Hyperbolic Key Patterns

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Abstract

For thousands of years key patterns have been created by artists from a number of cultures, almost exclusively on flat surfaces. There are different kinds of key patterns, but only the simplest have been analyzed mathematically. In this paper, we will extend the analysis of simple key patterns to hyperbolic key patterns and show examples of them. We will also show hyperbolic key patterns that are analogous to more complex Euclidean key patterns.

1 Introduction

Figure 1 shows an example of a key pattern that the Dutch artist M. C. Escher found in a Japanese pattern book and that he used in his lectures on repeating patterns. Figure 2 shows a hyperbolic version of that pattern. These patterns exhibit the characteristics of key patterns: they have rotation centers not lying on mirror lines, the motifs interlock, and they are made up of rectangular elements.

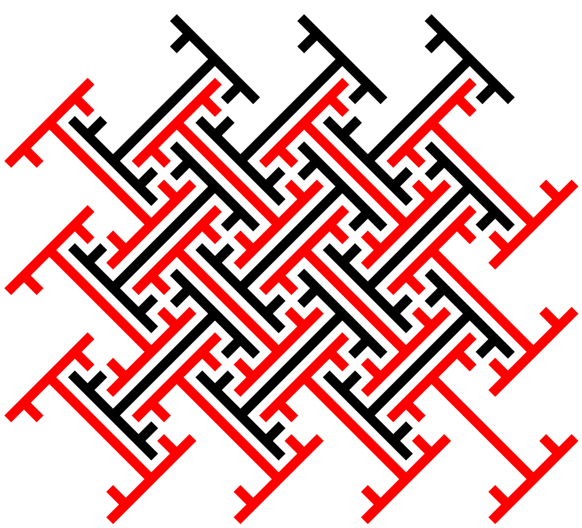

Figure 1: A Japanese key pattern.

Figure 2: A hyperbolic key pattern related to that of Figure 1.

We also note that key patterns may or may not have reflection symmetry. Finally, although key patterns are often used for border decorations, like frieze patterns, we will only discuss key patterns that are truly "2-dimensional" — that is they repeat in two different directions.

The history of key patterns goes back more than 15,000 years, although the term "key pattern" was only coined a century ago by J. Romilly Allen, after the rectangular stepping pattern of old fashioned keys [Sloss]. Sloss explains how to draw (Euclidean) key patterns by hand [Sloss]. Other terms that have been used include "spiral step patterns", "maze patterns", and Celtic fret patterns".

In the next section, we develop a theory of simple key patterns based on a square grid, as are almost all Celtic key patterns, for example. In Section 3 we generalize this theory, extending it to simple hyperbolic key patterns, and then in Section 4 we show a non-simple key pattern and its hyperbolic counterpart. We conclude by indicating directions of future work.

2 Simple Key Patterns on Square Grids

We define a *simple key pattern* to be one whose motif is an angular spiral. In this section the spirals will be based on a square grid. Figure 3 shows a right-turning angular spiral in black, with edge lengths of 1, 2, 3, 4, and 4.5 grid units. Figure 3 also shows a gray copy of that angular spiral rotated 180 degrees and translated so that the "stems" match up to form what Sloss calls an *S-curve* [Sloss] (really a reverse S curve). This is the reason the edge length of the stem is 4.5 grid units: it will be connected to another stem, forming a line segment of total length 9 units. Figure 4 shows black copies of this reverse S-curve linked to form diagonal chains; similar chains of gray S-curves interlock with them to form a repeating pattern.

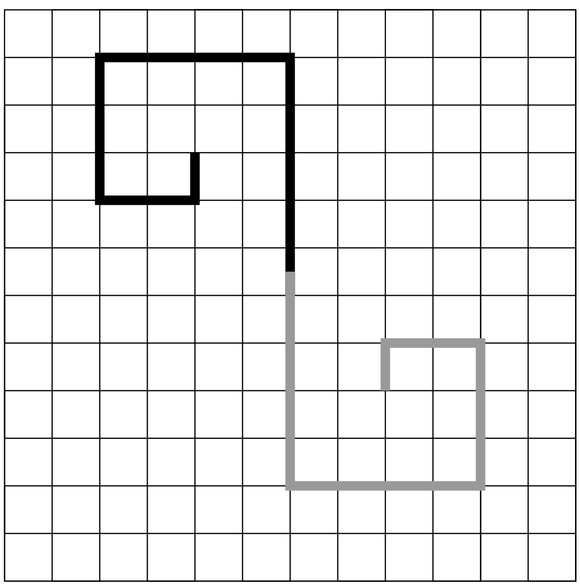

Figure 3: Two spiral segments forming an S-curve.

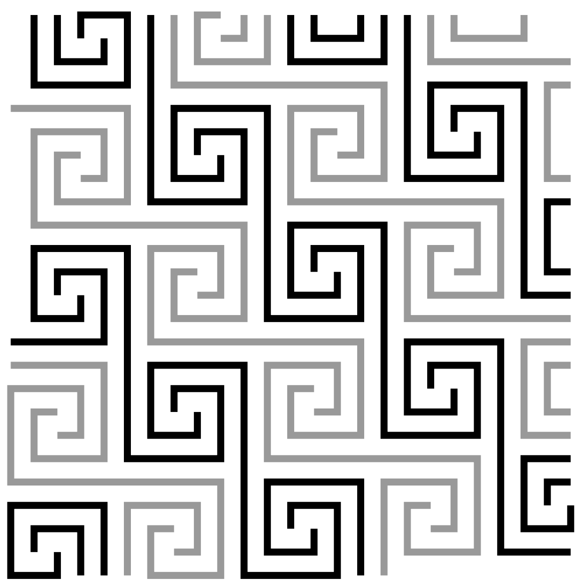

Figure 4: A repeating pattern of S-curves.

In turn, we define an *basic angular spiral* or simply a *basic spiral* based on a square grid as follows: it is a path that alternately proceeds straight along grid edges and turns right after traversing an increasing number of grid units. This is a minor abuse of terminology since a basic spiral is not a true spiral, which is a smooth, continually turning curve.

There are three factors that determine the geometry of a simple key pattern on a square grid: (1) the number of turns, n, in a basic spiral, (2) the number of basic spirals, the *spiral number*, that spiral in toward each other at the "spiral end" (containing the shortest segment), and (3) how the spiral segments are put together at the "stem end". This is all subject to the "one unit separation" rule: each line segment of a pattern should be exactly one grid unit from its neighbors. In Figures 3 and 4, the number of turns n is 4 (so there are five line segments), the spiral number is 2 since two spiral segments spiral about each other at their spiral ends, and two stems are joined by a 180 degree rotation. There is no limit on the number of turns, which is usually taken to be at least 2 in order to obtain non-trivial patterns. However, for a square grid the spiral number must be a divisor of 4 that is greater than 1, i.e. 2 or 4 (it is 2 in Figures 3 and 4).

In a square grid, there are three ways to attach new copies of the basic spiral to the stem of the first one: by rotating by multiples of 90 degrees about the end of the stem, by reflecting across the stem, and by reflecting across a perpendicular to the stem. We use the term *spiral group* for the set of all the copies of the basic spiral attached at their stems. In Figures 3 and 4 the spiral groups are the S-curves and the reverse S-curves.

There are four possible spiral groups: an S-curve (and its reversal), a *C-curve* formed by reflecting a basic spiral across its stem, what Sloss calls an *H-curve* formed by reflecting a C-curve about its "backbone", and a tetraskelion (or reverse swastika) formed by joining the stems of four basic spirals at 90 degree angles. The tetraskelion could be called the *X-curve*, since that letter comes closest to describing its symmetries. If the spiral number is 2, the edge lengths for an n-turn basic spiral for an S-curve, C-curve, H-curve, and X-curve are respectively: $[1, 2, \ldots, n, n + \frac{1}{2}]$, $[1, 2, \ldots, n, n + \frac{1}{2}]$, $[1, 2, \ldots, n, n]$, and $[1, 2, \ldots, n, n + 1]$, where the last length, the stem length, must break the numerical pattern so that the basic spirals match up properly to form their curves. If the spiral number is 4, the edge lengths are: $[1, 3, 5, \ldots, 2n - 1, 2n + \frac{1}{2}]$, $[1, 3, 5, \ldots, 2n - 1, 2n + \frac{1}{2}]$, $[1, 3, 5, \ldots, 2n - 1, 2n]$, and $[1, 3, 5, \ldots, 2n - 1, 2n + 1]$ respectively.

Figure 5 shows a pattern based on a C-curve with spiral number 4; if adjacent back-to-back C-curves were glued together in that pattern, one would get a pattern of H-curves — two such C-curves are shown in gray. The pattern of Figure 1 is closely related to an H-curve pattern with spiral number 4. C-curves by themselves seem to be rarely used in key patterns, but their related H-curves are heavily used. Figure 6 shows a key pattern of tetraskelions with spiral number 2. This finishes our discussion of key patterns based

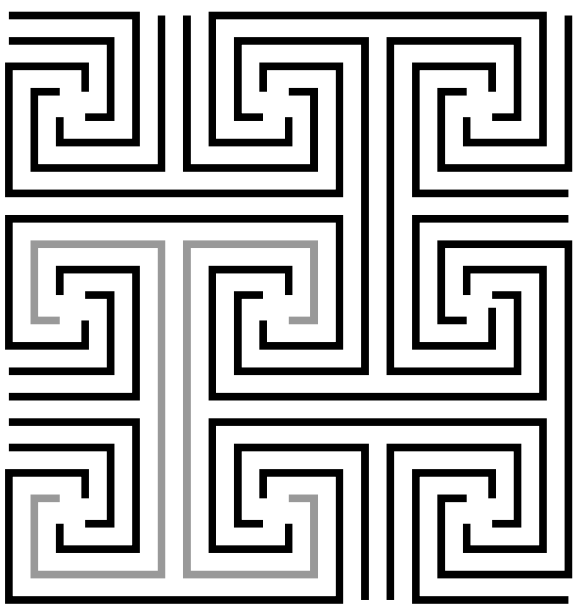

Figure 5: A C-curve key pattern.

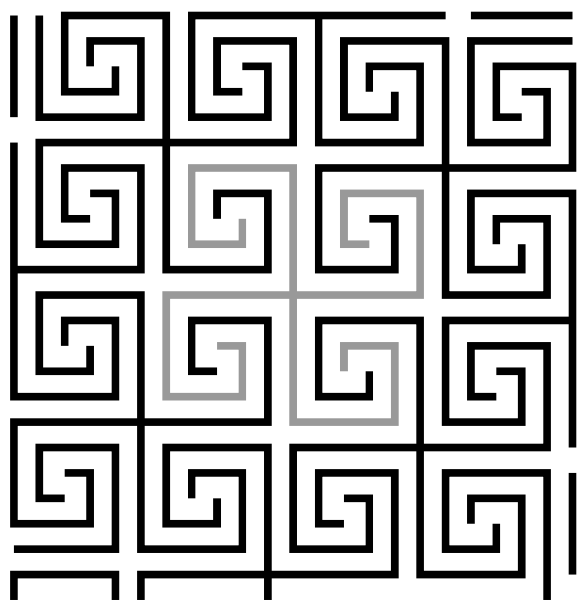

Figure 6: A tetraskelion key pattern.

on a square grid. In the next section we discuss key patterns on general grids.

Simple Key Patterns Based on General Grids

By a *general grid* we mean the pattern formed by the edges of a regular tessellation. The *regular tessellation*, $\{p,q\}$, consists of regular p-sided polygons with q of them meeting at each vertex. The edges of the $\{4,4\}$ tessellation form the familiar square grid. The values of p and q determine the geometry of the tessellation: the sphere, the Euclidean plane, or the hyperbolic plane depending on whether $(p-2)(q-2)$ is less than, equal to, or greater than 4. The Euclidean tessellations are $\{4,4\}$, $\{3,6\}$, and $\{6,3\}$, by squares, equilateral triangles, and regular hexagons respectively. The spherical tessellations, $\{3,3\}$, $\{3,4\}$, $\{3,5\}$, $\{4,3\}$, and $\{5,3\}$ are just the Platonic solids blown up onto their circumscribing spheres. There are an infinite number of regular hyperbolic tessellations since $(p-2)(q-2) > 4$ for infinitely many values of p and q.

In order to construct basic spiral on a general grid, we must be able to continue along a grid in one direction for more than one edge length. This means that q must be even. For example, the hexagon grid $\{6,3\}$ cannot be used to build even a basic spiral because after traversing one edge length, one must turn 60 degrees to the left or to the right in order to stay on the grid. Although the equilateral triangle tessellation $\{3,6\}$ could be used as a grid for a key pattern, this seems to have rarely been done.

Unlike the case of the square grid, if q is greater than 4, we have more than one choice for the turning angle, which is another parameter that must be taken into account in that case. For example, in the equilateral triangle tessellation $\{3,6\}$ all the right turns can either be 60 degrees or 120 degrees. In general, the turning angle can be any multiple of $360/q$ degrees up to $q/2-1$.

To generalize the other parameters, the spiral number must be a divisor of p, and the spiral group must be formed by applying a subgroup of the dihedral group D_q of reflections and rotations about the end of the stem of the basic spiral. This subgroup must contain either a $180\deg$ rotation or a reflection perpendicular to the stem so that the stem segment is continued across its end in a straight line. Also the subgroup cannot contain any reflection lines whose angle with the stem is an odd multiple of $180/q$ degrees.

Figure 7 shows a simple H-curve pattern based on the $\{6,4\}$ tessellation. Figure 8 shows a triskelion pattern with the rotational symmetries of the $\{3,8\}$ tessellation.

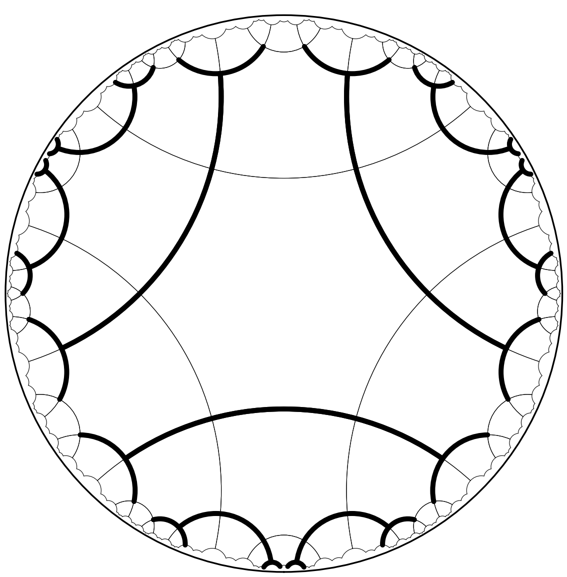

Figure 7: An H-curve based on the $\{6,4\}$ tessellation.

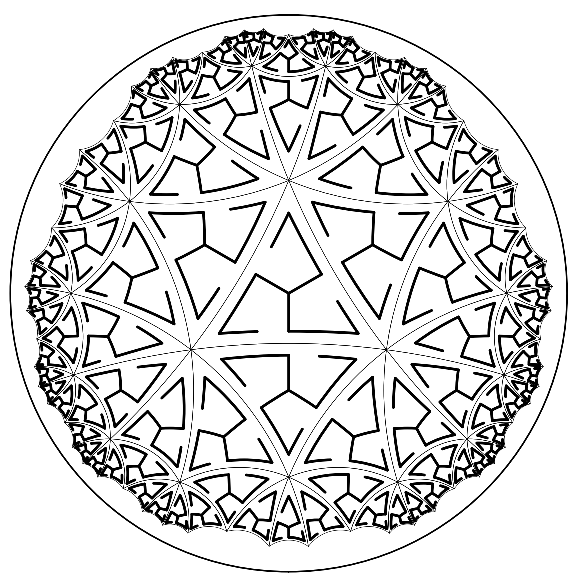

Figure 8: A triskelion key pattern within the $\{3,8\}$ tessellation.

In Figures 2, 7, and 8 above we have used the *Poincaré circle model* of hyperbolic geometry because

(1) it lies entirely within a *bounding circle* in the Euclidean plane (allowing the entire hyperbolic plane to be shown at once), and (2) it is *conformal* — the hyperbolic measure of an angle is equal to its Euclidean measure. In this model, the hyperbolic points are the interior points of the bounding circle and the hyperbolic lines are interior circular arcs perpendicular to the bounding circle, including diameters. We note that even with these properties, we can only easily see a small part of the simplest key patterns. Figure 8 shows a possible cure to this problem: removing the restriction that the basic spirals lie along grid edges.

To complete the discussion of simple key patterns on general grids, we note that the octahedral tessellation $\{3, 4\}$ is the only spherical tessellation that supports simple key patterns. Only three trivial key patterns are possible: three non-intersecting edges, two pairs of edges connected in V's on opposite faces, and a single S-curve consisting of 5 connected edges. In the next section, we look at a non-simple Islamic key pattern and its hyperbolic counterpart.

4 A Hyperbolic Islamic Key Pattern.

The art of several cultures has been enriched by key patterns. As mentioned above, the pattern of Figure 1 is Japanese, and the simple key patterns of Section 2 are very prevalent in Celtic art. Figure 9 shows an Islamic pattern that Escher copied from the mosque La Mesquita in Córdoba (page 32 of [Schattschneider]). It has the symmetries of a simple key pattern of H-curves with spiral number 4. Figure 10 shows a hyperbolic analogue of the pattern of Figure 9 with spiral number 6.

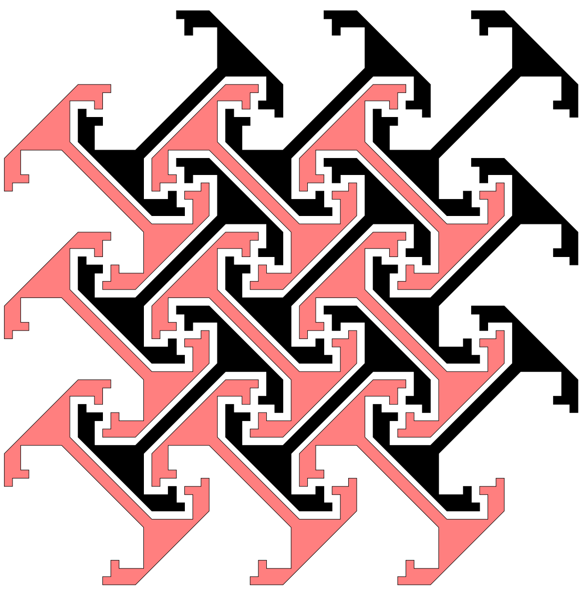

Figure 9: An Islamic key pattern.

Figure 10: A hyperbolic version of Figure 9.

5 Conclusions and Future Work

We have presented a theory of simple key patterns on a square grid and extended that theory to hyperbolic geometry. We see this as extending the understanding of the geometric theory behind symmetric patterns of other cultures as in [Dunham1, Dunham2].

One direction of research might be to relax the restriction that basic spirals lie along grid lines (as in Figure 8) — possibly allowing any unit length and turning angle as long as the basic spirals started and ended on a grid vertex. We have also showed some non-simple key patterns from several cultures and their hyperbolic analogues. It would be useful, although probably more difficult, to develop a theory of non-simple key patterns.

It would also be interesting to investigate the connection between simple key patterns and Krawczyk's spirolaterals [Krawczyk1, Krawczyk2, Krawczyk3], since their definitions are so similar. In fact, given the numerous writings on key patterns, there would seem to be many directions for further research.

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