ESMA 2010

M.C. Escher's Use of the Poincaré Models of Hyperbolic Geometry

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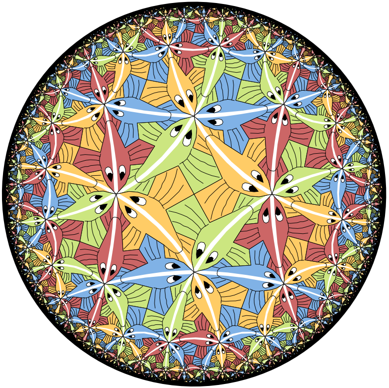


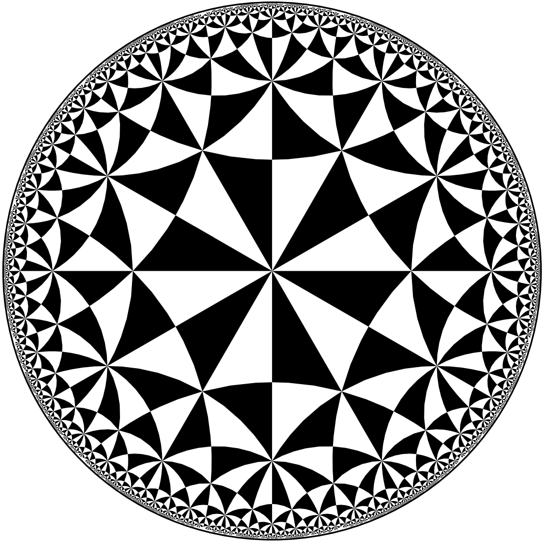
Outline

- Overview of Escher's use of Hyperbolic Geometry
- Repeating patterns and regular tessellations
- Symmetries and color symmetry
- Color symmetry of a family of fish patterns
- Escher's "Circle Limit" patterns and their color symmetry
- Escher's use of the Poincaré half-plane model
- Future research

Escher's use of Hyperbolic Geometry

- In 1957 the Canadian mathematician H.S.M. Coxeter sent Escher a reprint of a journal article containing a triangle pattern displayed in the Poincaré disk model of hyperbolic geometry.
- Escher had long sought to represent an infinite pattern in a finite area. Coxeter's triangle pattern gave Escher "quite a shock", since it instantly showed him how this could be done.
- At that time Escher had created two of what he called "line limit" patterns, which are based on the Poincaré half-plane model of hyperbolic geometry.
- Thus inspired by Coxeter, Escher went on to create his four "Circle Limit" patterns from 1958 to 1960.

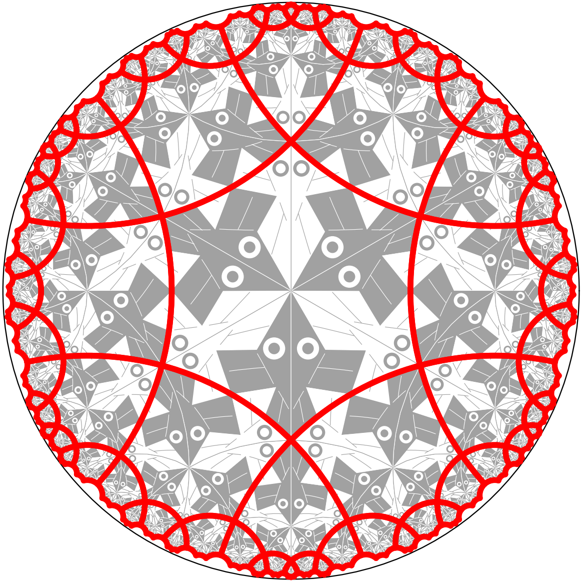
The Triangle Pattern that Coxeter sent Escher



Repeating Patterns and Regular Tessellations

- A *repeating pattern* in any of the 3 "classical geometries" (Euclidean, spherical, and hyperbolic geometry) is composed of congruent copies of a basic subpattern or *motif*.
- For example if we ignore color, one fish is a motif for *Circle Limit III*.
- The *regular tessellation*, $\{p, q\}$, is an important kind of repeating pattern composed of regular p-sided polygons meeting q at a vertex.
- If $(p-2)(q-2) < 4$, $\{p, q\}$ is a spherical tessellation (assuming $p > 2$ and $q > 2$ to avoid special cases).
- If $(p-2)(q-2) = 4$, $\{p, q\}$ is a Euclidean tessellation.
- If $(p-2)(q-2) > 4$, $\{p, q\}$ is a hyperbolic tessellation. The next slide shows the $\{6, 4\}$ tessellation.
- Escher based his 4 "Circle Limit" patterns, and many of his spherical and Euclidean patterns on regular tessellations.

The Regular Tessellation $\{6, 4\}$ Superimposed on Circle Limit I



A Table of the Regular Tessellations

q \ p	3	4	5	6	7	8	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
8	$*$	$*$	$*$	$*$	$*$	$*$	$\cdots$
7	$*$	$*$	$*$	$*$	$*$	$*$	$\cdots$
6	□	$*$	$*$	$*$	$*$	$*$	$\cdots$
5	○	$*$	$*$	$*$	$*$	$*$	$\cdots$
4	○	□	$*$	$*$	$*$	$*$	$\cdots$
3	○	○	○	□	$*$	$*$	$\cdots$

□ - Euclidean tessellations

○ - spherical tessellations

$*$ - hyperbolic tessellations

Symmetries and Color Symmetry

- A *symmetry* of a repeating pattern is an isometry that maps the pattern onto itself. Thus each motif goes onto another copy of the motif.
- There are 4-fold rotation symmetries about the right fins in *Circle Limit III*, and 3-fold rotations about the left fins and noses.
- A *color symmetry* of a pattern of colored motifs is a symmetry of the uncolored pattern that takes all motifs of one color to motifs of a single color — that is, it permutes the colors of the motifs.
- A pattern has (perfect) color symmetry if the mapping from the symmetry group of the uncolored pattern to the permutation group of the colors is onto a transitive subgroup of the color permutation group.

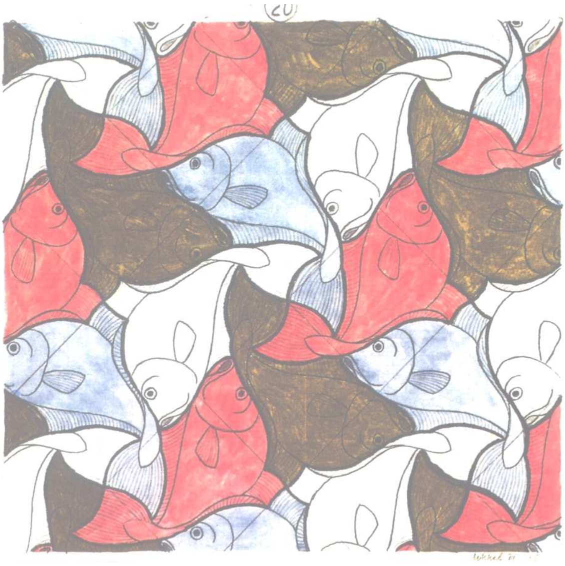
History of the Theory of Color Symmetry

- H.J. Woods analyzed 2-color symmetry in "Counterchange Symmetry in Plane Patterns" in *Journal of the Textile Institute* (Manchester) in 1936.
- In 1961, B.L. Van der Waerden and J.J. Burckhardt defined what we now call (perfect) color symmetry in "Farbgrupen" in *Zeitschrift für Kristallographie*.
- Color symmetry is one of the hallmarks of Escher patterns, another is that the motifs tile the plane without gaps or overlaps. Of course in coloring patterns, Escher also adhered to the map-coloring principle.
- Escher created patterns with 2- and 3-color in the mid 1920's, and in the late 1930's he created patterns with 4-color symmetry.

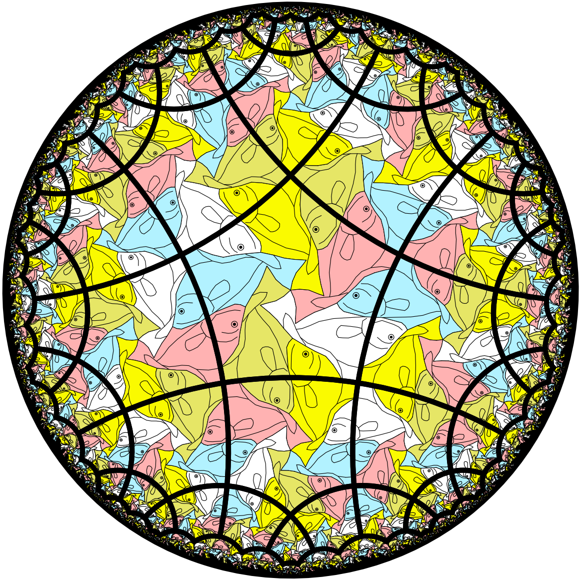
A Family of Fish Patterns

- In 1938 Escher created his Regular Division Drawing Number 20, with a fish motif and 4-color symmetry.
- This pattern is just one of a theoretically infinite family of fish patterns based on $\{p, q\}$ tessellations.

Escher's Regular Division Drawing Number 20 with 4-color symmetry (1938)



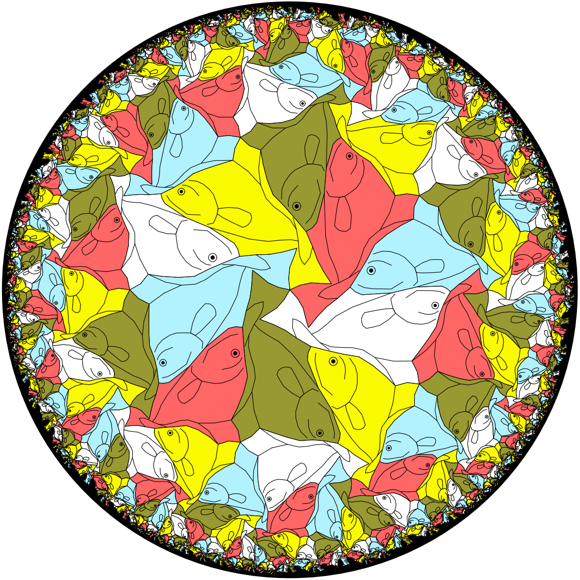
A Hyperbolic Fish Pattern Based on $\{5, 4\}$



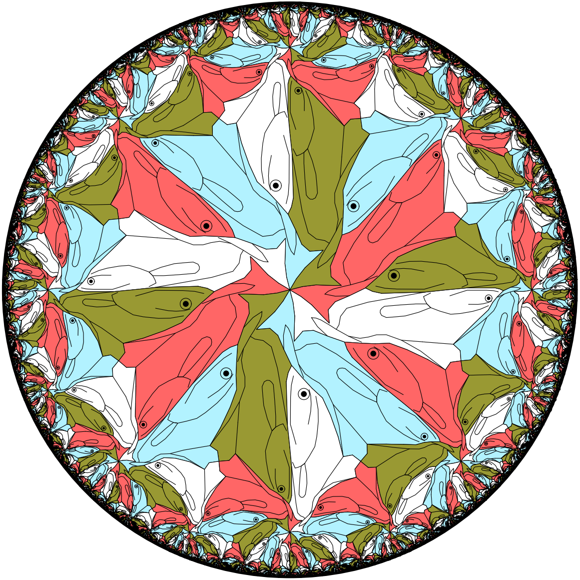
The Color Symmetry of Fish Patterns

- Theoretically, we can create a fish pattern based on $\{p, q\}$ like the one above for any values of p and q provided $p \geq 3$ and $q \geq 3$.
- For these patterns, p is the number of fish that meet their tails and q is the number of fish that meet at their dorsal fins.
- This family of fish patterns is based on Escher's 4-colored Notebook Drawing Number 20 above, which is based on the Euclidean "square" tessellation $\{4, 4\}$.
- For Notebook Drawing Number 20, at least four colors are needed for color symmetry and to satisfy the map-coloring principle.
- The hyperbolic fish pattern based on the $\{5, 4\}$ tessellation requires at least five colors for color symmetry since five is prime.

A 5-colored Fish Pattern Based on $\{5, 5\}$



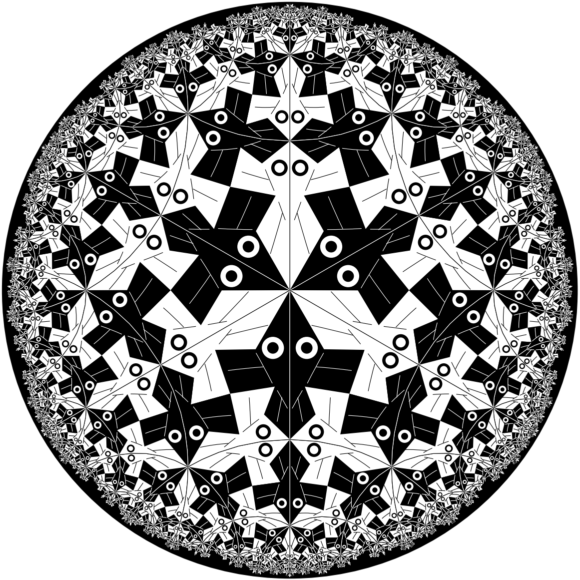
A 4-colored Pattern of (distorted) Fish Based on $\{8, 4\}$



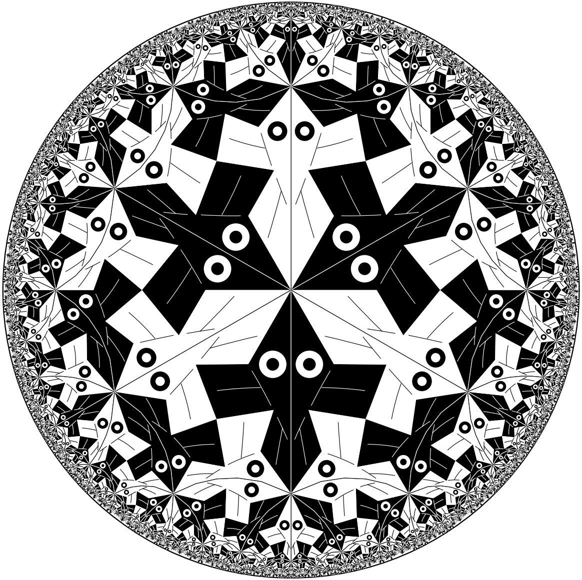
Color Symmetry of Escher's "Circle Limits"

- *Circle Limit I* does not have color symmetry, but related patterns do. For a pattern in the *Circle Limit I* family, p and q must be even due to reflection lines across the backbones of the fish. To obtain 2-color symmetry, p must equal q.
- *Circle Limit II* has 3-color symmetry.
- *Circle Limit III* has 4-color symmetry, and cannot be symmetrically colored with fewer colors.
- Patterns in the *Circle Limit IV* family cannot have color symmetry.

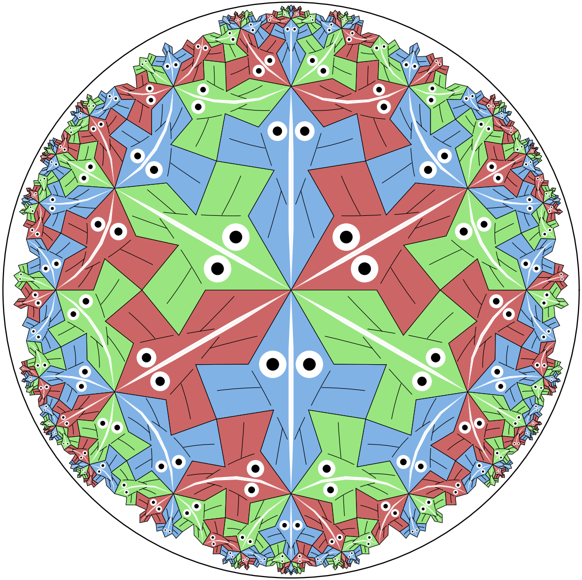
Escher's Circle Limit I $\{6, 4\}$ Pattern
No color symmetry



A 2-colored Circle Limit I Pattern Based on the $\{6, 6\}$ Tessellation

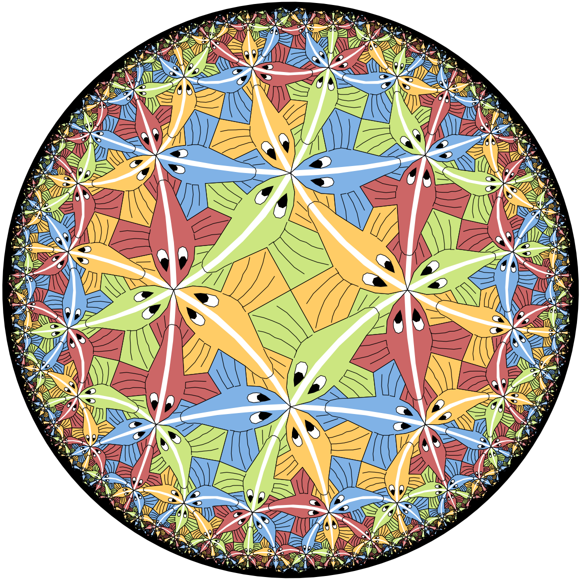


A 3-colored Circle Limit I Pattern
Based on the $\{6, 6\}$ Tessellation

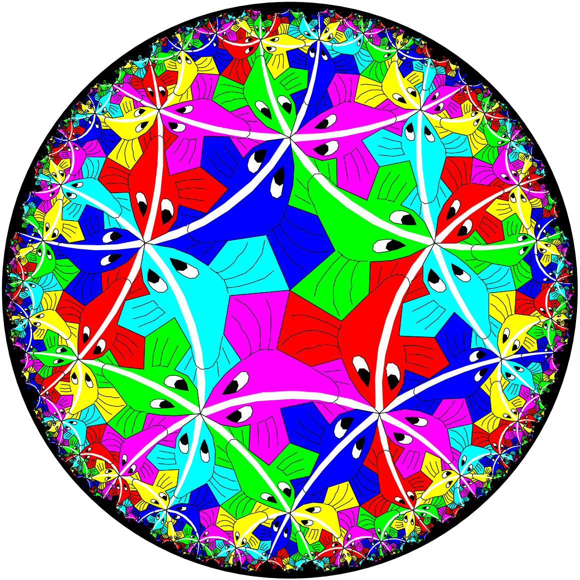


Escher's Circle Limit II — 3-colored

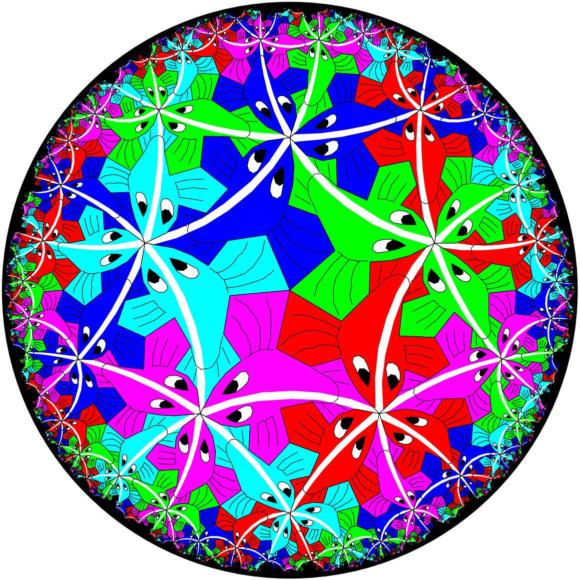
Escher's Circle Limit III
4 colors are necessary for color symmetry



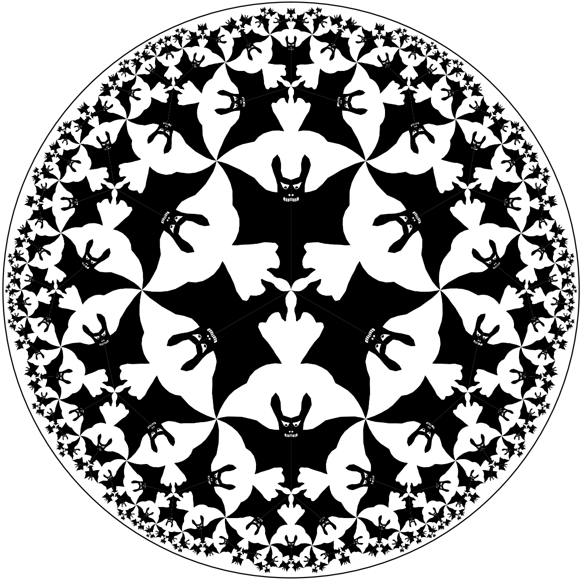
A $(5, 3, 3)$ Circle Limit III pattern
Needs 6 colors to maintain colors on backbones.



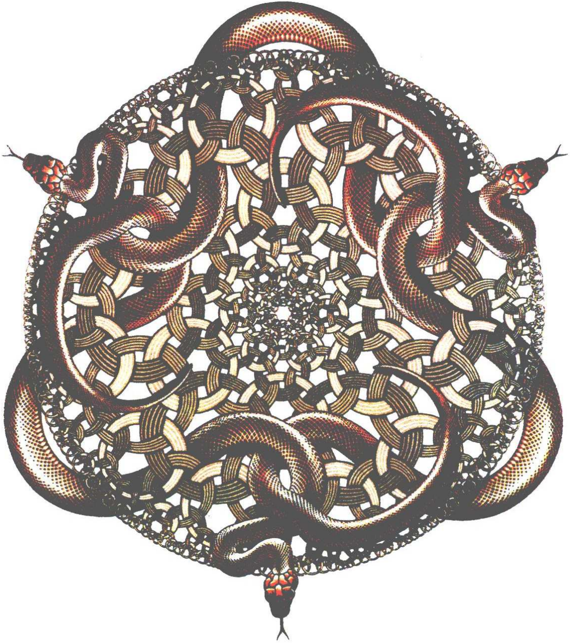
A 5-coloring of the $(5, 3, 3)$ pattern
Colors on backbone lines alternate



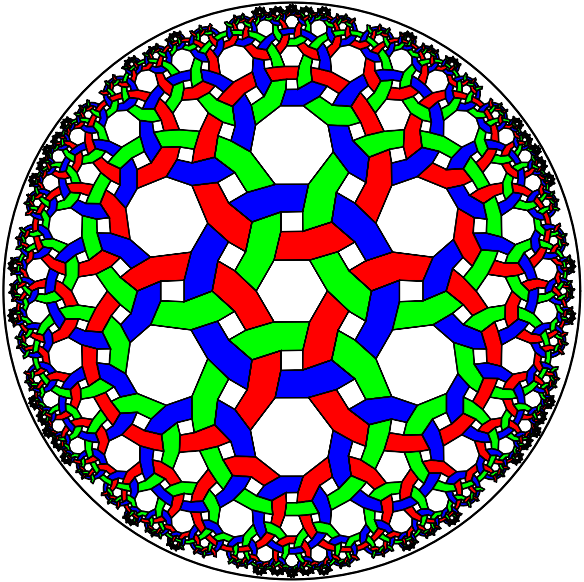
Escher's Circle Limit IV Pattern
No color symmetry



Snakes — Another (partial) Circle Limit Pattern



The Hyperbolic Pattern of Rings in Snakes with 2-color Symmetry

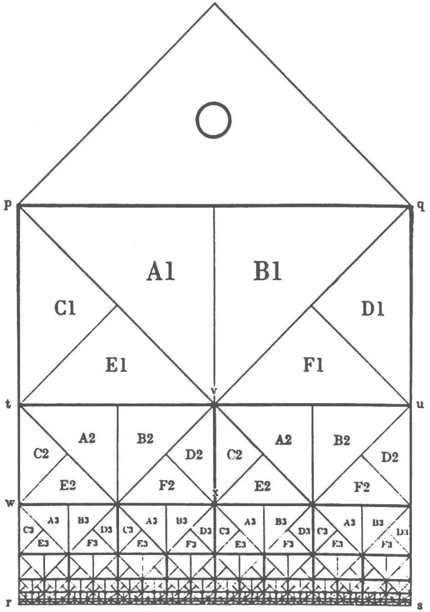


Escher Use of the Poincaré Half-plane Model

Regular Division Drawing 101 (1956) — No color symmetry

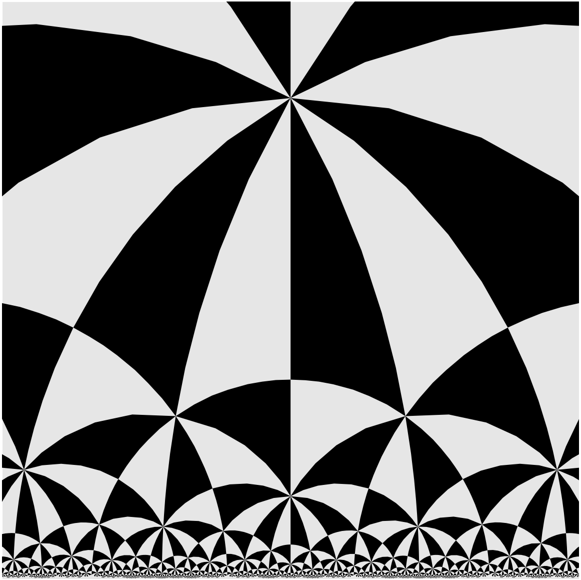
Regular Division of the Plane VI (1957)
No color symmetry

Study for Regular Division of the Plane VI

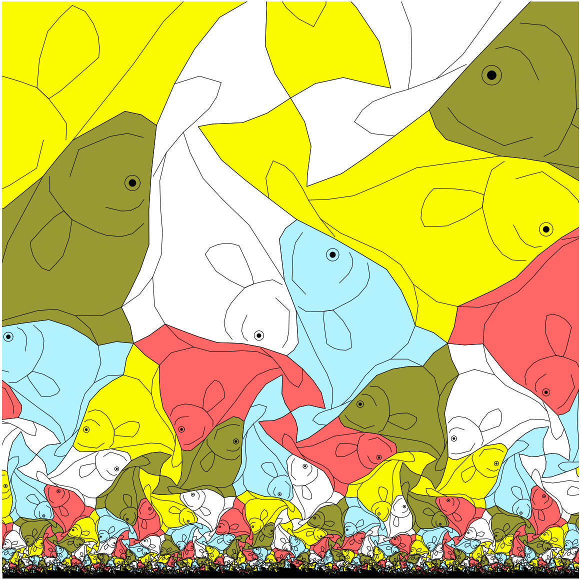


Square Limit (1964)
only 2-color Central Rotation Symmetry

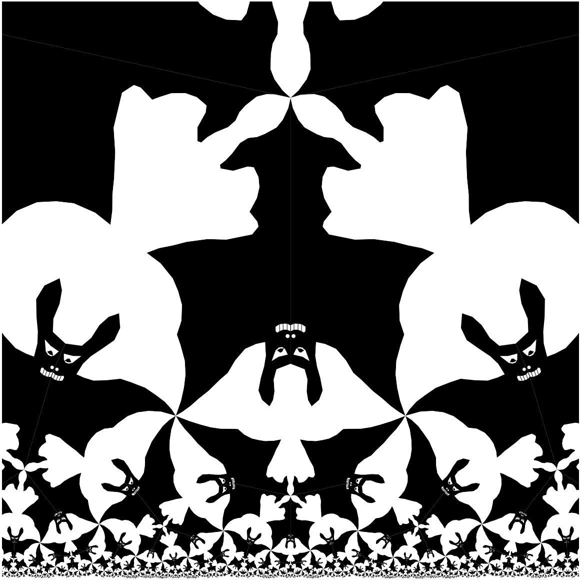
A Half-plane Version of the (6,4,2) Triangle Pattern



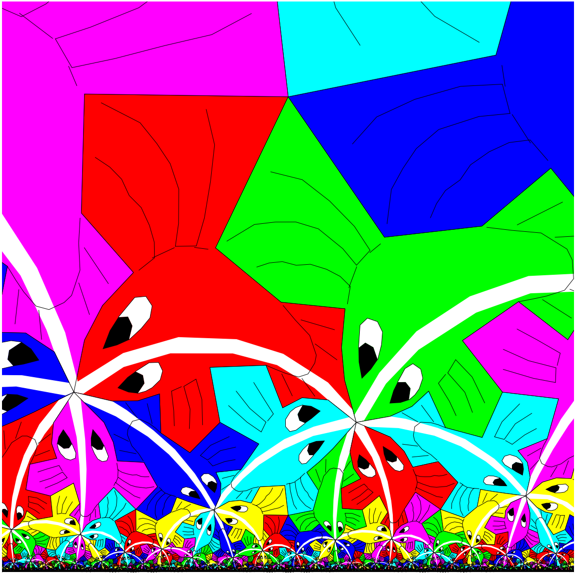
A Half-plane Version of a $\{4, 5\}$ Fish Pattern



A Half-plane Version of Circle Limit IV



A Half-plane Version of the $(5, 3, 3)$ Pattern



Future Work

- Automatically generate the colors so that the pattern is symmetrically colored. Currently this must be done manually for each pattern in a family.
- For any pattern, generate a coloring with the minimum number of colors.
- Extend such a generation algorithm so that it can handle additional restrictions, such as using the same color for fish along each backbone line of a *Circle Limit III* pattern.

Thank You!