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Angels and Devils on Triply Periodic Polyhedra

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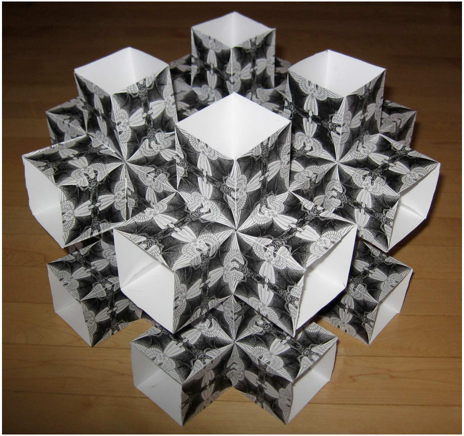


Outline

- Escher's Angels and Devils
- Triply periodic polyhedra
- Hyperbolic geometry and regular tessellations
- Relation between periodic polyhedra and regular tessellations
- An angels and devils pattern on the $\{4, 6|4\}$ polyhedron
- An angels and devils pattern on a $\{4, 5\}$ polyhedron
- An angels and devils pattern on another $\{4, 5\}$ polyhedron
- A Pattern of butterflies on a $\{3, 8\}$ polyhedron
- A fish pattern on the $\{6, 6|3\}$ polyhedron
- Future research

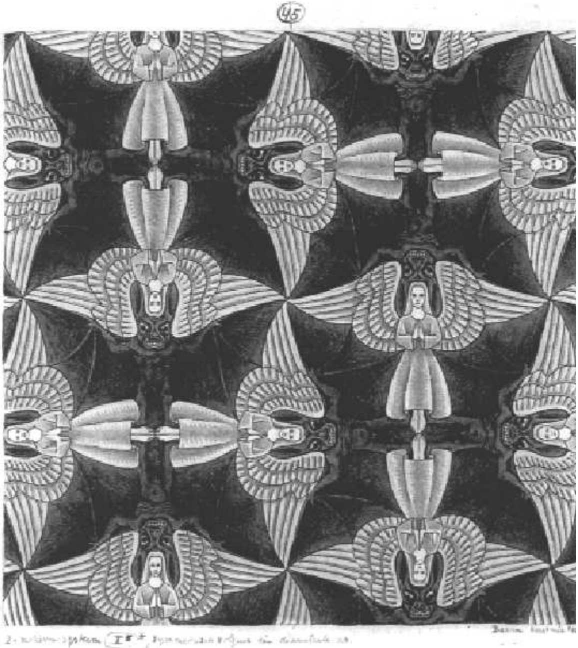
Escher's Angels and Devils

M.C. Escher created angels and devils patterns in each of the three "classical" geometries:

- Regular Division Drawing Number 45 — a repeating pattern in the Euclidean plane (1941).
- A carved maple ball — a spherical pattern (1942).
- His print *Circle Limit IV* — a repeating pattern in the hyperbolic plane (1960).

Recently I have placed angels and devils patterns on triply periodic polyhedra, as shown on the title slide.

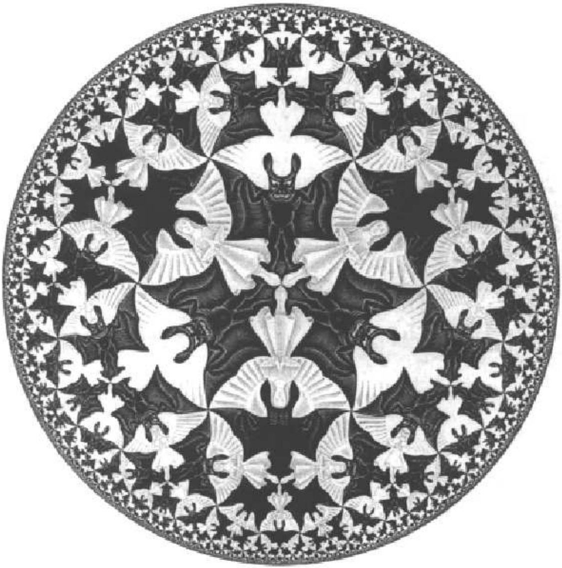
Escher's Regular Division Drawing Number 45



Escher's Carved Maple Sphere with Angels and Devils

"Heaven and Hell" carved sphere, 1942. Maple, stained in two colors, diameter 235 mm.

Escher's Hyperblic Circle Limit IV Print



Triply Periodic Polyhedra

- A *triply periodic polyhedron* is a (non-closed) polyhedron that repeats in three different directions in Euclidean 3-space.
- The most symmetric ones are the *regular skew polyhedra*, which are "flag-transitive" — there is a symmetry of the polyhedron that takes any vertex, edge containing that vertex, and face containing that edge to any other such (vertex, edge, face) combination. These are natural analogs of the Platonic solids.
- In 1926 John Petrie and H.S.M. Coxeter proved that there are exactly three regular skew polyhedra, which Coxeter called $\{4, 6|4\}$, $\{6, 4|4\}$, and $\{6, 6|3\}$, where $\{p, q|r\}$ denotes a polyhedron made up of p-sided regular polygons meeting q at a vertex, and with regular r-sided holes.

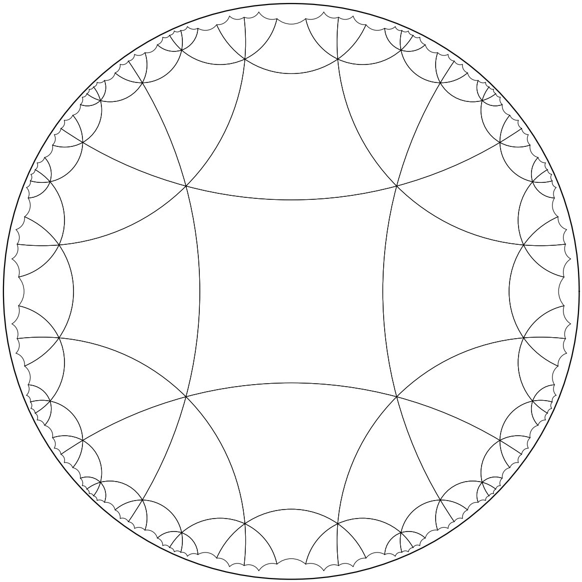
Hyperbolic Geometry and Regular Tessellations

- In 1901, David Hilbert proved that, unlike the sphere, there was no isometric (distance-preserving) embedding of the hyperbolic plane into ordinary Euclidean 3-space.
- Thus we must use *models* of hyperbolic geometry in which Euclidean objects have hyperbolic meaning, and which must distort distance.
- One such model is the *Poincaré disk model*. The hyperbolic points in this model are represented by interior point of a Euclidean circle — the *bounding circle*. The hyperbolic lines are represented by (internal) circular arcs that are perpendicular to the bounding circle (with diameters as special cases).
- This model is appealing to artists since (1) angles have their Euclidean measure (i.e. it is conformal), so that motifs of a repeating pattern retain their approximate shape as they get smaller toward the edge of the bounding circle, and (2) it can display an entire pattern in a finite area.

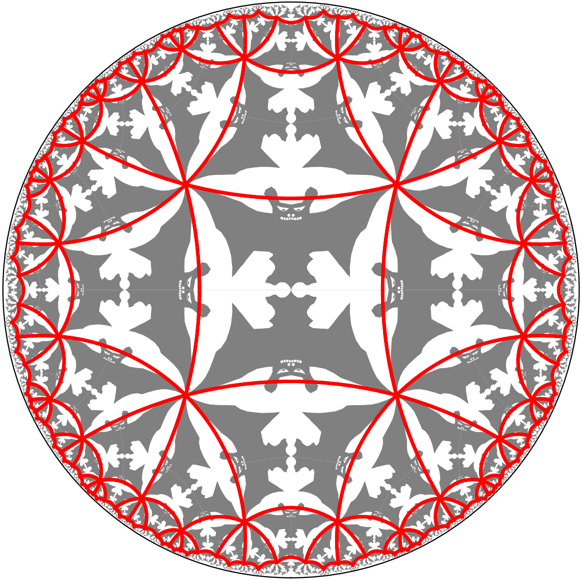
Repeating Patterns and Regular Tessellations

- A *repeating pattern* in any of the 3 "classical geometries" (Euclidean, spherical, and hyperbolic geometry) is composed of congruent copies of a basic subpattern or *motif*.
- The *regular tessellation*, $\{p, q\}$, is an important kind of repeating pattern composed of regular p-sided polygons meeting q at a vertex.
- If $(p-2)(q-2) < 4$, $\{p, q\}$ is a spherical tessellation (assuming $p > 2$ and $q > 2$ to avoid special cases).
- If $(p-2)(q-2) = 4$, $\{p, q\}$ is a Euclidean tessellation.
- If $(p-2)(q-2) > 4$, $\{p, q\}$ is a hyperbolic tessellation. The next slide shows the $\{6, 4\}$ tessellation.
- Escher based his 4 "Circle Limit" patterns, and many of his spherical and Euclidean patterns on regular tessellations.

The Regular Tessellation $\{4, 6\}$



The tessellation $\{4, 6\}$ superimposed on the pattern of angels and devils of the title slide pattern



Relation between periodic polyhedra and regular tessellations — a 2-Step Process

- (1) Some triply periodic polyhedra approximate triply periodic minimal surfaces (TPMS's).

  As a bonus, some triply periodic polyhedra contain embedded Euclidean lines which are also lines embedded in the corresponding TPMS. These lines form skew rhombi for the regular skew polyhedra. If these skew rhombi are spanned by "soap films", one obtains the corresponding TPMS.

- (2) As a minimal surface, a TPMS has negative curvature (except for isolated points of zero curvature), and so its universal covering surface also has negative curvature and thus has the same large-scale geometry as the hyperbolic plane.

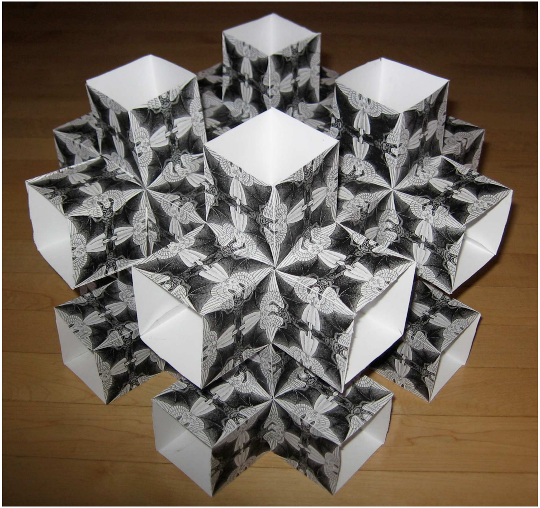
  So the polygons of the triply periodic polyhedron can be transferred to the polygons of a corresponding regular tessellation of the hyperbolic plane, and similarly a pattern on such a polyhedron can be "lifted" to a *universal covering pattern* in the hyperbolic plane.

An Angels and Devils Pattern on the $\{4, 6|4\}$ Polyhedron

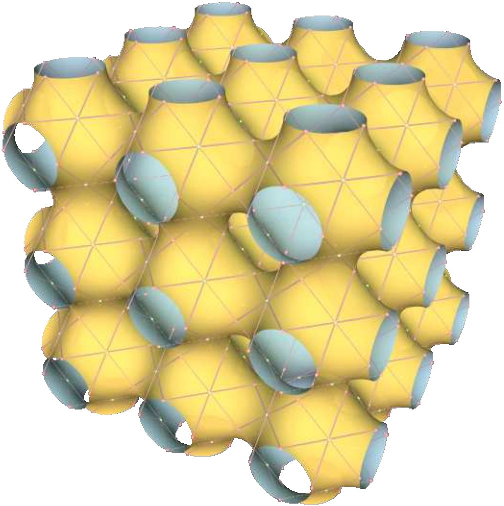
The $\{4, 6|4\}$ polyhedron is easiest to understand. It consists of invisble "hub" cubes connected by "strut" cubes on all 6 faces of the hubs. We show the 2-step relation between the patterned $\{4, 6|4\}$ polyhedron and its "universal covering pattern" as follows:

- The pattern of the Title Slide, which we have seen.
- Schwarz's P-surface, the TPMS that is approximated by the $\{4, 6|4\}$ polyhedron, showing its embedded lines.
- A close-up of Schwarz's P-surface showing the skew rhombi.
- The hyperbolic "universal covering pattern" of the Title Slide polyhedron.

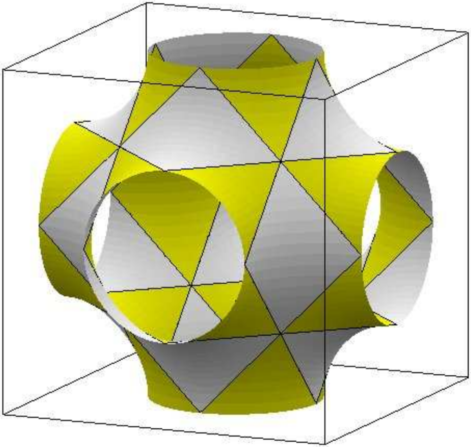
The triply periodic polyhedron of the Title Slide



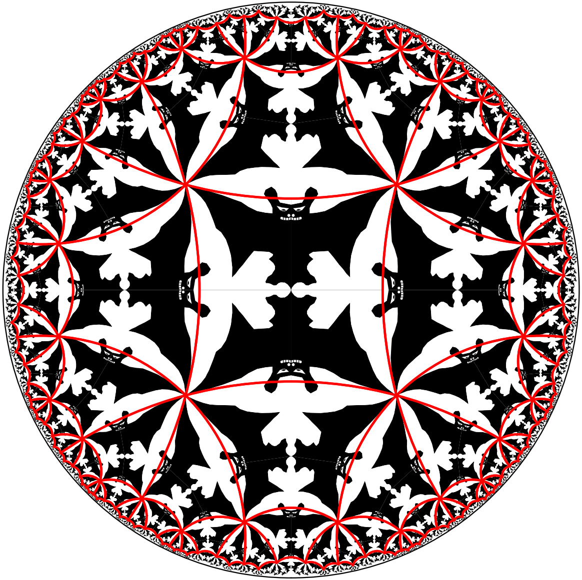
Schwarz's P-surface — approximated by the previous triply periodic polyhedron, and showing embedded lines



A close-up of Schwarz's P-surface showing corresponding embedded lines and "skew rhombi"



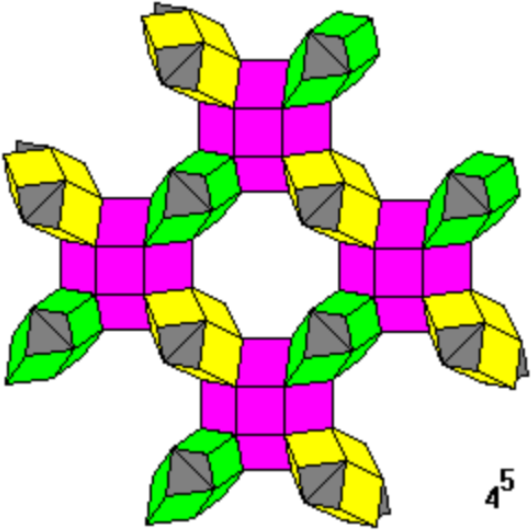
The pattern of the Title Slide "lifted" to its hyperbolic "universal covering pattern". — showing hyperbolic 4-gons corresponding to the square faces of the polyhedron.



Filling a "Gap" — a $\{4, 5\}$ Pattern

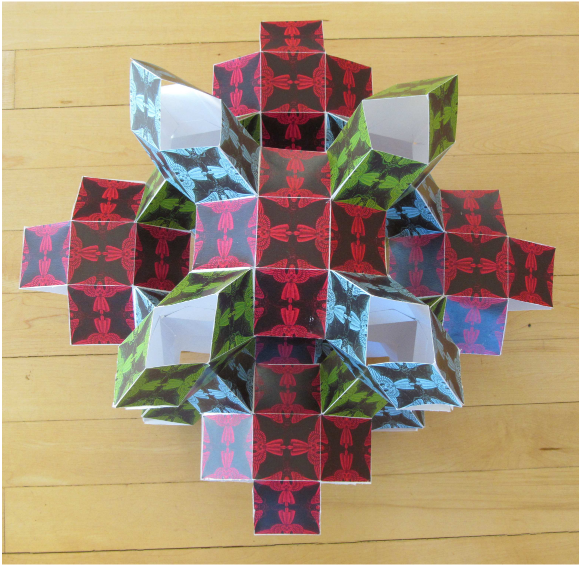
- We have seen the progression from Escher's carved spherical pattern based on the $\{4, 3\}$ tessellation, to Notebook Pattern 45, based on the $\{4, 4\}$ tessellaion, and *Circle Limit IV* based on the $\{6, 4\}$ tessellaion.
- But Escher did not create an "angels and devils" pattern based on the $\{4, 5\}$ tessellation.
- We fill that "gap" by displaying and angels and devils pattern on a $\{4, 5\}$ polyhedron.

A $\{4,5\}$ Polyhedron

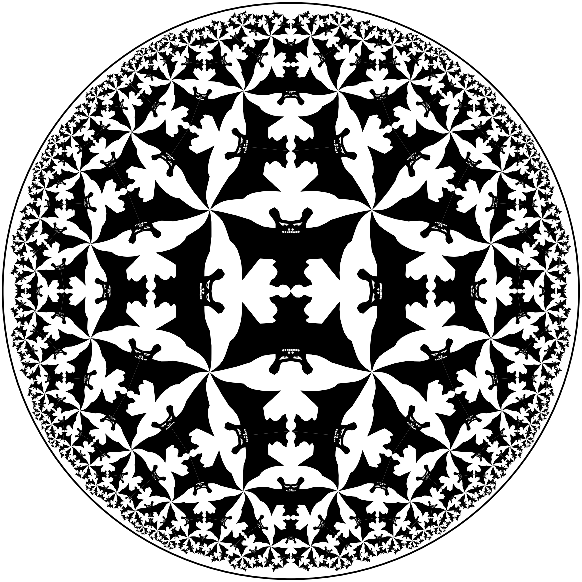


A "Construction Unit" for the $\{4, 5\}$ Polyhedron

The $\{4, 5\}$ Polyhedron with Angels and Devils



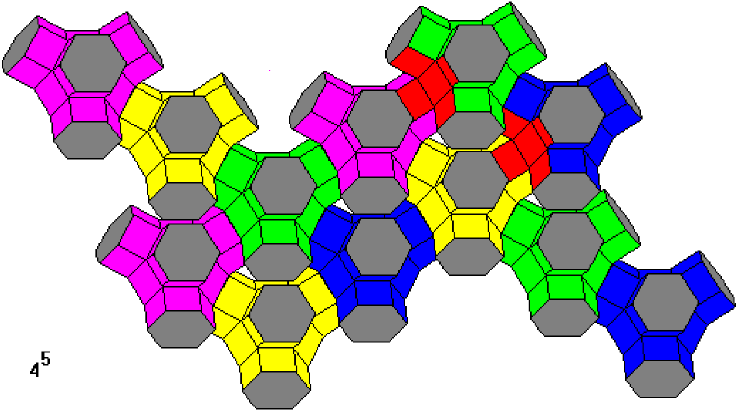
The Hyperbolic “Universal Covering Pattern”



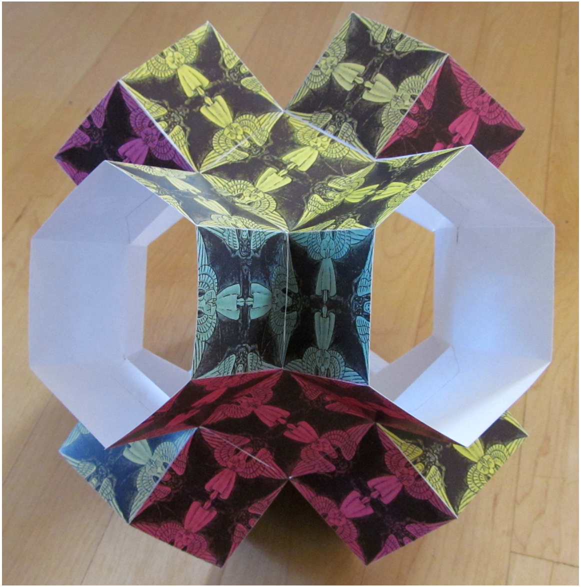
The “Complement” of the $\{4, 5\}$ Polyhedron

- The complement of the $\{4, 5\}$ polyhedron is perhaps easier to understand.
- It is composed of truncated octahedral “hubs” having their hexagonal faces connected by regular hexagonal prisms as “struts”.
- The next slide shows a piece of the $\{4, 5\}$ polyhedron.
- The following slide shows one of the hubs with the struts that are connected to it, decorated with angels and devils.

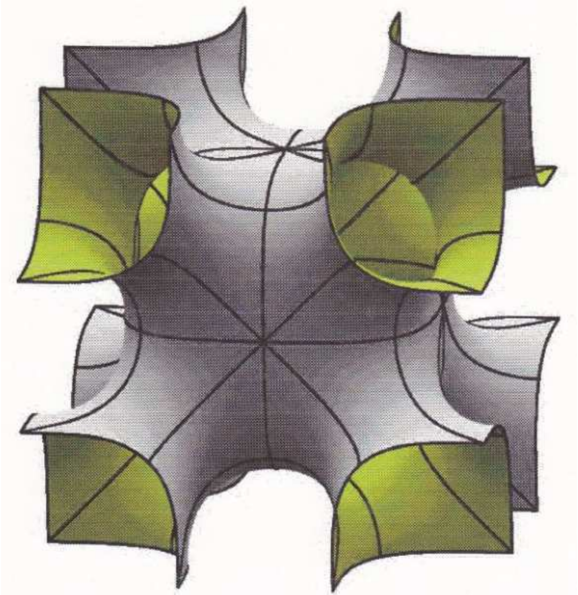
A piece of the $\{4, 5\}$ Complement Polyhedron



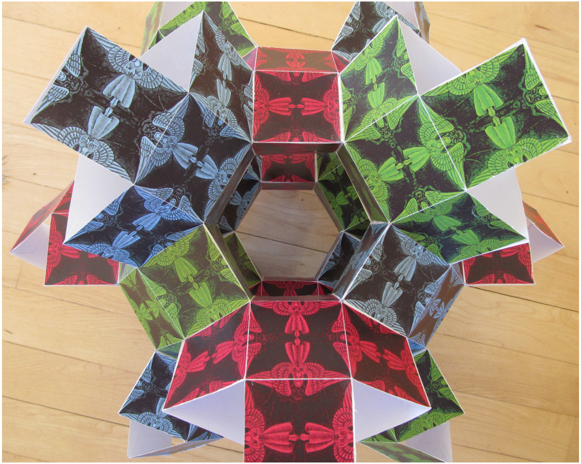
A "Hub" and 8 "struts" of the $\{4, 5\}$ Complement Polyhedron



A Piece of the TPMS IWP Surface Corresponding to the Hub



A View of the Original $\{4, 5\}$ Polyhedron Looking down a Hexagonal Hole

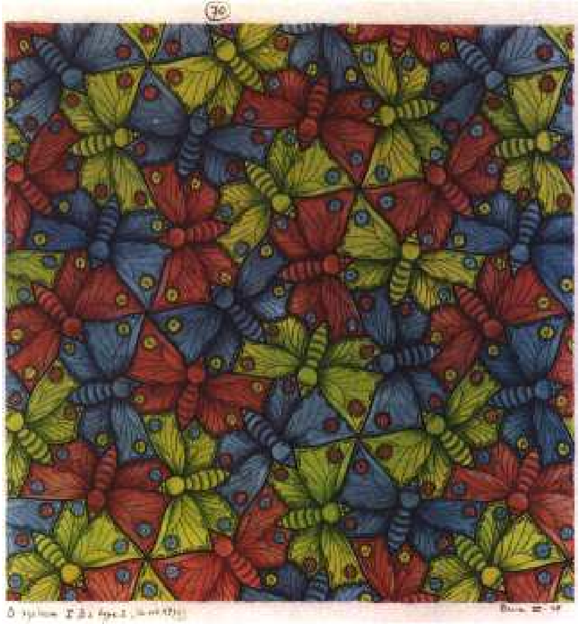


A Pattern of Butterflies on a $\{3, 8\}$ Polyhedron

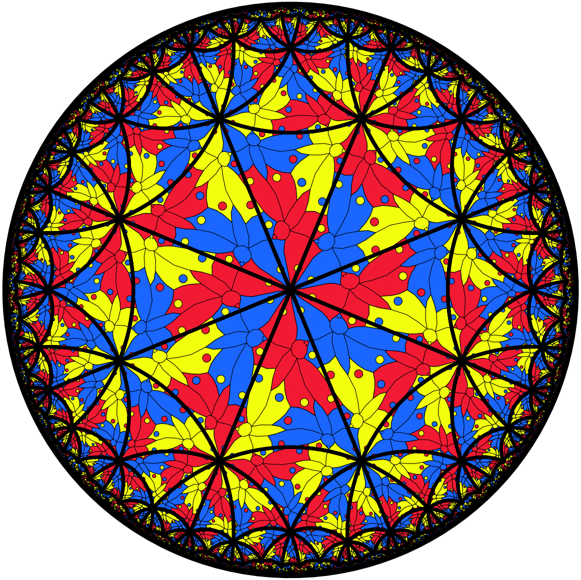
The butterfly pattern on the triply periodic $\{3, 8\}$ polyhedron of the title slide was inspired by Escher's Regular Division Drawing Number 70. This polyhedron is related to Schwarz's D-surface, a TPMS with the topology of a thickened diamond lattice. We show:

- Escher's Regular Division Drawing # 70.
- A hyperbolic pattern of butterflies based on the $\{3, 8\}$ tessellation — the "universal covering pattern" of the patterned polyhedron.
- A construction unit of a $\{3, 8\}$ polyhedron consisting of a regular octahedral "hub" and four octahedral "struts" placed on alternate faces of the hubs.
- Part of Schwarz's D-surface corresponding to the construction unit.
- Another view of the patterned $\{3, 8\}$ polyhedron of the title slide down one of its "tunnels".
- A close-up of a vertex of the patterned polyhedron.

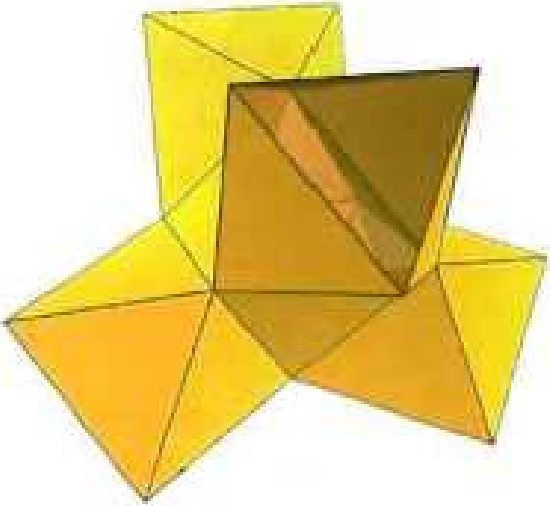
Escher's Regular Division Drawing # 70



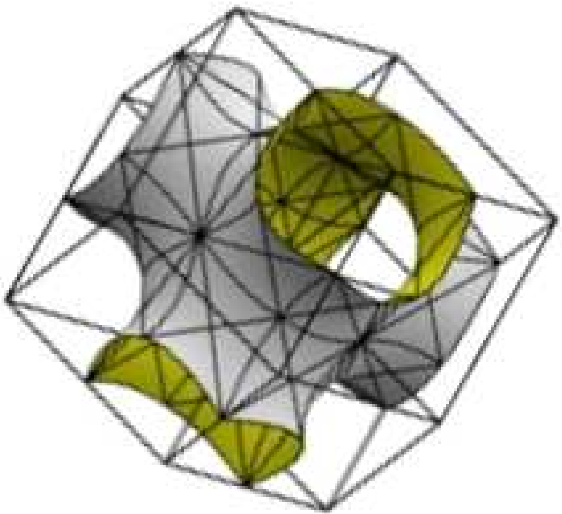
A pattern of butterflies based on the $\{3, 8\}$ tessellation — the "universal covering pattern" for the polyhedron.



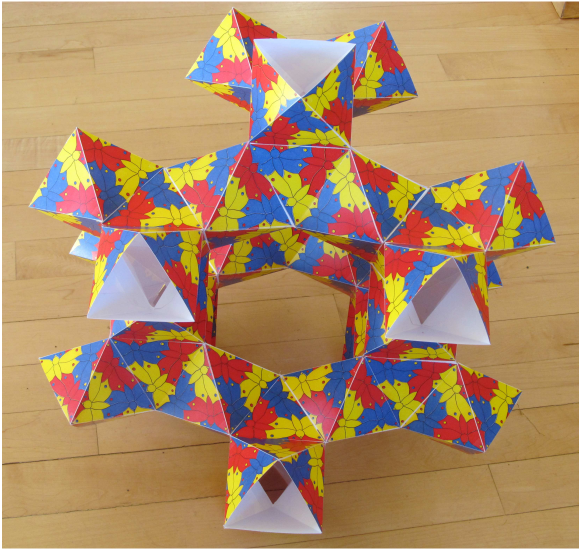
A “construction unit” of the triply periodic polyhedron



A corresponding piece of Schwarz's D-surface



A view down one of the "tunnels" of the $\{3, 8\}$ polyhedron.



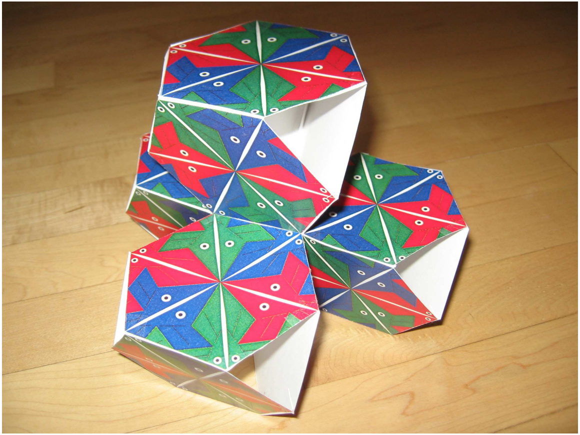
A close-up of a vertex of the patterned polyhedron.

A Pattern of Fish on the $\{6,6|3\}$ Polyhedron

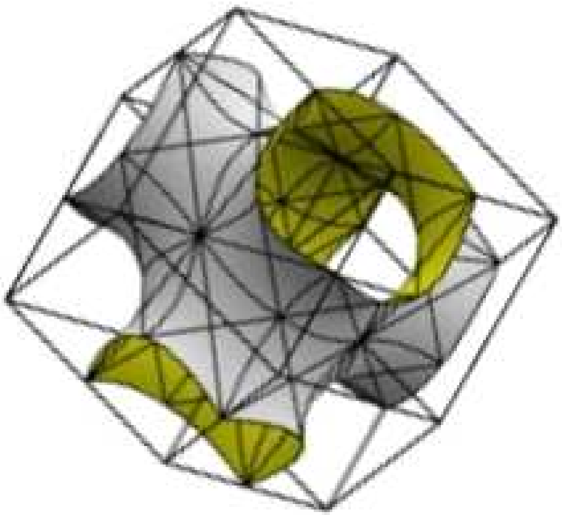
The $\{6,6|3\}$ polyhedron is self-dual. It consists of truncated tetrahedra, four of which share (invisible) equilateral triangular faces with an invisible small regular tetrahedron. The embedded backbone lines of the fish also form skew rhombi (but different than for the $\{4,6|4\}$ and $\{6,4|4\}$ polyhedra). If we span these skew rhombi with "soap films", we obtain the corresponding TPMS, Schwarz's D-surface which has the topology of a thickened diamond lattice. For this polyheron we show:

- The pattern of fish on the $\{6,6|3\}$ polyhedron.
- A "construction unit" of Schwarz's D-surface within a rhombic dodecahedron. Since rhombic dodecahedra tile space, this gives the entire D-surface.
- A top view of the patterned polyhedron that shows a vertex.
- The hyperbolic "universal covering pattern" of the patterned polyhedron.

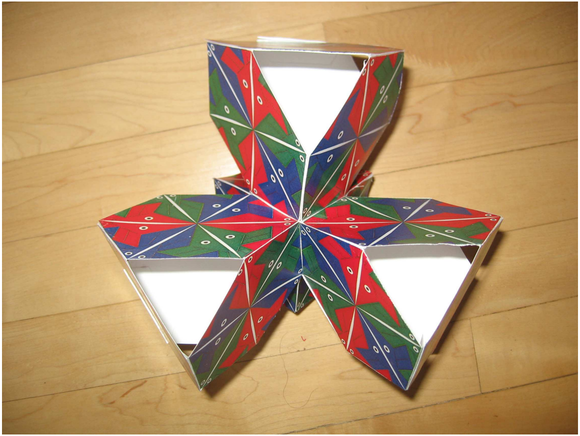
The Pattern of Fish on the $\{6, 6|3\}$ Polyhedron — showing an invisible tetrahedral hub with 4 truncated tetrahedral "struts"



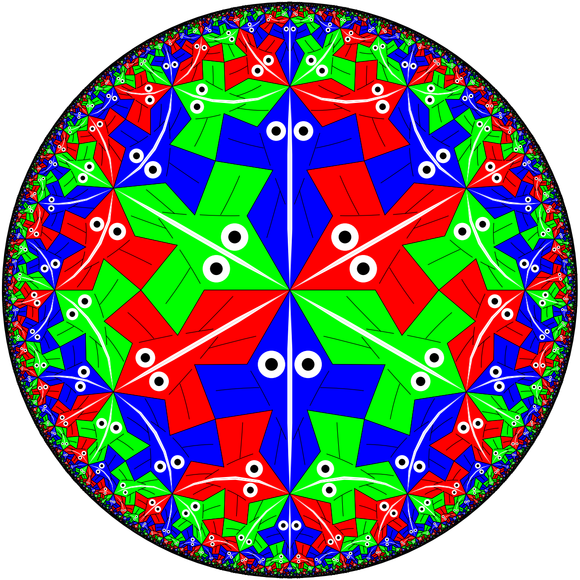
A piece of Schwarz's D-surface showing embedded lines



A top view of the fish on the $\{6, 6|3\}$ polyhedron — showing a vertex



The corresponding universal covering pattern of fish — based on the $\{6,6\}$ tessellation

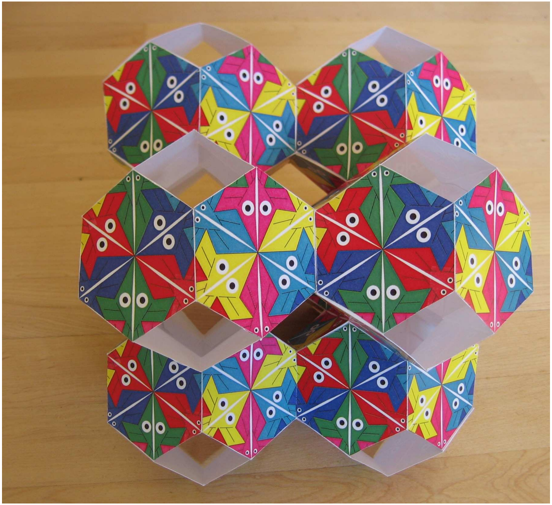


A Fish Pattern on the $\{6,4|4\}$ Polyhedron

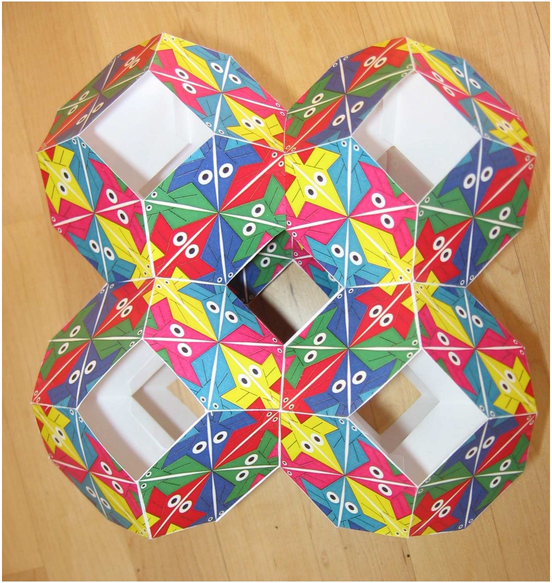
The $\{6,4|4\}$ polyhedron is dual to the $\{4,6|4\}$ polyhedron of the title slide. Thus they both approximate the same TPMS, Schwarz's P-surface. The $\{4,6|4\}$ polyhedron consists of truncated octahedra in a cubic lattice arrangement and connected on their (invisible) square faces. For this polyheron we show:

- The pattern of fish on the $\{6,4|4\}$ polyhedron.
- A top view of the patterned polyhedron that shows how fish of a single color line up along backbone lines.
- The hyperbolic "universal covering pattern" of the patterned polyhedron.

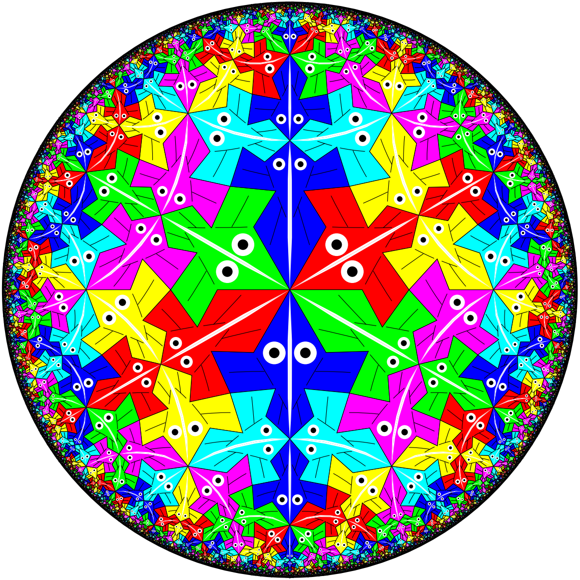
The Pattern of Fish on the $\{6, 4|4\}$ Polyhedron



A top view of the fish on the $\{6,4|4\}$ polyhedron — showing fish along embedded lines



The hyperbolic universal covering pattern of fish — a version of Escher's Circle Limit I pattern with 6-color symmetry



Future Work

- Put other patterns on the regular skew polyhedra.
- Place patterns on non-regular, uniform triply periodic polyhedra.
- Put patterns on non-uniform triply periodic polyhedra — especially those that more closely approximate triply periodic minimal surfaces.
- Draw patterns on TPMS's — the gyroid, for example.

Thank You!

Claude, Renzo and *all* the other ESMA Conference organizers

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