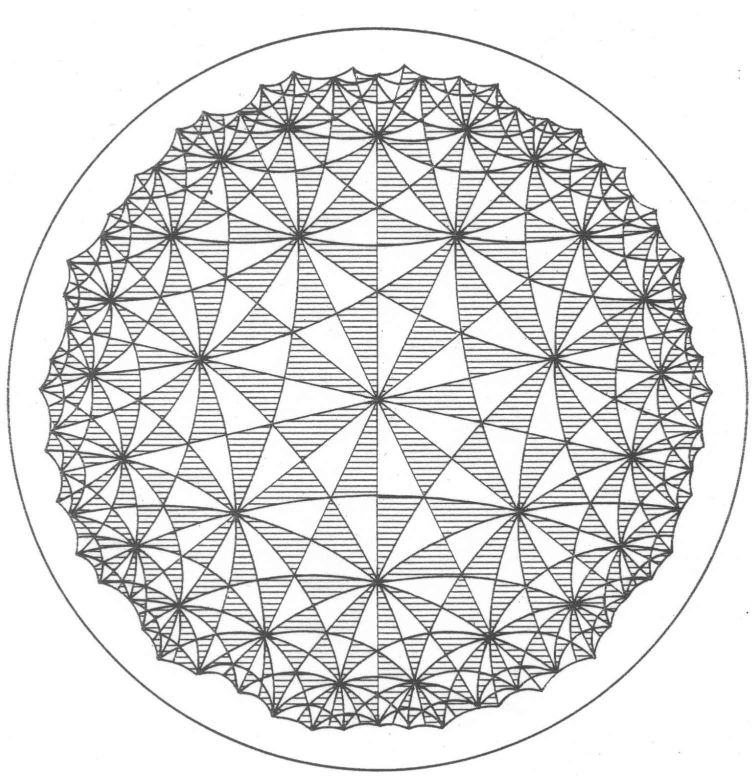
1. History

1. Pre-Escher

2. Escher's patterns

3. Post-Escher = Dunham, Ferguson, Sazdanovic, etc.

Triangle group (7,3,2) tessellation
Originally in *Vorlesungen über die Theorie der elliptischen Modulfunctionen*
F. Klein and R. Fricke, 1890.



H.S.M. Coseter’s Figure 7
in *Crystal Symmetry and Its Generalizations*
Trans. Royal Soc. of Canada, 1957.

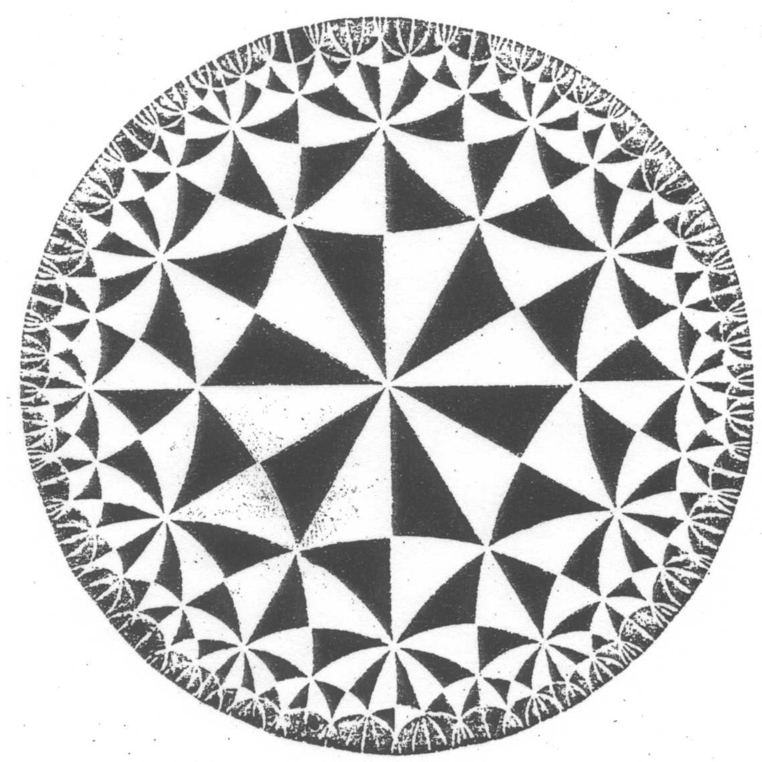
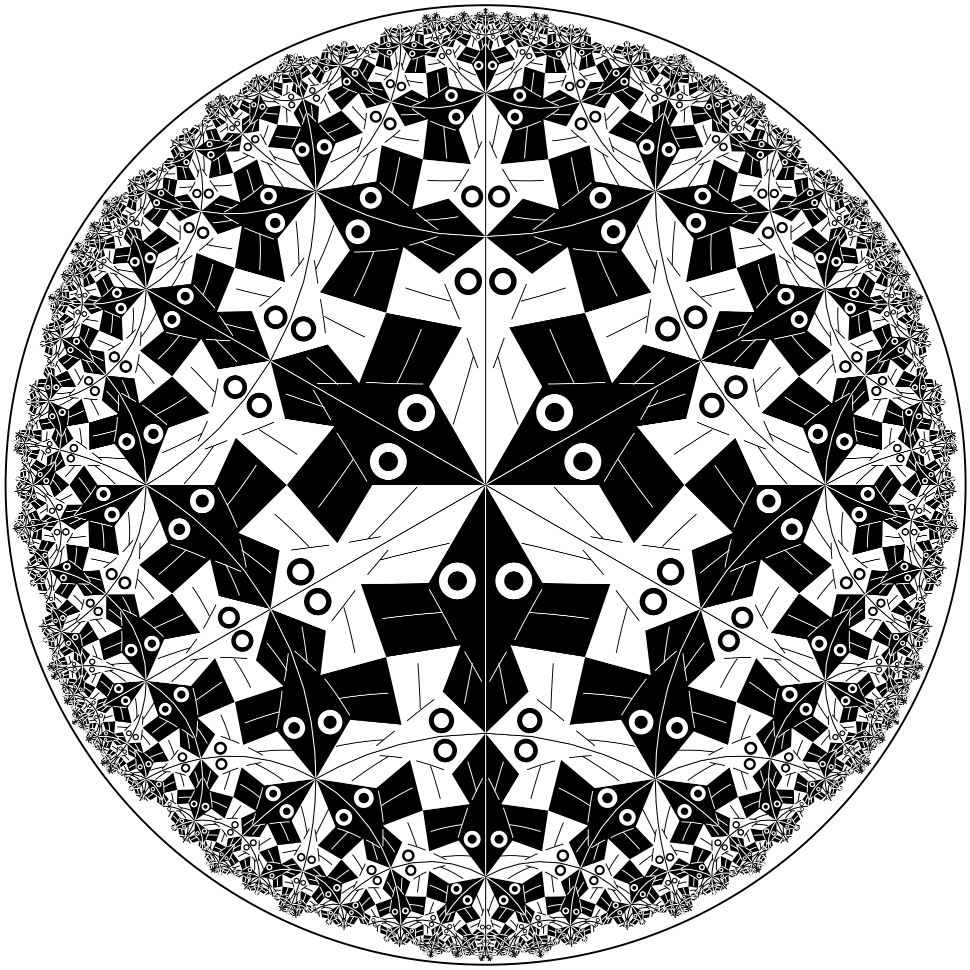


FIGURE 7

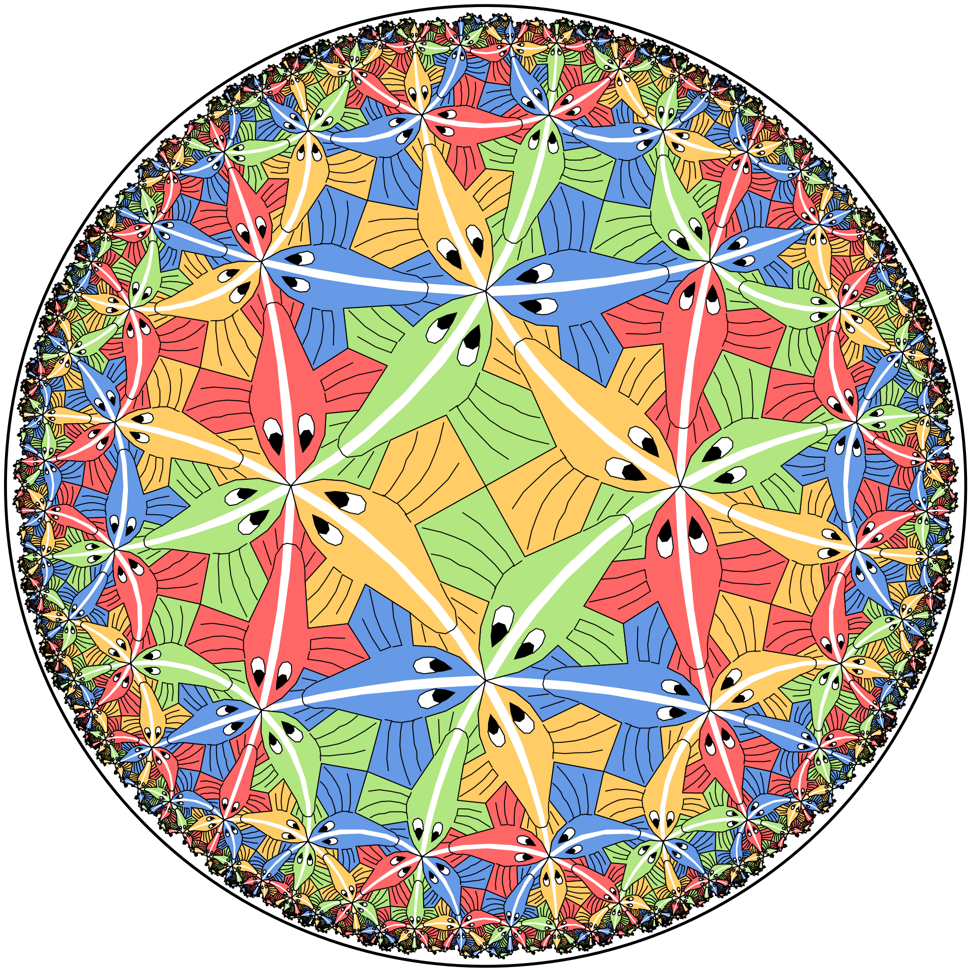
M.C. Escher’s “Circle Limit” Patterns

Circle Limit I

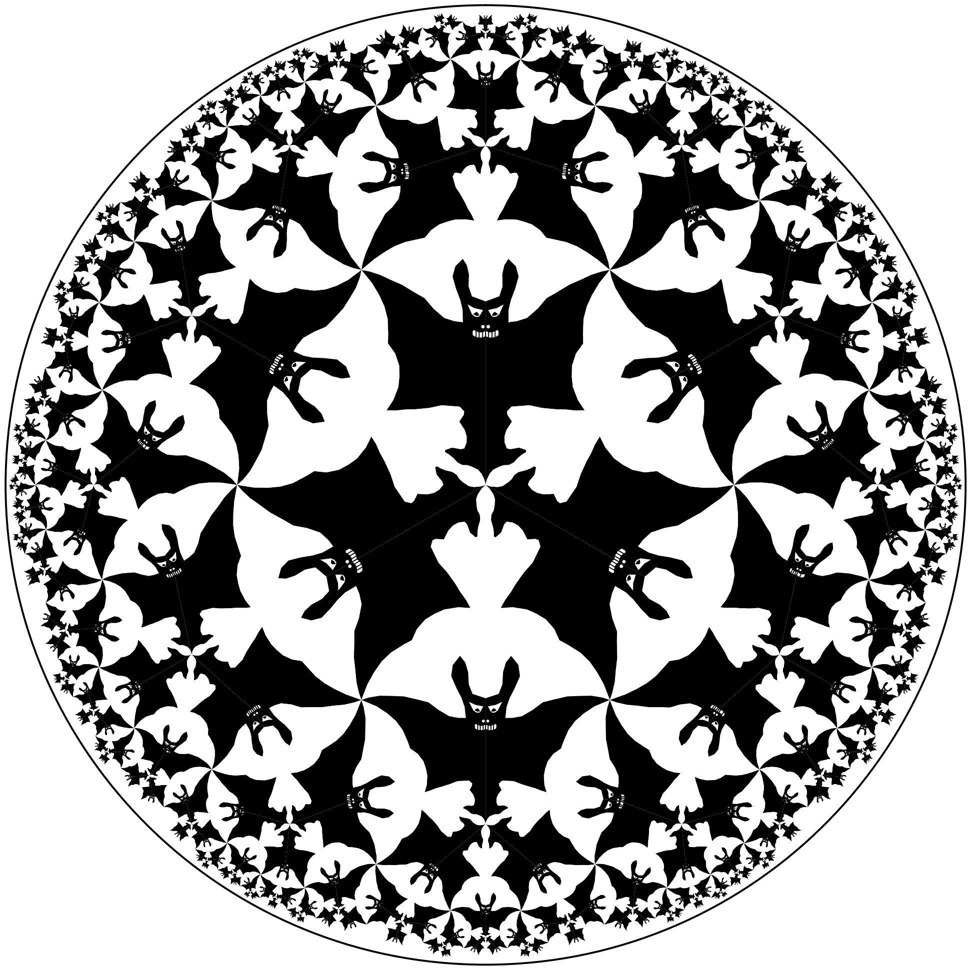


Circle Limit II

Circle Limit III

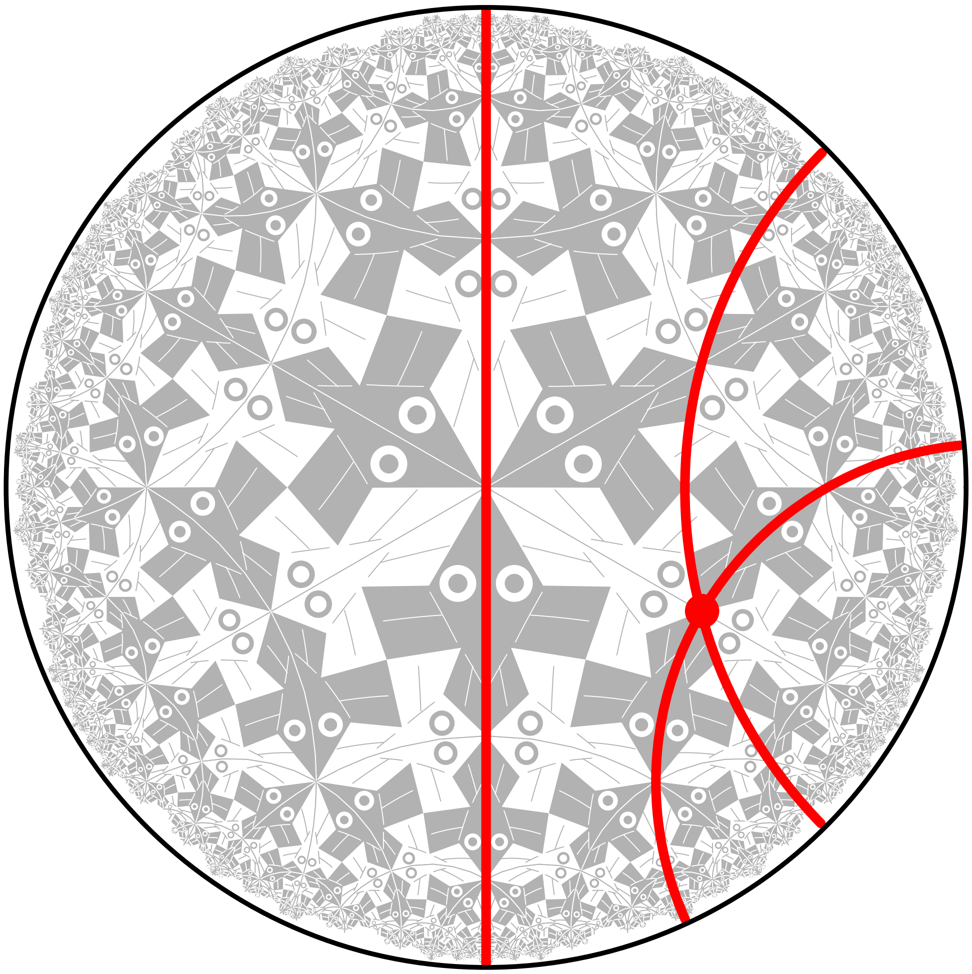


Circle Limit IV



2. Generation of Repeating Hyperbolic Patterns

Following Escher, we use the Poincaré disk model of hyperbolic geometry.



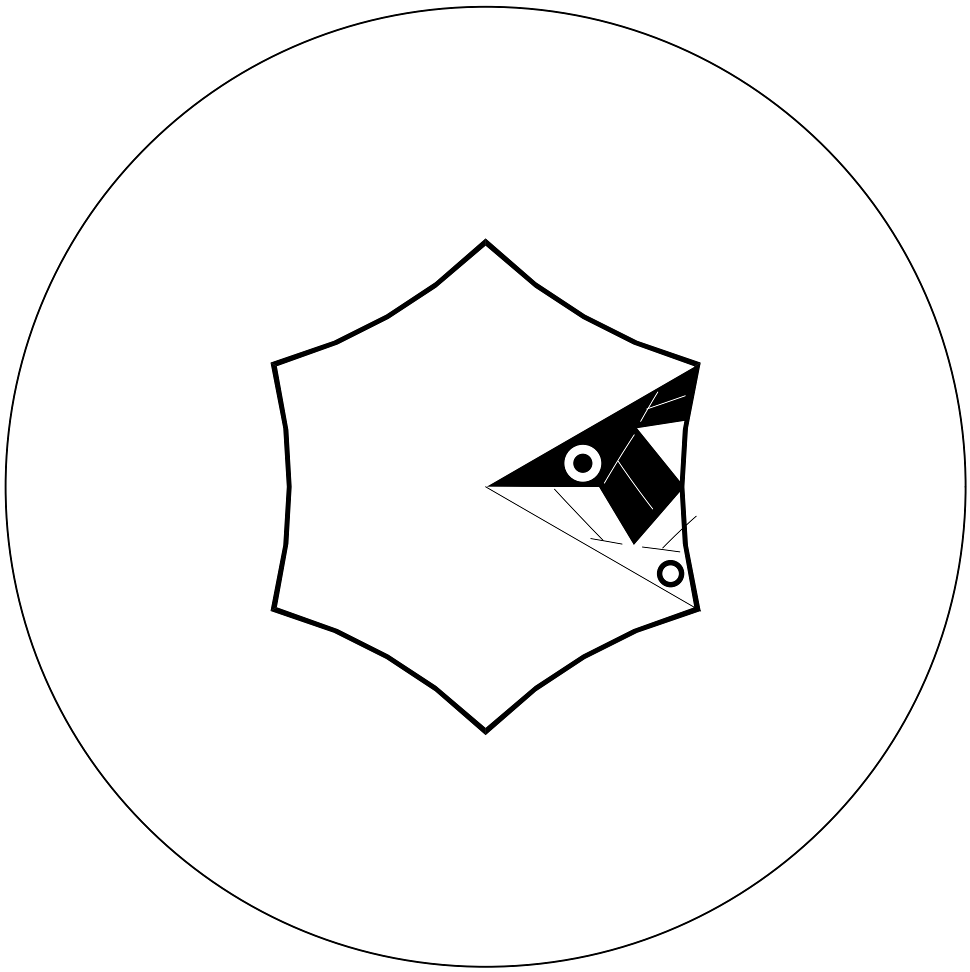
The Pattern Generation Process

Consists of two steps:

1. **Design the basic subpattern or *motif* — done by a hyperbolic drawing program.**
2. **Transform copies of the motif about the hyperbolic plane: *replication***

Repeating Patterns

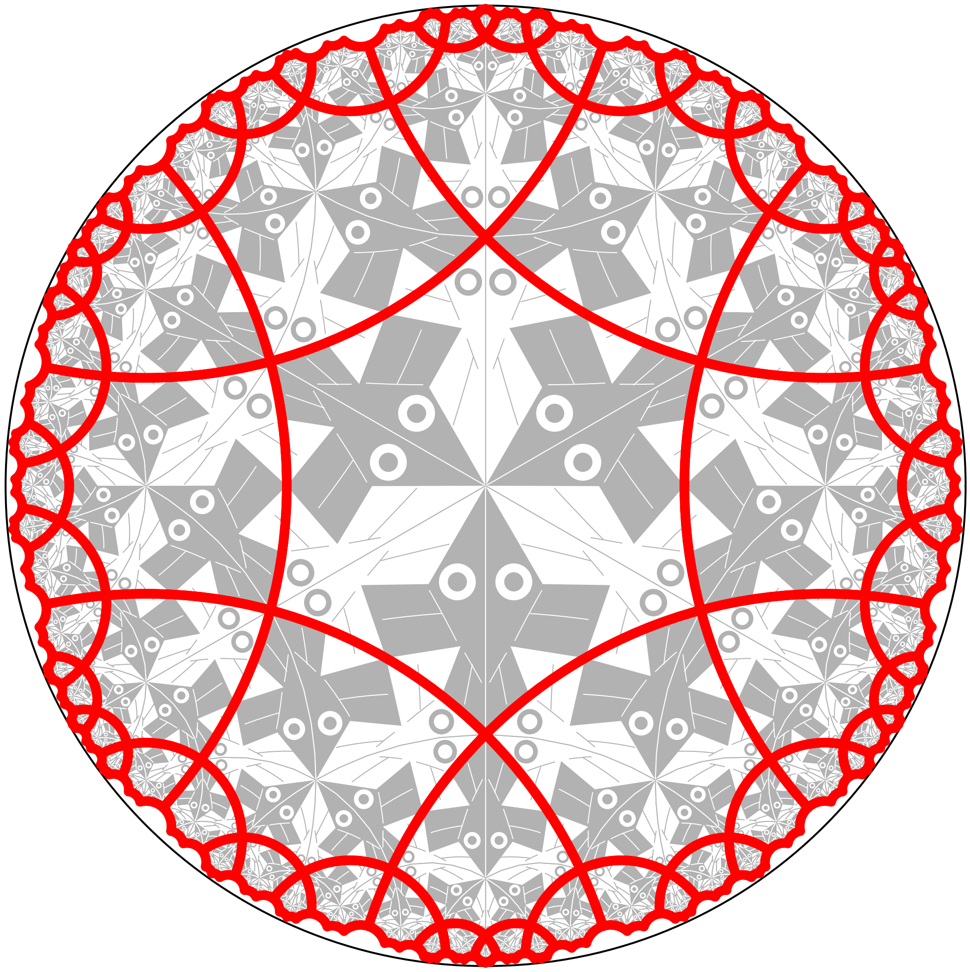
A *repeating pattern* is composed of congruent copies of the motif. A motif for *Circle Limit I*.



The Regular Tessellations $\{p, q\}$

- **Escher based his four "Circle Limit" patterns (and many of his Euclidean and spherical patterns) on regular tessellations.**
- **The *regular tessellation* $\{p, q\}$ is a tiling composed of regular p-sided polygons, or *p-gons* meeting q at each vertex.**
- **It is necessary that $(p-2)(q-2) > 4$ for the tessellation to be hyperbolic.**
- **If $(p-2)(q-2) = 4$ or $(p-2)(q-2) < 4$ the tessellation is Euclidean or spherical respectively.**

The Regular Tessellation $\{6, 4\}$



A Table of the Regular Tessellations

q											
⋮			⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
11			*	*	*	*	*	*	*	*	*	⋯
10			*	*	*	*	*	*	*	*	*	⋯
9			*	*	*	*	*	*	*	*	*	⋯
8			*	*	*	*	*	*	*	*	*	⋯
7			*	*	*	*	*	*	*	*	*	⋯
6			□	*	*	*	*	*	*	*	*	⋯
5			○	*	*	*	*	*	*	*	*	⋯
4			○	□	*	*	*	*	*	*	*	⋯
3			○	○	○	□	*	*	*	*	*	⋯
2												
1												
	1	2	3	4	5	6	7	8	9	10	11	⋯

p

□ - Euclidean tessellations

○ - spherical tessellations

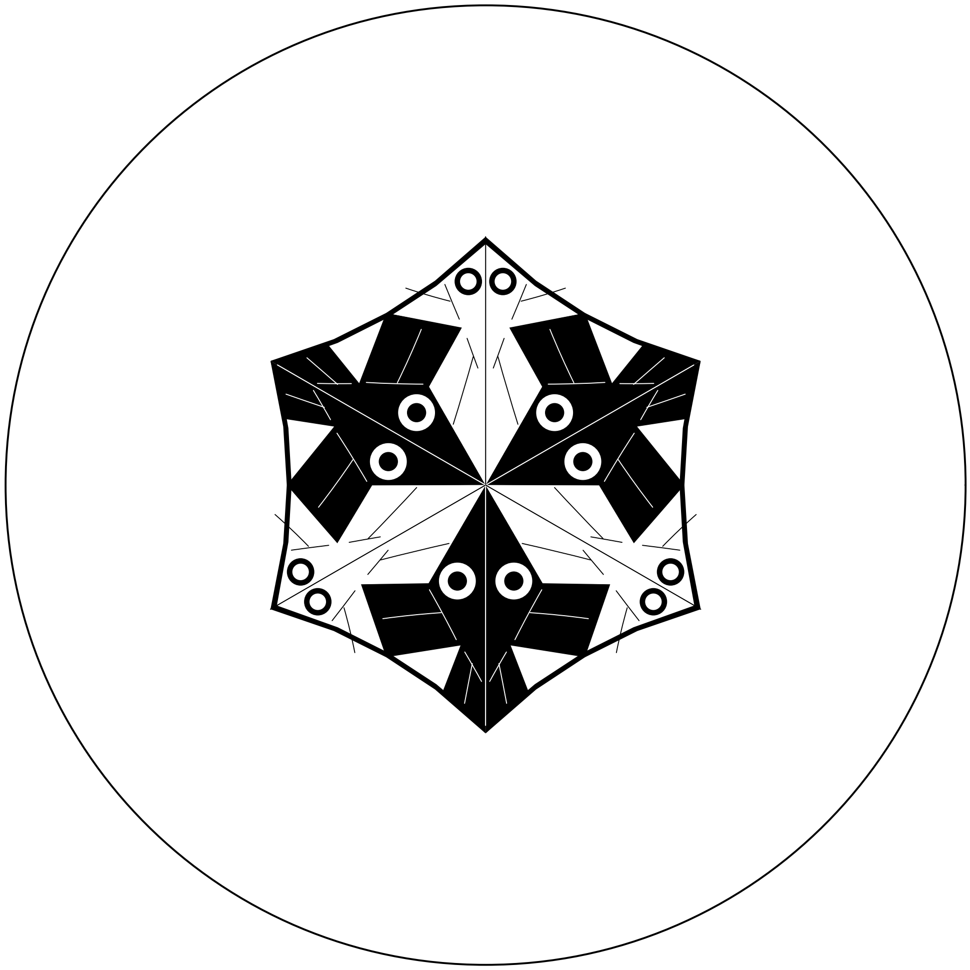
* - hyperbolic tessellations

Replicating the Pattern

In order to reduce the number of transformations and to simplify the replication process, we form the *p-gon pattern* from all the copies of the motif touching the center of the bounding circle.

- **Thus to replicate the pattern, we need only apply transformations to the p-gon pattern rather than to each individual motif.**
- **Some parts of the p-gon pattern may protrude from the enclosing p-gon, as long as there are corresponding indentations, so that the final pattern will fit together like a jigsaw puzzle.**
- **The p-gon pattern is often called the *translation unit* for repeating Euclidean patterns.**

The p-gon pattern for *Circle Limit I*



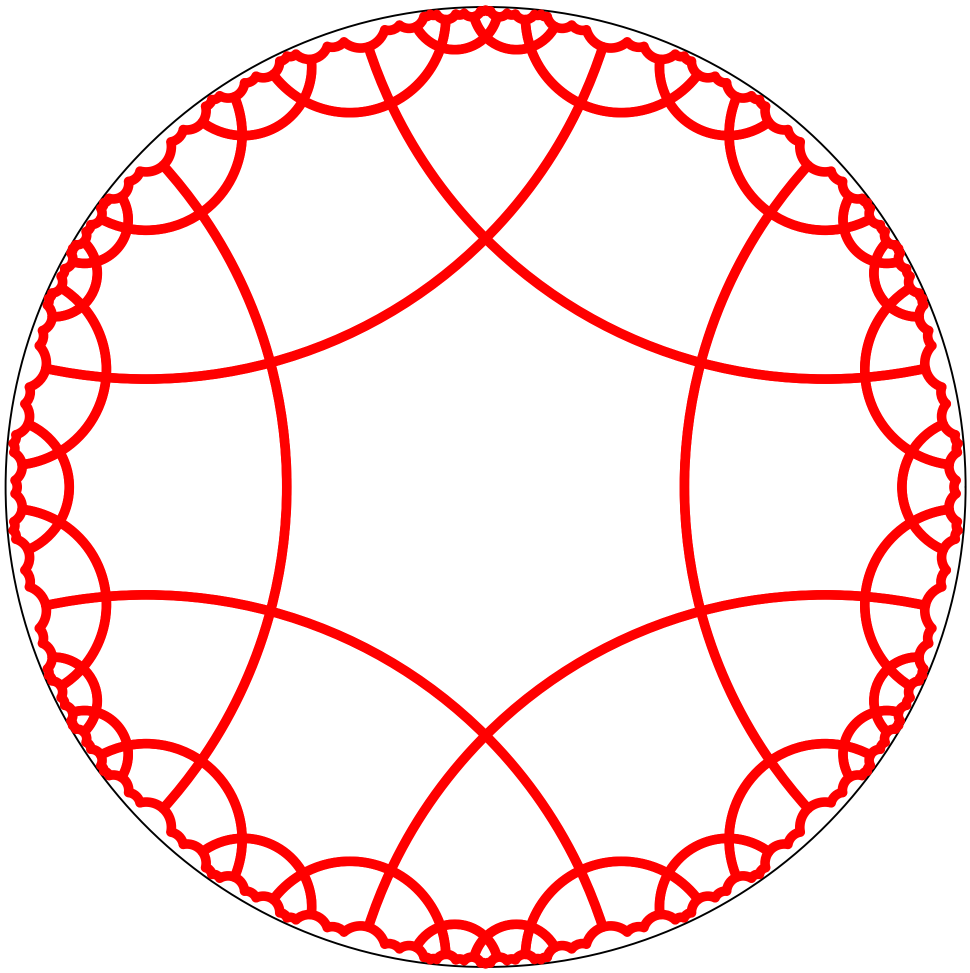
Layers of p-gons

We note that the p-gons of a $\{p, q\}$ tessellation are arranged in *layers* as follows:

- **The first layer is just the central p-gon.**
- **The $k + 1^{st}$ layer consists of all p-gons sharing and edge or a vertex with a p-gon in the k^{th} layer (and no previous layers).**
- **Theoretically a repeating hyperbolic pattern has an infinite number of layers, however if we only replicate a small number of layers, this is usually enough to appear to fill the bounding circle to our Euclidean eyes.**

The Regular Tessellation $\{6, 4\}$ — Revisited

To show the layer structure and exposure of p-gons.



Exposure of a p-gon

We also define the exposure of a p-gon in terms of the number of edges it has in common with the next layer.

- **A p-gon has *minimum exposure* if it has the fewest edges in common with the next layer, and thus shares an edge with the previous layer.**
- **A p-gon has *maximum exposure* if it has the most edges in common with the next layer, and thus only shares a vertex with the previous layer.**
- **In the pseudo-code, we abbreviate these values as `MAX_EXP` and `MIN_EXP` respectively.**

The Replication Algorithm

The replication algorithm consists of two parts:

1. **A top-level "driver" routine** `replicate()` **that draws the first layer, and calls a second routine,** `recursiveRep()`**, to draw the rest of the layers.**
2. **A routine** `recursiveRep()` **that recursively draws the rest of the desired number of layers.**

A tiling pattern is determined by how the p-gon pattern is transformed across p-gon edges. These transformations are in the array `edgeTran[]`

The Top-level Routine `replicate()`

```
Replicate ( motif ) {

  drawPgon ( motif, IDENT ) ;  // Draw central p-gon

  for ( i = 1 to p ) {    // Iterate over each vertex

    qTran = edgeTran[i-1]

    for ( j = 1 to q-2 ) {  // Iterate about a vertex

      exposure = (j == 1) ? MIN_EXP : MAX_EXP ;
      recursiveRep ( motif, qTran, 2, exposure ) ;
      qTran = addToTran ( qTran, -1 ) ;

    }
  }
}
```

The function `addToTran()` is described next.

The Function `addToTran()`

Transformations contain a matrix, the orientation, and an index, `pPosition`, of the edge across which the last transformation was made (`edgeTran[i].pPosition` is the edge matched with edge `i` in the tiling). Here is `addToTran()`

```
addToTran ( tran, shift ) {
  if ( shift % p == 0 ) return tran ;
  else return computeTran ( tran, shift ) ;
}
```

where `computeTran()` is:

```
computeTran ( tran, shift ) {
  newEdge = (tran.pPosition +
             tran.orientation * shift) % p ;
  return tranMult(tran, edgeTran[newEdge]) ;
}
```

and where `tranMult ( t1, t2 )` multiplies the matrices and orientations, sets the `pPosition` to `t2.pPosition`, and returns the result.

The Routine `recursiveRep()`

```
recursiveRep ( motif, initialTran, layer, exposure ) {

   drawPgon ( motif, initialTran ) ;  // Draw p-gon pattern

   if ( layer < maxLayers ) {          // If any more layers
      pShift = ( exposure == MIN_EXP ) ? 1 : 0 ;
      verticesToDo = ( exposure == MIN_EXP ) ? p-3 : p-2 ;

      for ( i = 1 to verticesToDo ) {  // Iterate over vertices
         pTran = computeTran ( initialTran, pShift ) ;
         qSkip = ( i == 1 ) ? -1 : 0 ;
         qTran = addToTran ( pTran, qSkip ) ;
         pgonsToDo = ( i == 1 ) ? q-3 : q-2 ;

         for ( j = 1 to pgonsToDo ) { // Iterate about a vertex
            newExposure = ( i == 1 ) ? MIN_EXP : MAX_EXP ;
            recursiveRep(motif, qTran, layer+1, newExposure);
            qTran = addToTran ( qTran, -1 ) ;
         }
         pShift = (pShift + 1) % p ;  // Advance to next vertex
      }
   }
}
```

Special Cases

The algorithm above works for $p > 3$ and $q > 3$.

If $p = 3$ or $q = 3$, the same algorithm works, but with different values of `pShift, verticesToDo, qSkip,` **etc.**

The case $p = 3$

In `replicate()` **the calculation of** `exposure` **in the inner loop is the same as the general case.**

In `recursiveRep()`**:**

- `pShift = 1` **regardless of exposure.**
- `verticesToDo = 1` **regardless of exposure.**
- `qSkip` **is** `-1` **for** `MIN_EXP` **and** `0` **for** `MAX_EXP`**.**
- `pgonsToDo` **is** `q - 4` **for** `MIN_EXP` **and** `q - 3` **for** `MAX_EXP`**.**
- `newExposure` **is the same as the general case.**

In both `replicate()` **and** `recursiveRep()` **at the last iteration of the inner loop, the call to** `recursiveRep()` **is replaced by a non-recursive call to** `drawPgon()`**.**

The case $q = 3$

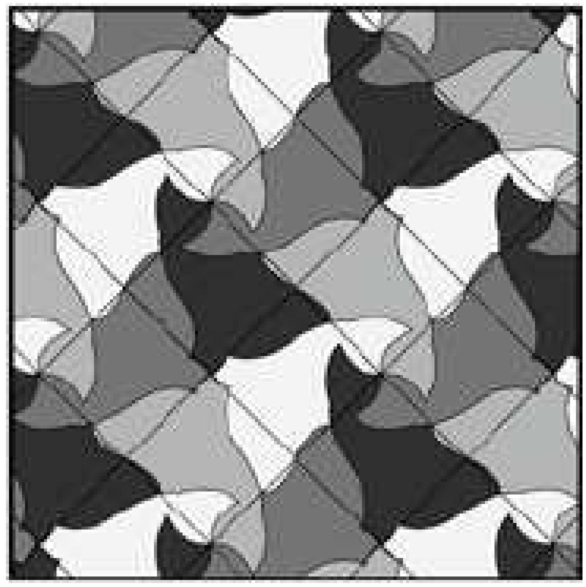
In `replicate()`**,** `exposure = MAX_EXP` **in the inner loop regardless of whether it is the first iteration or not.**

In `recursiveRep()`**:**

- `pShift` **is** `3` **for** `MIN_EXP` **and** `2` **for** `MAX_EXP`**.**
- `verticesToDo` **is** `p - 5` **for** `MIN_EXP` **and** `p - 4` **for** `MAX_EXP`**.**
- `qSkip = 0` **for all cases.**
- `pgonsToDo = 1` **for all cases.**
- `newExposure` **is** `MIN_EXP` **if** `i = 1` **and** `MAX_EXP` **if** `i > 1`**.**

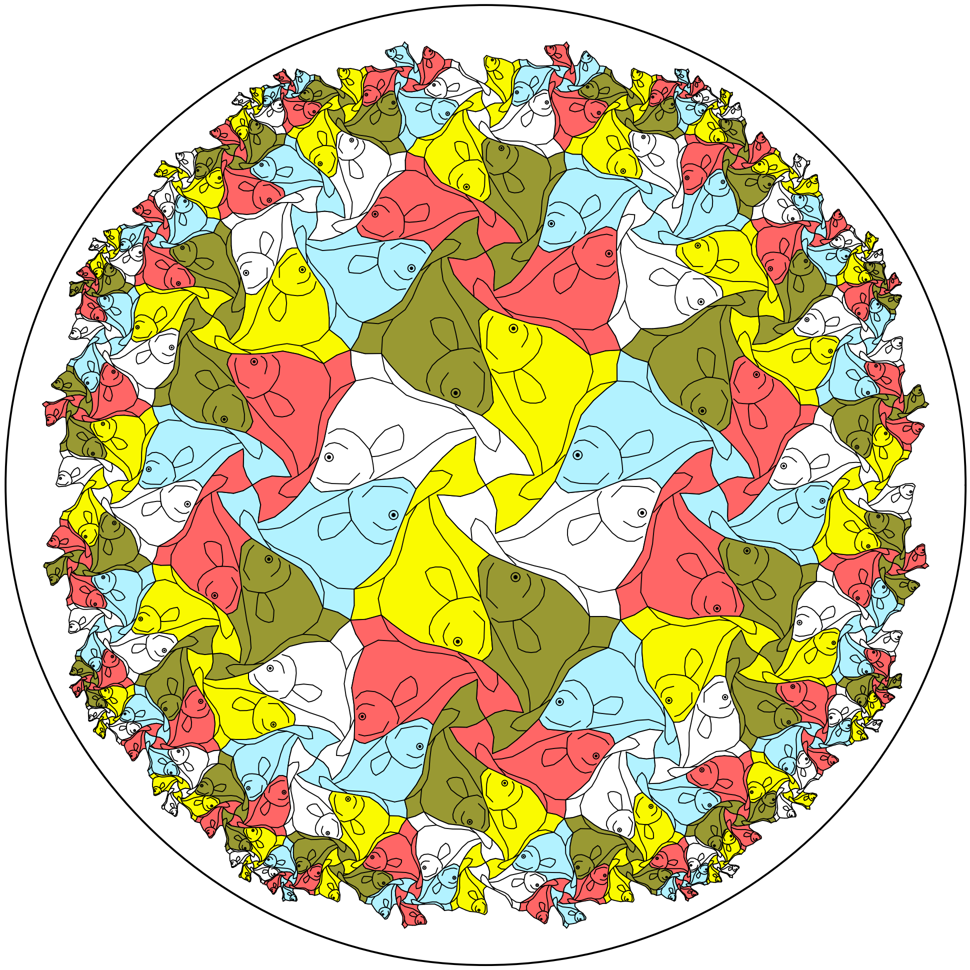
Some New Hyperbolic Patterns

Escher's Euclidean Notebook Drawing 20, based on the $\{4, 4\}$ tessellation.

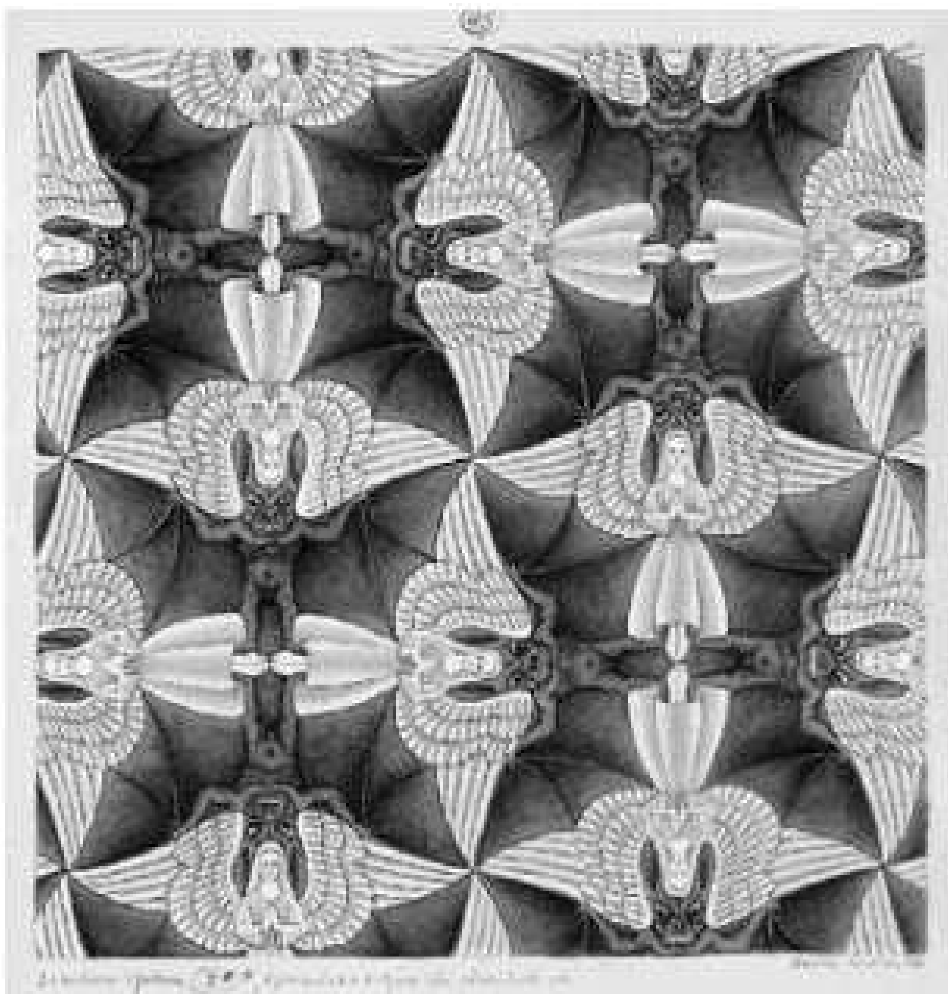


Escher's Spherical Fish Pattern Based on $\{4, 3\}$

A Hyperbolic Fish Pattern Based on $\{4, 5\}$

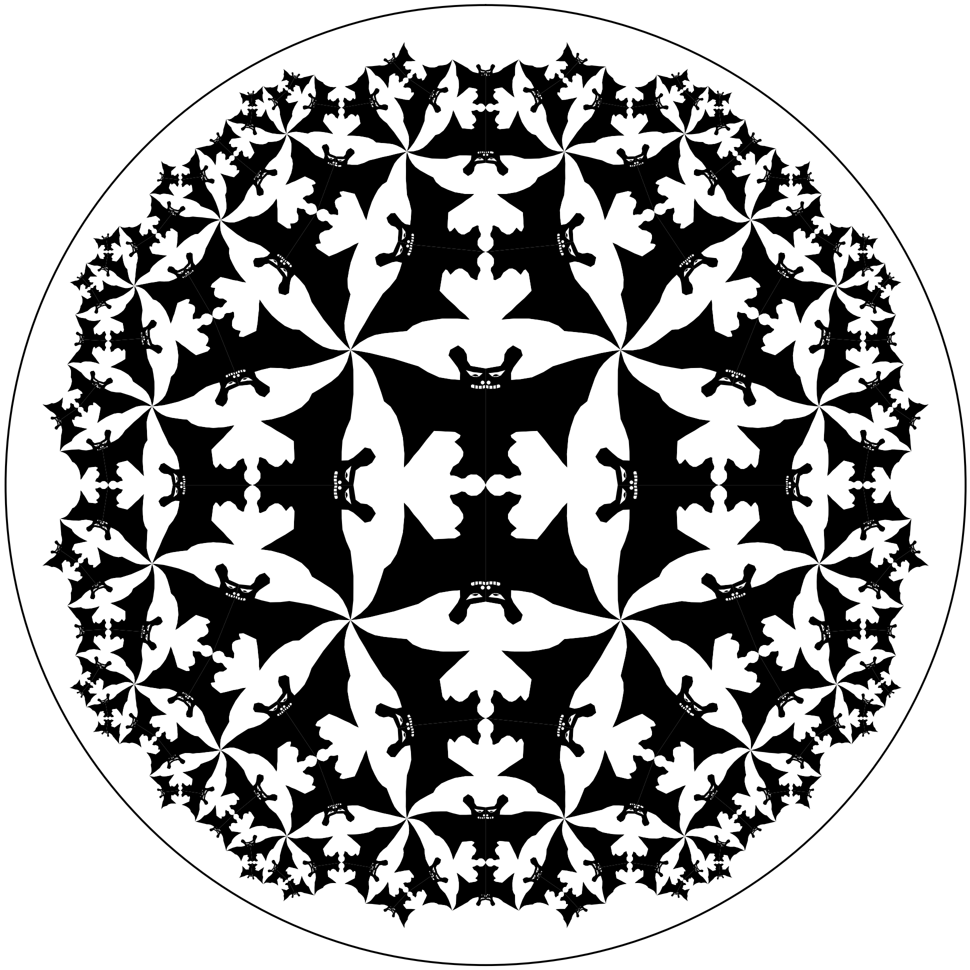


Escher's Euclidean Notebook Drawing 45, based on the $\{4, 4\}$ tessellation.

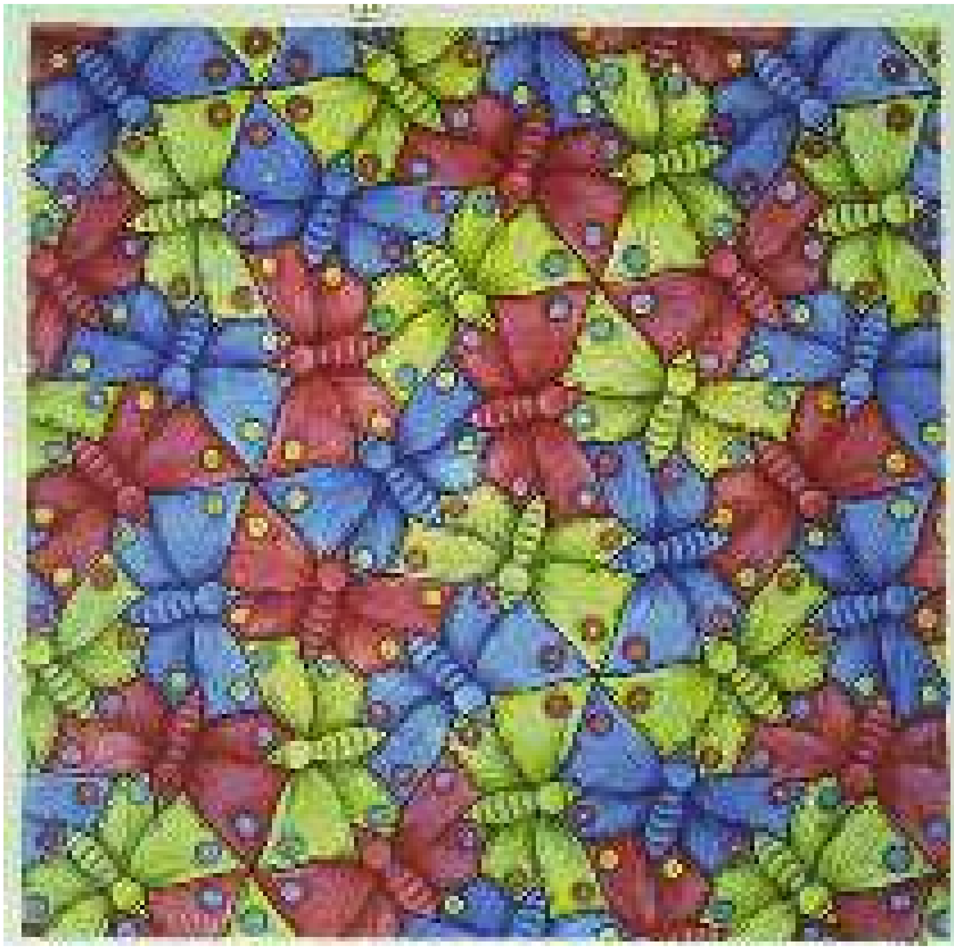


Escher's Spherical "Heaven and Hell" Based on $\{4, 3\}$

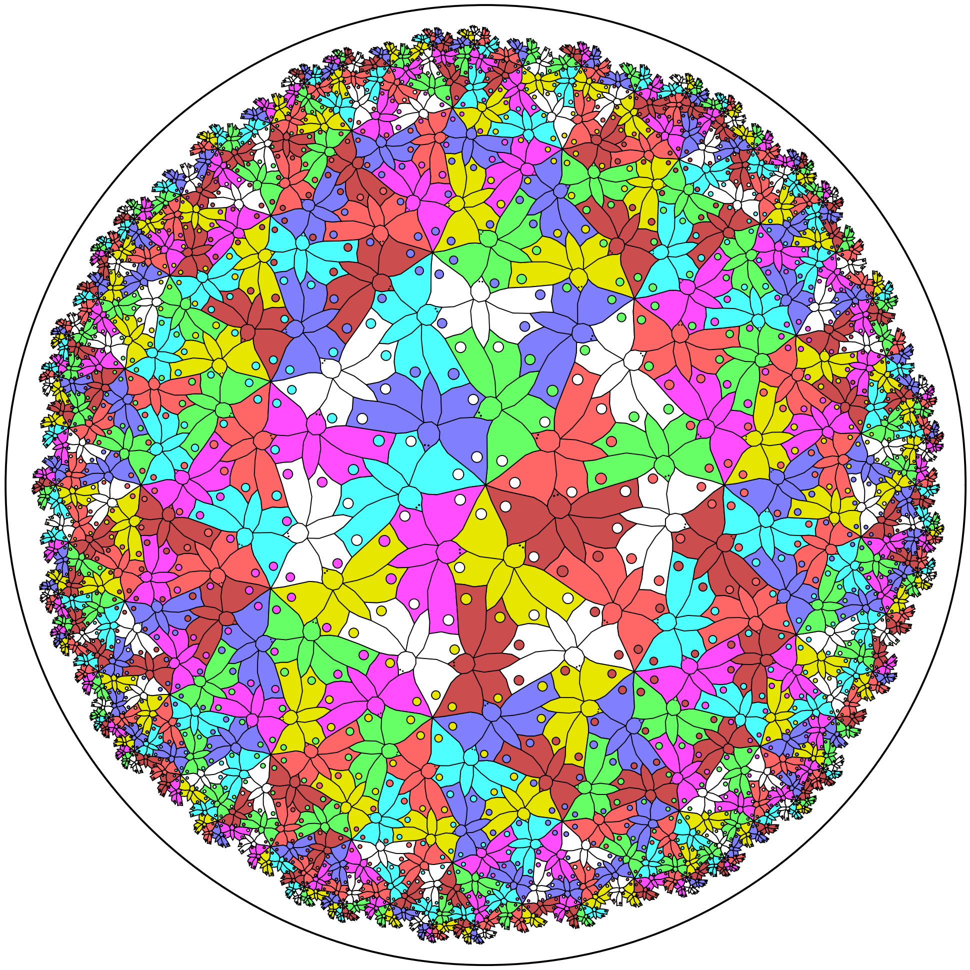
A Hyperbolic "Heaven and Hell" Pattern Based on $\{4, 5\}$



Escher's Euclidean Notebook Drawing 70, based on the $\{6,3\}$ tessellation.



A Hyperbolic Butterfly Pattern Based on $\{7, 3\}$



3. Pattens Based on Non-Regular Polygon Tessellations

A non-regular p-sided polygon with $q_1, q_2, \ldots, q_p$ copies around the respective vertices forms a hyperbolic tessellation provided

$$\sum_{i=1}^{p} \frac{1}{q_i} < \frac{p}{2} - 1$$

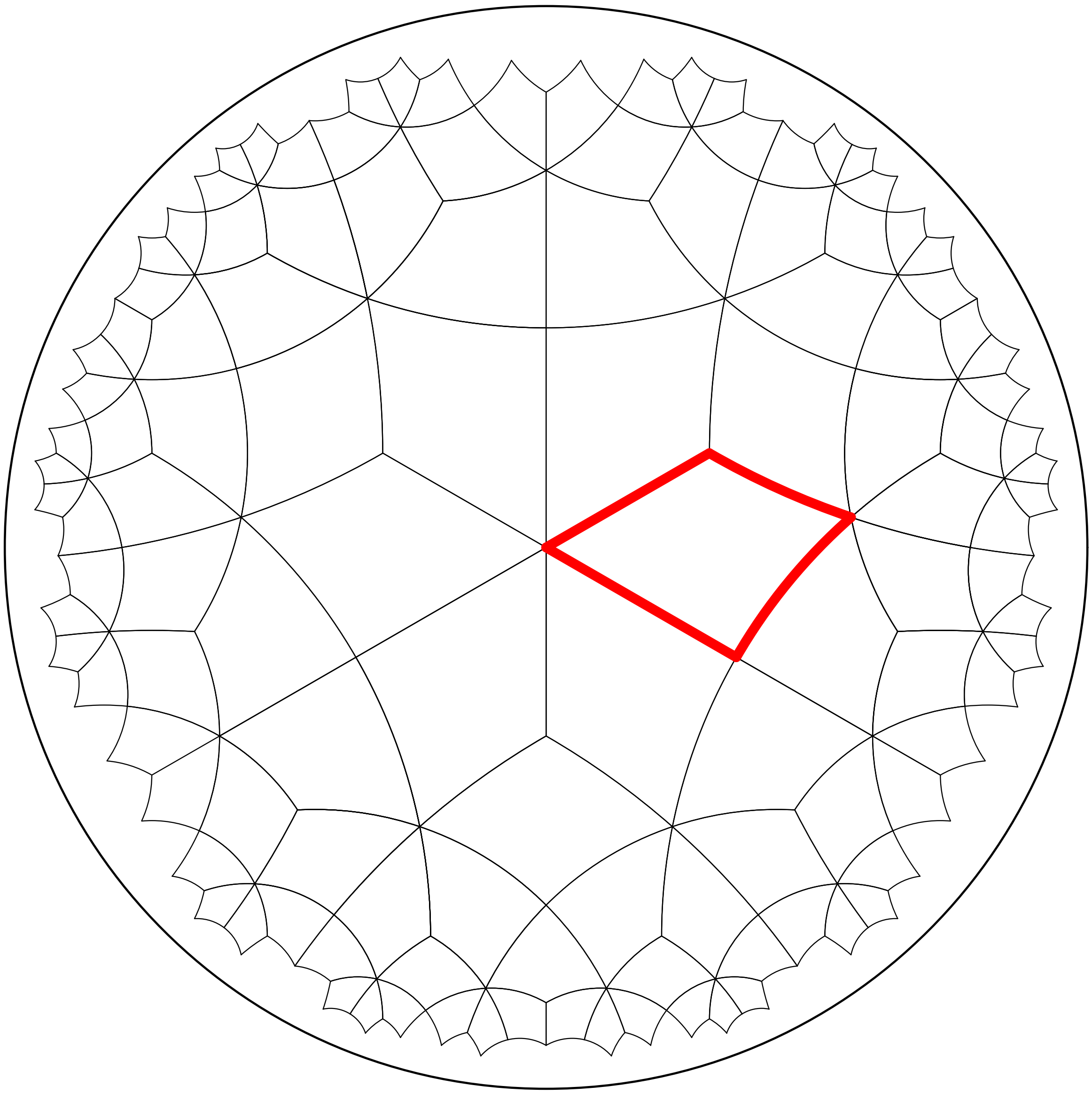
(so the interior angle at the i^{th} vertex is $2\pi/q_i$).

This tessellation is denoted $\{p; q_1, q_2, \ldots, q_p\}$

The pattern drawing algorithm is similar to the case for regular tessellations: a non-recursive "driver", `replicate()` calls a recursive routine `replicateMotif()`.

Unfortunately this algorithm draws multiple copies of the motif if $p = 3$ or if any of the $q_i = 3$. There are only a few duplications near the center, but the number of them grows exponentially in the number of layers.

A $\{4; 6, 3, 6, 4\}$ Polygon Tessellation



The Top-level "Driver" `replicate()`

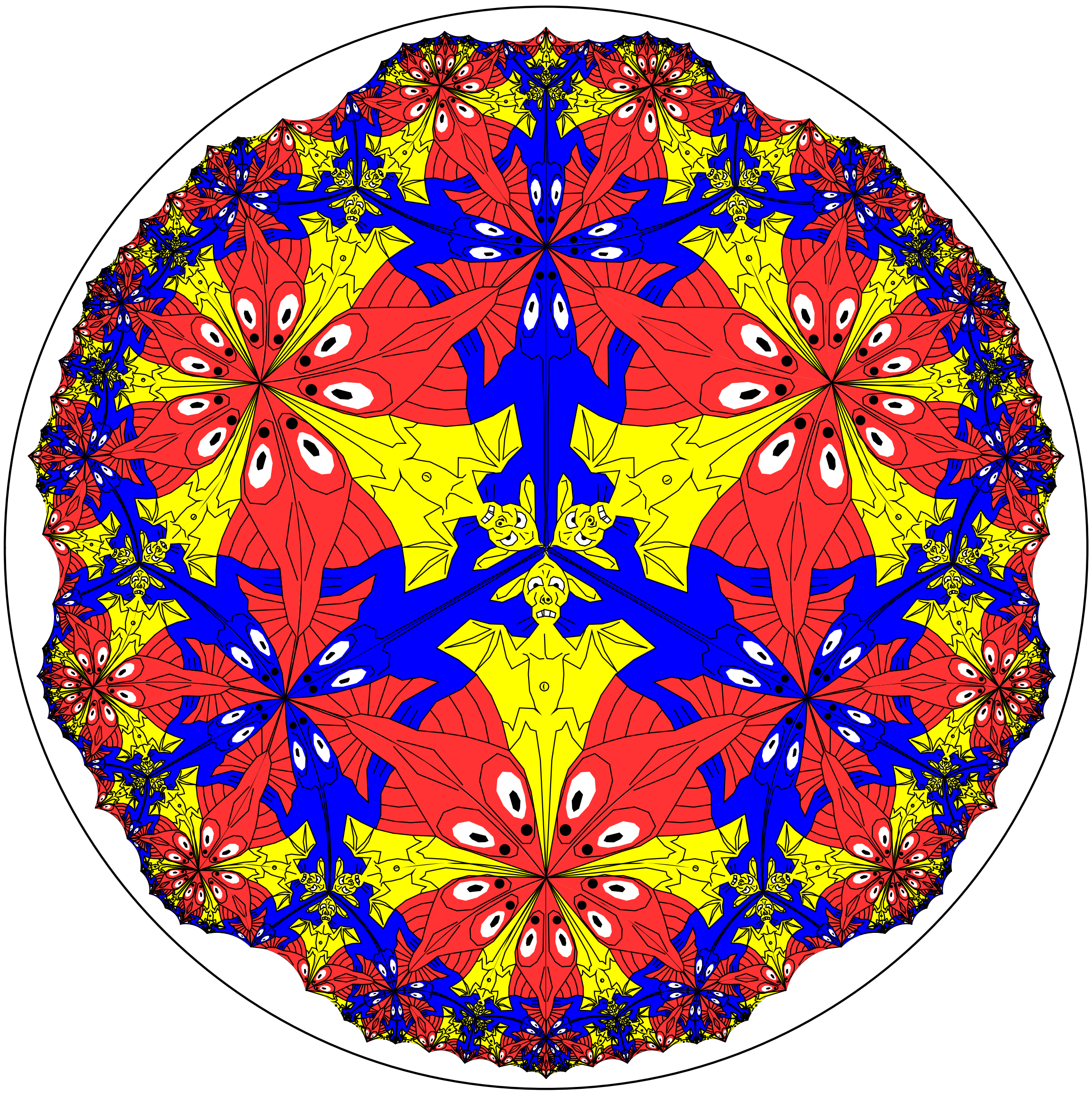
The replication process starts with the following top-level "driver", which calls the recursive routine `replicateMotif()` to create the rest of the pattern.

```
replicate ( motif )
{
  for ( j = 1 to q[1] )
  {
    qTran = edgeTran[1] ;

    replicateMotif(motif,qTran,2,MAX_EXP);

    qTran = addToTran ( qTran, -1 ) ;
  }
}
```

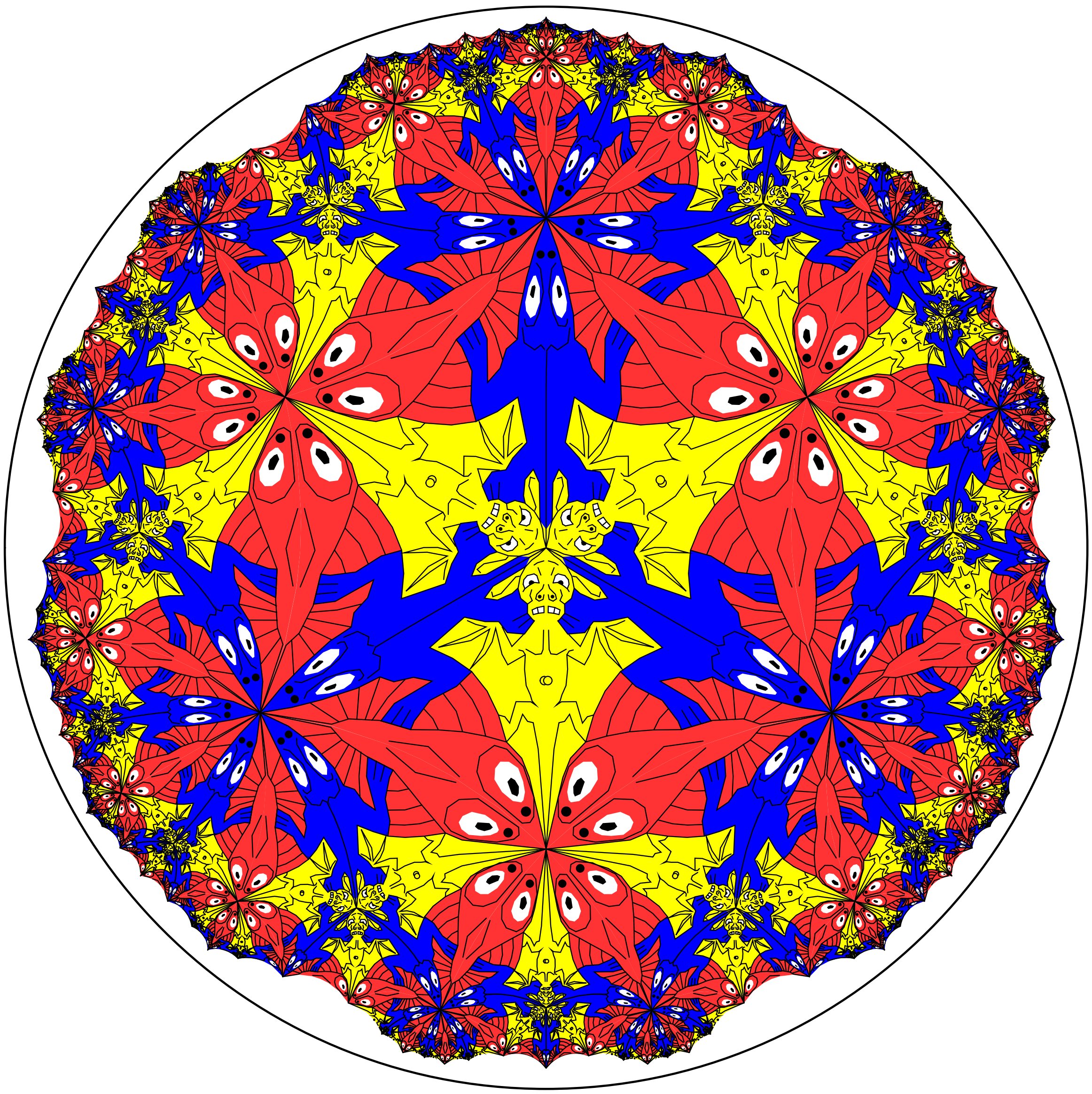
The Recursive Routine `replicateMotif()`

```
replicateMotif(motif, inTran, layer, exposure)
{
  drawMotif ( motif, inTran ) ;
  if ( layer < maxLayers )
  {
    pShift = pShiftArray[exposure] ;
    verticesToDo = p -
                  verticesToSkipArray[exposure] ;

    for ( i = 1 to verticesToDo )
    {
      pTran = computeTran(initialTran, pShift) ;
      first_i = ( i == 1 ) ;
      qTran = addToTran(pTran, qShiftArray[first_i]) ;
      if ( pTran.orientation > 0 )
         vertex = (pTran.pposition-1) % p ;
      else
         vertex =  pTran.pposition ;
      polygonsToDo = q[vertex] -
                  polygonsToSkipArray[first_i] ;

      for ( j = 1 to polygonsToDo )
      {
        first_j = ( j == 1 ) ;
        newExpose = exposureArray[first_j] ;

        replicateMotif(motif, qTran, layer+1, newExpose) ;
        qTran = addToTran ( qTran, -1 ) ;
      }
      pShift = (pShift + 1) % p ;
    }
  }
}
```

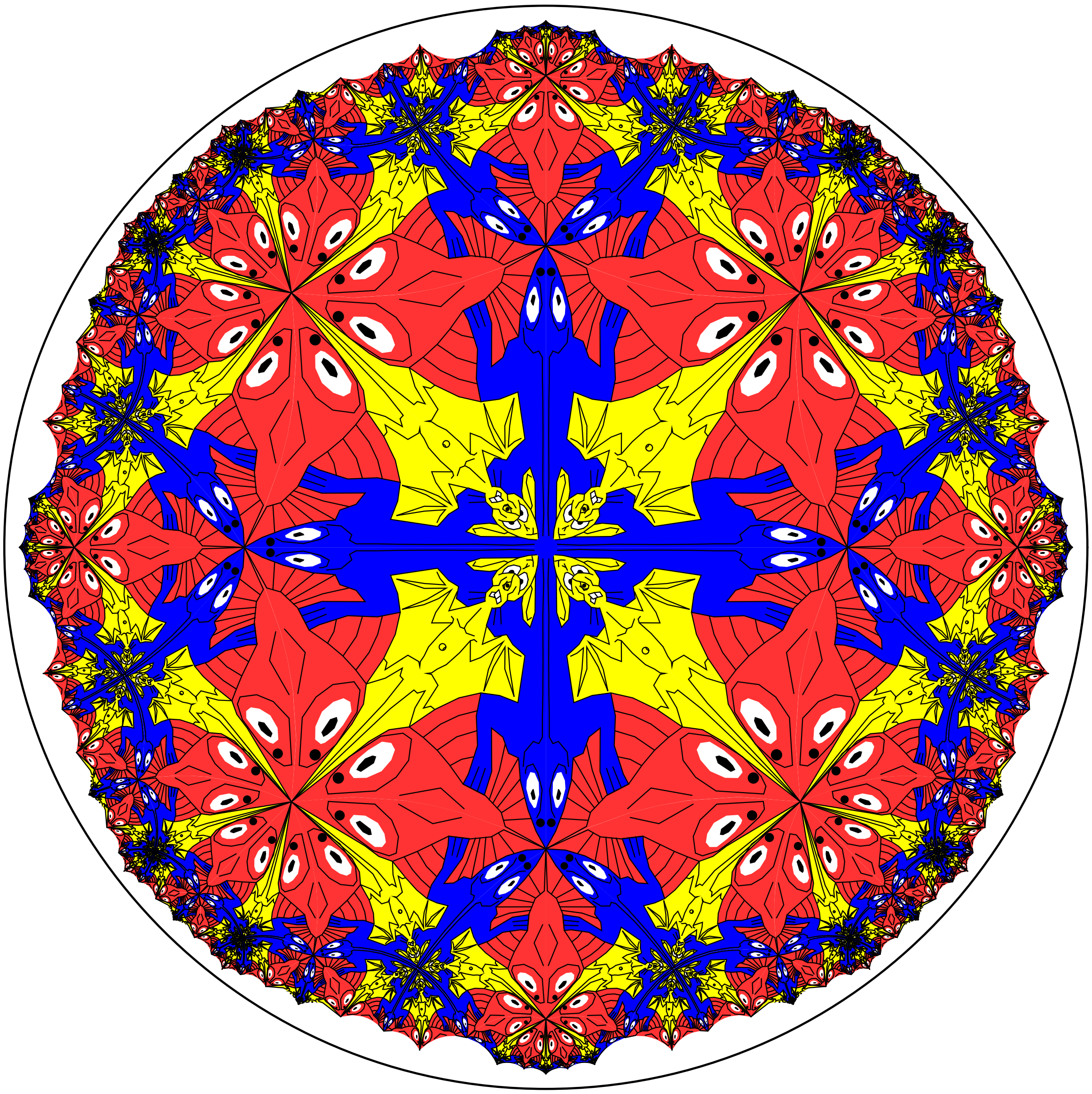
A "Three Element" Pattern with Different Numbers of Animals Meeting at their Heads



A "Three Element" Pattern with 3 Bats, 5 Lizards, and 4 Fish Meeting at their Heads



A “Three Element” Pattern with 3 Bats, 5 Lizards, and 4 Fish Meeting at their Heads



4. Future Work

- **Fix the non-regular polygon tessellation algorithm so that it does not make duplicate copies of the motif at some locations.**
- **Allow some or all of the vertices of the fundamental polygon to lie on the bounding circle.**
- **Automatically generate patterns with color symmetry.**

The End

I hope not!

Escher's Euclidean Notebook Drawing 42, based on the $\{4, 4\}$ tessellation.

A Hyperbolic Shell Pattern Based on $\{4, 5\}$