USE OF MODELS OF HYPERBOLIC GEOMETRY IN THE CREATION OF HYPERBOLIC PATTERNS

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ABSTRACT: In 1958, the Dutch artist M.C. Escher became the first person to create artistic patterns in hyperbolic geometry. He used the Poincaré circle model of hyperbolic geometry. Slightly more than 20 years later, my students and I implemented a computer program that could draw repeating hyperbolic patterns in this model. The program made substantial use of the Weierstrass model of hyperbolic geometry as an intermediate step. We have also made use of the Klein model of hyperbolic geometry, both for approximating hyperbolic lines and for transforming motifs from one set of combinatorial values to another.

Keywords: Hyperbolic geometry, geometric models, mathematical art.

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1.INTRODUCTION

For over 100 years mathematicians have created repeating patterns in the hyperbolic plane, but the Dutch artist M.C. Escher was the first person to create such patterns for artistic purposes. In 1958 he was inspired by an article he received from the Canadian mathematician H.S.M. Coxeter who had used one of Escher's patterns in that article. The article contained a figure displaying a triangle pattern in the Poincaré circle model of hyperbolic geometry. Figure 1 shows a copy of that pattern. In Escher's words, this figure gave him "quite a shock" because it showed him how to construct an infinite pattern with a circular limit – that is with motifs that got smaller towards the edge of a circular disk – and it also gave Escher's hyperbolic patterns their "Circle Limit" names.

Escher created his patterns as wood block prints, so he had to laboriously carve each one and print it by hand. It seemed to me that computer graphics would provide a much easier solution to the problem of creating such hyperbolic patterns.

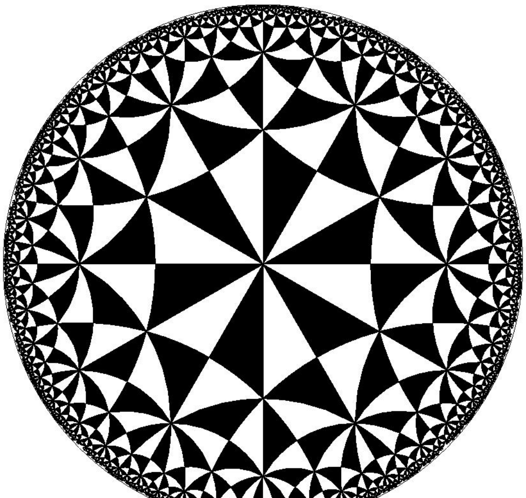

Figure 1: Coxeter's figure.

In 1980 two students and I set about to design and implement a computer program that could draw repeating hyperbolic patterns as Escher had done. Of course we wanted to display the output in the Poincaré circle model, but it turned out that the transformations were easier to represent in the Weierstrass model of hyperbolic geometry. Since there are simple mappings between the two models, the Weierstrass model was also incorporated into our program.

Later, I also made use of the Klein model of hyperbolic geometry. It was used in two ways: first to obtain simple approximations to circular arcs that represented hyperbolic lines, and second to transform motifs from one combinatorial pattern to another.

In the following sections, I will start with a review of some concepts from hyperbolic geometry, including the Poincaré and Weierstrass models of it. Then I will discuss repeating patterns and regular tessellations. Next, I will describe the pattern generation algorithm. Then I will show how the Klein model was used in creating patterns. Finally, I will summarize the results.

2. HYPERBOLIC GEOMETRY AND THE POINCARÉ AND WEIERSTRASS MODELS

By its definition, hyperbolic geometry satisfies the negation of the Euclidean parallel axiom together with all the other axioms of (plane) Euclidean geometry. Specifically, the hyperbolic parallel axiom states that given a line and a point not on that line, there is more than one line through the point that does not meet the original line. In contrast to the Euclidean plane and the sphere, there is no isometric (distance preserving) embedding of the hyperbolic plane in Euclidean 3-space, which was proved by David Hilbert in 1901 [4]. Thus we must rely on *models* of hyperbolic geometry: constructions in which Euclidean objects have hyperbolic interpretations, and which must perforce distort distances from their Euclidean values.

As an example, in the *Poincaré circle model,* the hyperbolic points are just the points in the interior of a Euclidean bounding circle, and the hyperbolic lines are represented by circular arcs within the bounding circle that are perpendicular to it (with diameters as special cases). The Poincaré circle model appealed to Escher and has appealed to other artists who wished to create repeating hyperbolic patterns for two reasons: (1) it can show an entire pattern in a bounded region, and (2) is is *conformal*, that is, angles have their Euclidean measure, which means that copies of motifs retain the same approximate shape regardless of size. Distances are measured in such a way that equal hyperbolic distances correspond to ever smaller Euclidean distances as one approaches the bounding circle. Figure 2 shows a rendition of Escher's first hyperbolic pattern *Circle Limit I.*

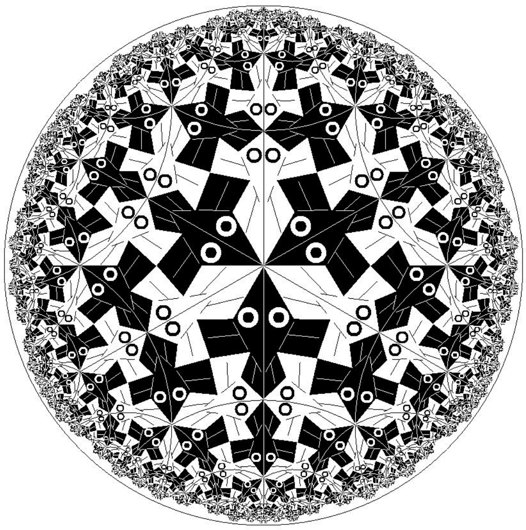

Figure 2: A rendition of Escher's hyperbolic pattern *Circle Limit I.*

In *Circle Limit I* the circular arcs forming the backbone lines of the fish are hyperbolic lines in the Poincaré model and trailing edges of the fishes' fins also lie along hyperbolic lines.

The points of the Weierstrass model of hyperbolic geometry are points on the upper sheet of the "unit" hyperboloid of two sheets in Euclidean 3-space: $z^2-x^2-y^2=1$, $z>0$. Each line of the Weierstrass model is a hyperbola that is the intersection of a plane through the origin with the upper sheet of the hyperboloid. There is a simple mapping from the Weierstrass model to the Poincaré model considered as the unit disk in the xy-plane – it is "stereographic" projection toward the point $(0,0,-1)$ (the vertex of the lower sheet of the unit hyperboloid). It is given by the formula $(x,y,z) \rightarrow (x/(1+z), y/(1+z), 0)$, the intersection of the projector with the xy-plane. The inverse mapping is also simple – its formula is given by $(x,y) \rightarrow (2x/(1-s),2y/(1-s),(1+s)/(1-s))$, where $s=x^2+y^2$.

Poincaré also devised his *upper half-plane model* whose points are those in the xy-plane with $y>0$. This model is also conformal, but since it could not be displayed in a finite area, it does not have as much appeal to artists. Neither have I found a use for this model in creating hyperbolic patterns. The book by Greenberg [3] has an extensive discussion of hyperbolic geometry, including the various models of it.

3. REPEATING PATTERNS, REGULAR TESSELLATIONS, AND SYMMETRIES

A *repeating pattern* is a pattern made up of congruent copies of a basic sub-pattern or *motif*. If we disregard color, a triangle is a motif for the pattern of Figure 1 (taking color into account, the triangle patterns has what is called perfect 2-color symmetry, however). In Figure 2, the motif can be taken to be half a black fish together with half of an adjacent white fish. Note that in that pattern, the black fish are not congruent to the white fish, since three black fish meet nose-to-nose whereas only two white fish meet nose-to-nose. The same definition of a repeating pattern applies to each of the three "classical" geometries: the sphere, the Euclidean plane, and the hyperbolic plane (which have constant positive, zero, and negative curvatures respectively). One important kind of repeating pattern is the *regular tessellation*, denoted $\{p,q\}$, consisting of regular p-sided polygons or *p-gons* meeting q at a vertex. If $(p-2)(q-2) > 4$, the tessellation $\{p,q\}$ is a tessellation of the hyperbolic plane; otherwise if $(p-2)(q-2) = 4$ or $(p-2)(q-2) < 4$, the tessellation is Euclidean or spherical respectively. Figure 3 shows the {6,4} tessellation superimposed on the *Circle Limit I* pattern of Figure 2.

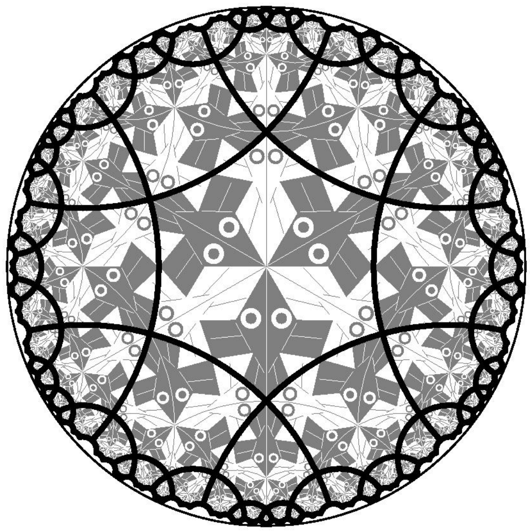

Figure 3: The {6,4} tessellation superimposed on a rendition of the *Circle Limit I* pattern.

Regular tessellations form the basis for all four of Escher's Circle Limit patterns, with {6,4} also forming the basis for *Circle Limit IV,* and the {8,3} tessellation forming the basis for *Circle Limit II* and *Circle Limit III.* Figure 4 shows the {8,3} tessellation superimposed on a rendition of the *Circle Limit III* pattern. Regular tessellations also form the basis for all of Escher's spherical patterns, and many of his Euclidean repeating patterns.

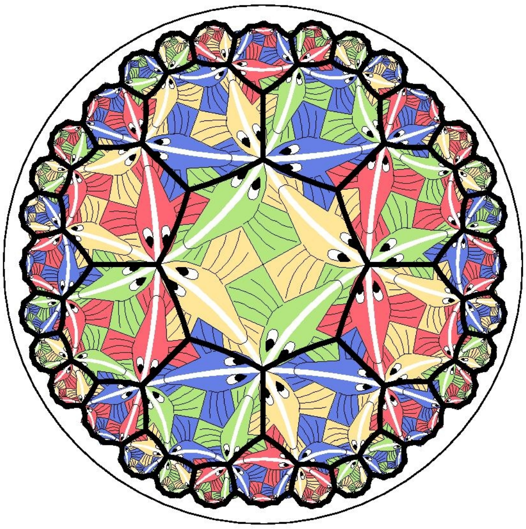

Figure 4: The {8,3} tessellation superimposed on a rendition of the *Circle Limit III* pattern .

A *symmetry* of a repeating pattern is an isometry (distance-preserving) transformation that takes the pattern onto itself. For example reflections across the sides of the triangles of Figure 1 (ignoring color) are symmetries of that pattern. Reflections across a line in the Poincaré model are inversions in the circular arcs representing the lines. Reflections across the backbone lines are symmetries of the pattern of Figure 2. Other symmetries of that pattern include rotations by 60 degrees about meeting points of noses of the black fish, and translations by four fish-lengths along backbone lines. In hyperbolic geometry, as in Euclidean geometry, successive reflections across two lines having a common perpendicular results in a translation, and successive reflections across intersecting lines results in a rotation about the intersection point by twice the angle of intersection.

A repeating pattern has *n-color symmetry* if it is colored with *n* colors and each symmetry of the uncolored pattern maps all motifs of one color to a single color, that is, a symmetry of the uncolored pattern permutes the colors. It is also required that colors be permuted transitively so that all the colors get mixed around. This concept is usually called *perfect* color symmetry. The pattern of Figure 1 has 2-color symmetry; in Figure 4, the Circle Limit III pattern has 4-color symmetry. The pattern in Figure 2 does not have color symmetry since there is no symmetry that maps the black fish to the white fish because, as mentioned above, the black and white fish are not congruent.

4. THE PATTERN GENERATION ALGORITHM

The patterns that our algorithm can generate are based on the regular tessellations $\{p,q\}$. We start by interactively constructing a motif within the central p-gon. That motif is then copied by using reflections across diameters or rotations about the center of the bounding circle (or both), depending on the symmetries of the desired pattern, to form what we call the *super-motif*. Thus the super-motif of the *Circle Limit III* pattern consists of four fish. In theory, the super-motif can be successively transformed across the sides of the p-gons until they fill the disk. Since there are an infinite number of p-gons, we stop after the disk has been sufficiently filled to give an idea of what the infinite pattern would look like. There are different ways to choose which p-gon to fill next. There is an overview of these choices in [2].

In the Weierstrass model, reflections can be represented by 3-by-3 matrices that map the upper sheet of the hyperboloid onto itself. For example, reflection across a line that makes an angle θ with the x-axis is represented by reflection in a plane that makes an angle θ with the x-axis:

$$\begin{pmatrix} \cos(2\theta) & \sin(2\theta) & 0 \\ \sin(2\theta) & -\cos(2\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1)$$

Reflection across a hyperbolic line/ circular arc in the Poincaré model that is perpendicular to the x-axis (and the bounding circle) is represented in the Weierstrass model by:

$$\begin{pmatrix} -\cosh(2d) & 0 & \sinh(2d) \\ 0 & 1 & 0 \\ -\sinh(2d) & 0 & \cosh(2d) \end{pmatrix} \quad (2)$$

where d is the hyperbolic distance from the origin to the intersection of the circular arc and the x-axis. The product of two reflections as in (1) is a rotation about the z-axis. If we apply such a rotation, then the reflection represented in (2), then the inverse of the rotation, we can obtain the matrix that represents reflection across any line. Since any isometry can be built from at most three reflections, the matrices above can be used to generate the isometry. The reason for using 3-by-3 real matrices instead of complex linear fractional mappings is that the latter cannot easily represent orientation-reversing transformations.

Once the motif has been created in the Poincaré model, the points that describe it are mapped up onto the Weierstrass model. The resulting points in 3-space are then transformed around the Weierstrass model using the 3-by-3 matrices as described above. Finally the transformed copies of the motif are mapped back down onto the Poincaré disk to generate the whole pattern. The details are treated in [1]. Figure 5 shows a rendition of Escher's *Circle Limit IV* pattern that was generated by this process.

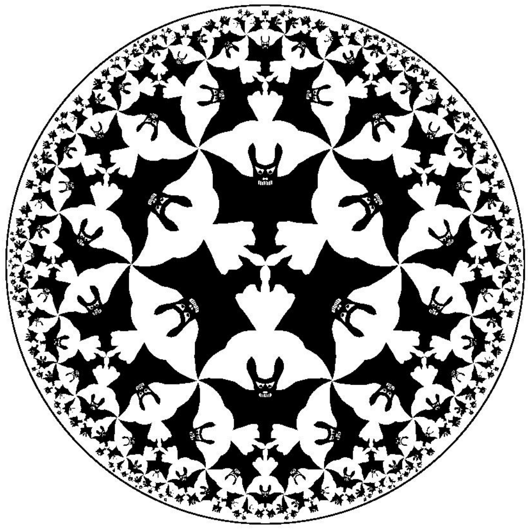

Figure 5: A rendition of Escher's hyperbolic pattern *Circle Limit IV*.

5. THE KLEIN MODEL AND ITS USES

As in the Poincaré model, the points in the are the interior points of the unit circle. But unlike the Poincaré model, hyperbolic lines are represented by chords. There is a simple relation between the models: the orthogonal circular arc and the chord that represent a hyperbolic line have the same endpoints on the bounding circle.

There is a simple mapping from the Poincaré model to the Klein model: $(x,y) \to (2x/(1+s), 2y/(1+s))$, where $s = x^2+y^2$. The mapping $(x,y) \to$ (x/d, y/d)

is its inverse, where $d = 1 + \sqrt{1 - x^2 - y^2}$.
One use of the Klein model is to obtain an approximation of a segment of a hyperbolic line/orthogonal circular arc in the Poincaré model by Euclidean line segments. This is useful when the orthogonal circular arc is close to the center of the bounding circle, and thus has a large radius of curvature. Some graphics systems do not handle this situation well. The approximation works by mapping the endpoints of the arc segment to the Klein model, obtaining a Euclidean line segment which can easily be finely subdivided, mapping the subdivision points back to the Poincaré model and connecting them with short Euclidean line segments.

The Klein model also proved useful in another situation. After a motif has been created for a pattern based on a $\{p,q\}$ tessellation, we would like to transform it so that it works with a $\{p',q'\}$ tessellation, thus giving a new pattern. Consider the subdivision of the central p-gon into p isosceles triangles with angles $2\pi/p, \pi/q$, and π/q. If the motif for the pattern based on $\{p,q\}$ fits in such an isosceles triangle, we proceed as follows. First map the motif inside the triangle to the Klein model, which would be a Euclidean isosceles triangle. That isosceles triangle can be deformed by a differentially scaling along the x-axis and y-axis to obtain the Euclidean/Klein isosceles triangle associated with the $\{p',q'\}$ tessellation. This last isosceles triangle and the motif within it can then be mapped to the Poincaré model.

As an example of this process, we start with Escher's *Circle Limit I* pattern and then create a pattern based on the {6,6} tessellation in which the black fish are congruent to the white fish, and thus has 2-color symmetry. Figure 6 shows the Poincaré isosceles triangle pattern associated with the {6,4} tessellation.

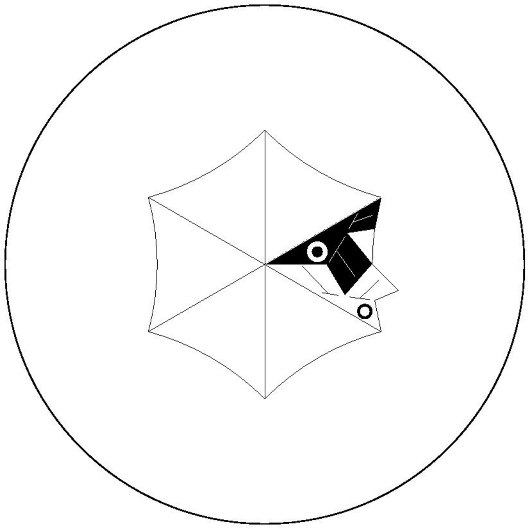

Figure 6: The isosceles triangle motif associated with the {6,4} tessellation.

Figure 7 shows the transformed isosceles triangle motif associated with the {6,6} tessellation. Note that since $p' = p$ and $q' > q$ the $\{p',q'\}$ motif extends farther toward the bounding circle.

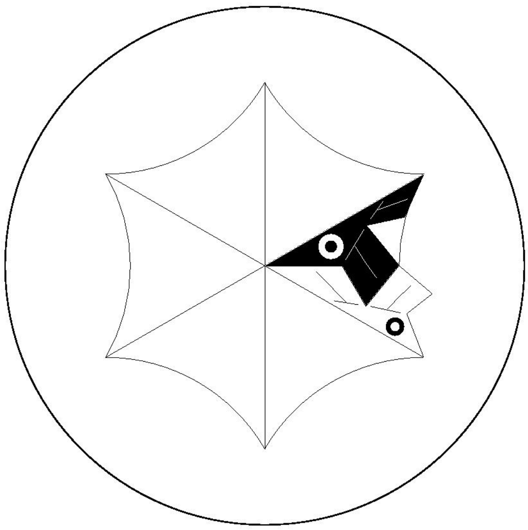

Figure 7: The isosceles triangle motif associated with the {6,6} tessellation.

Finally, we use the transformed motif

to generate the whole pattern, as shown in Figure 8.

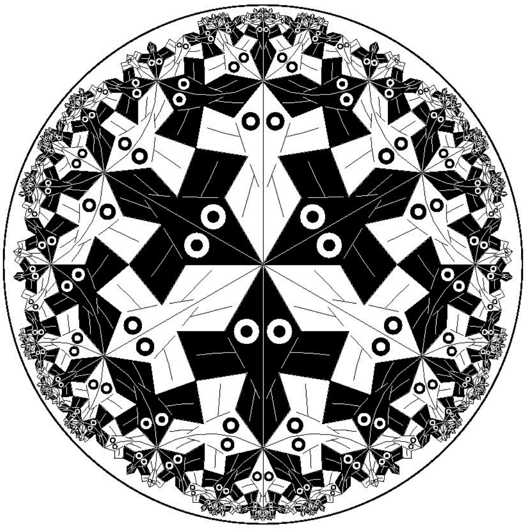

Figure 8: The *Circle Limit I* pattern transformed to the {6,6} tessellation.

As another example of this process, we transform the 4-arm cross pattern of Escher's *Circle Limit II* print to a corresponding 5-arm cross pattern. Figure 9 shows a rendition of Escher's *Circle Limit II* pattern.

Figure 9: A rendition of Escher's hyperbolic pattern *Circle Limit II.*

Figure 10 shows the corresponding 5-arm cross pattern that is based on the {10,3} tessellation.

We note that this transformation process only works if the motif is associated with the isosceles triangle described above. For example in *Circle Limit III*, which is based on the {8,3} tessellation, there are four fish in the super-motif, which therefore must be contained in two of the isosceles triangles. However, by designing a new fish motif like that of *Circle Limit III*, we can create a new pattern based on the {10,3} tessellation, as shown in Figure 11.

Figure 10: A 5-arm cross pattern based on the {10,3} tessellation.

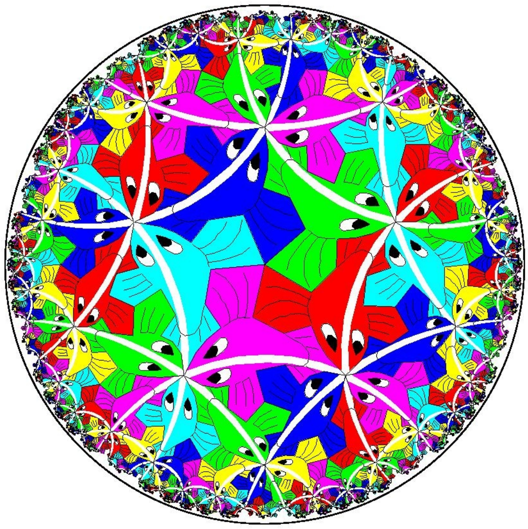

Figure 11: A fish pattern based on the {10,3} tessellation.

6. CONCLUSIONS

After discussing basic concepts, we have shown how different models of hyperbolic geometry play important roles in the generation of repeating patterns of the hyperbolic plane. First, the Poincaré circle model is useful for actually displaying such patterns because: (1) it is finite, so that viewers can see the entire pattern at once, and (2) it is conformal, so that copies of the motif retain their approximate shapes and are thus recognizable even when small. Next, the Weierstrass model plays a key role in the pattern generation process, since the transformations are easy to represent in that model. Finally, the Klein model can be used to create new motifs from existing motifs and thus to create new patterns.

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