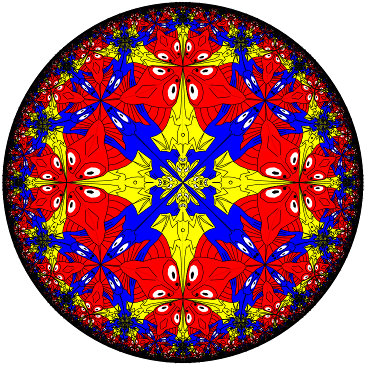
ICGG 2010

A Family of “Three Element” M.C. Escher Patterns

Douglas Dunham
University of Minnesota Duluth
Duluth, Minnesota
USA



Outline

- A brief history of "Three Element" patterns
- A review of hyperbolic geometry
- Repeating patterns
- Regular tessellations
- Families of patterns
- The family of Escher's *Circle Limit I* patterns
- The family of "Three Element" patterns
- Future work

History

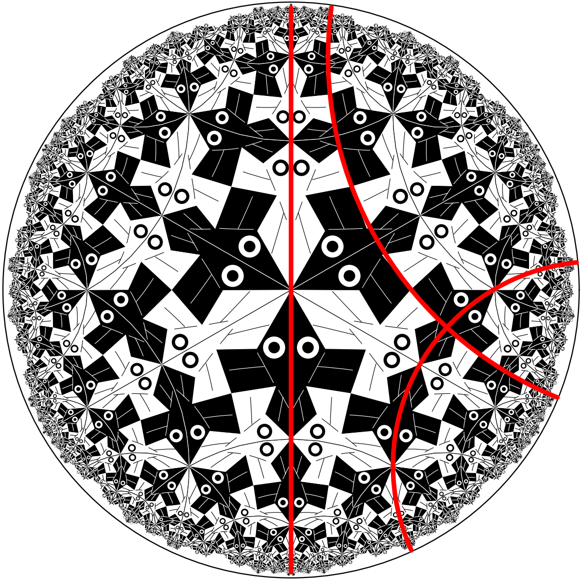
- In 1952 Escher created his Regular Division Drawing Number 85, consisting of fish, lizards, and bats.
- Shortly after that he drew that pattern on a rhombic dodecahedron.
- In 1963 Escher and C.V.S. Roosevelt commissioned the netsuke artist Masatoshi to carve the "three element" pattern on an ivory sphere.
- In 1977 Doris Schattschneider and Wallace Walker had the pattern printed on the net for a regular octahedron in their book *M.C. Escher Kaleidocycles.*
- In the early 1980's my students and I implemented our first hyperbolic pattern drawing program.
- Shortly after year 2000, I used a program to draw a "three element" pattern in the hyperbolic plane, the only one of the three classical geometries Escher did not use for "three element" patterns.

M.C. Escher's Regular Division Drawing 85 (1952)

Hyperbolic Geometry

- In 1901, David Hilbert proved that, unlike the sphere, there was no isometric (distance-preserving) embedding of the hyperbolic plane into ordinary Euclidean 3-space.
- Thus we must use *models* of hyperbolic geometry in which Euclidean objects have hyperbolic meaning, and which must distort distance.
- One such model, used by Escher, is the *Poincaré disk model*.
- The hyperbolic points in this model are represented by interior point of a Euclidean circle — the *bounding circle*.
- The hyperbolic lines are represented by (internal) circular arcs that are perpendicular to the bounding circle (including diameters as special cases).

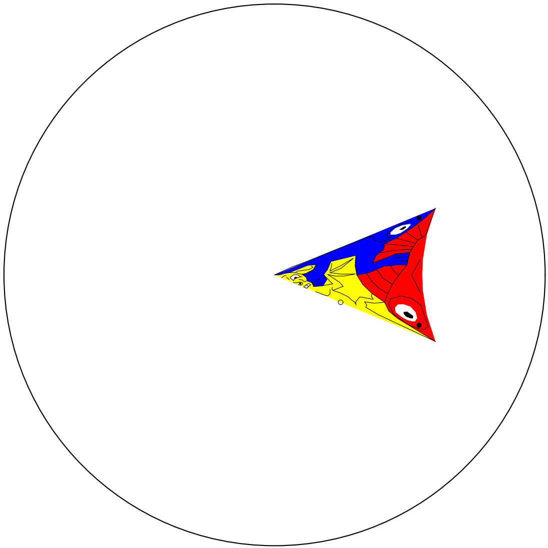
Escher's Circle Limit I Showing Hyperbolic Lines.



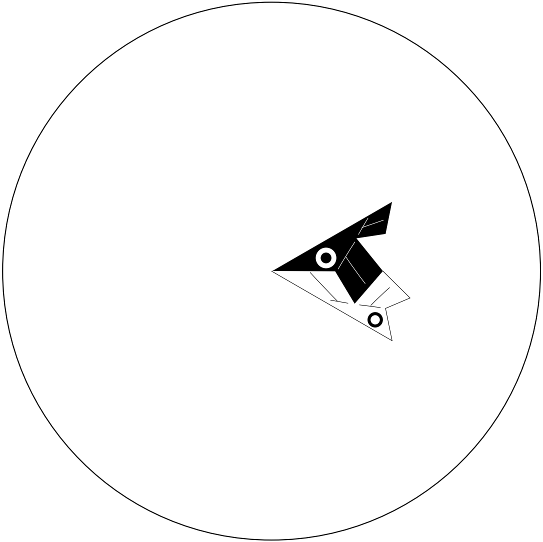
Repeating Patterns

- A *repeating pattern* in any of the 3 “classical geometries” is composed of congruent copies of a basic subpattern or *motif*.
- For example half a black fish plus half an adjacent white fish forms a motif for *Circle Limit I*
- Similarly, a triangle containing half a fish, half a lizard, and half a bat, forms a motif for a “three element” pattern.

A Motif for the Title Slide Three Element Pattern



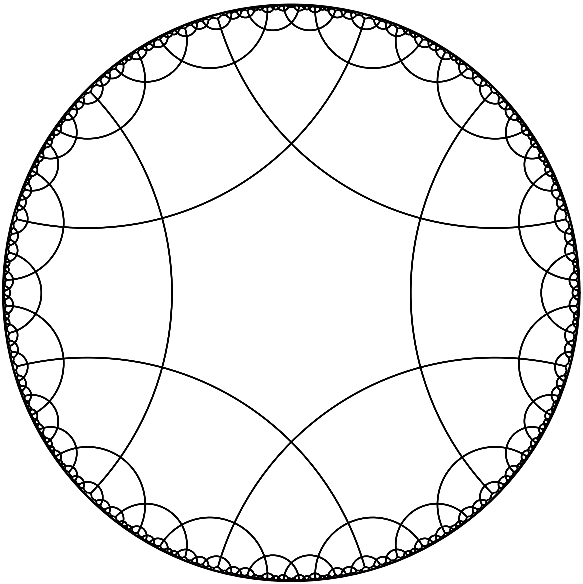
A Motif for Circle Limit I



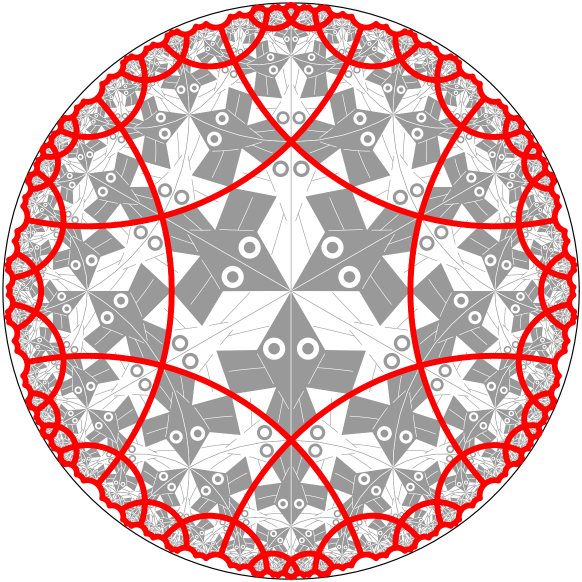
Regular Tessellations

- The *regular tessellation*, $\{p, q\}$, is an important kind of repeating pattern composed of regular p-sided polygons meeting q at a vertex.
- If $(p-2)(q-2) < 4$, $\{p, q\}$ is a spherical tessellation (assuming $p > 2$ and $q > 2$ to avoid special cases).
- If $(p-2)(q-2) = 4$, $\{p, q\}$ is a Euclidean tessellation.
- If $(p-2)(q-2) > 4$, $\{p, q\}$ is a hyperbolic tessellation. The next slide shows the $\{6, 4\}$ tessellation.
- Escher based his 4 "Circle Limit" patterns, and many of his spherical and Euclidean patterns on regular tessellations.

The $\{6, 4\}$ Tessellation.



The $\{6, 4\}$ Tessellation Underlying Circle Limit I



A Table of the Regular Tessellations

q \ p	3	4	5	6	7	8	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
8	*	*	*	*	*	*	$\cdots$
7	*	*	*	*	*	*	$\cdots$
6	□	*	*	*	*	*	$\cdots$
5	○	*	*	*	*	*	$\cdots$
4	○	□	*	*	*	*	$\cdots$
3	○	○	○	□	*	*	$\cdots$

□ - Euclidean tessellations

○ - spherical tessellations

* - hyperbolic tessellations

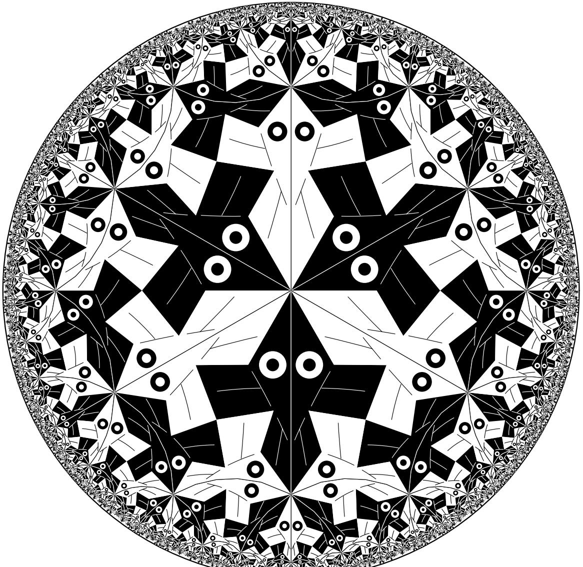
Families of Patterns

- If a pattern is based on an underlying $\{p, q\}$ tessellation, we can conceive of other patterns with the same motif (actually slightly distorted) based on a different tessellation $\{p', q'\}$.
- This observation leads us to consider an whole *family* of such patterns indexed by p and q.

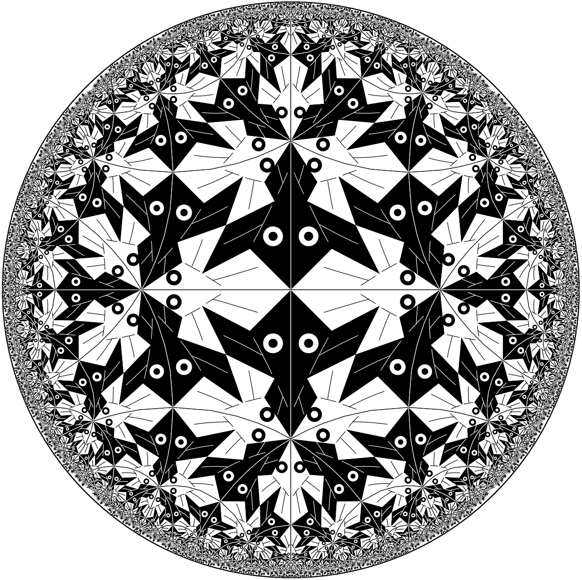
The *Circle Limit I* Family of Patterns

- For a *Circle Limit I* pattern based on a $\{p, q\}$ tessellation, both p and q must be even, since the backbone lines are axes of reflection symmetry.
- For these patterns, $p/2$ is the number of black fish meeting at their noses and $q/2$ is the number of white fish that meeting at noses.
- For this family, we let $(p/2, q/2)$ denote the pattern based on the $\{p, q\}$ tessellation.
- So *Circle Limit I* would be $(3, 2)$ in this notation.

A $(3,3)$ Circle Limit I Pattern.



A $(2, 3)$ Circle Limit I Pattern.

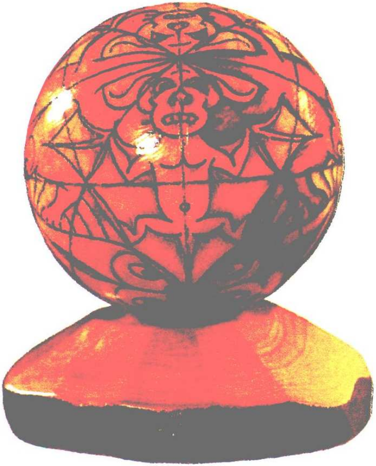


The Family of "Three Element" Patterns

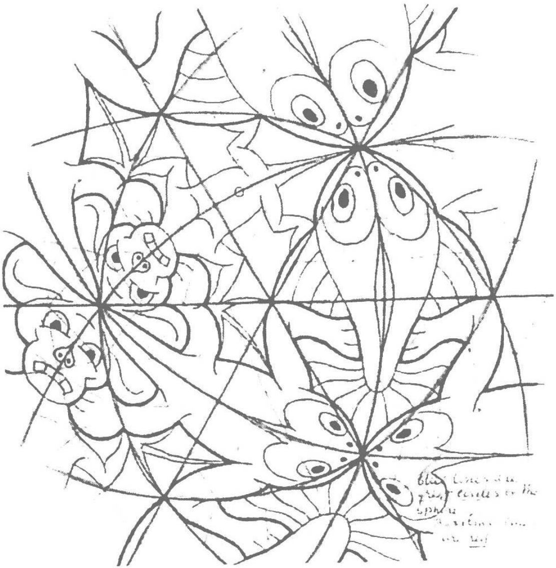
- In 1952 Escher created his Regular Division Drawing Number 85, consisting of fish, lizards, and bats, which represent the three "elements" water, earth, and air.
- The patterns of this family depend on three numbers, p, q, and r, which represent respectively the number of fish, lizards, and bats meeting at their heads. We let (p, q, r) denote such a pattern.
- Thus Regular Division Drawing would be denoted $(3, 3, 3)$, and the title slide would be $(4, 4, 4)$.
- If $1/p + 1/q + 1/r > 1$, the pattern will be spherical.
- If $1/p + 1/q + 1/r = 1$, the pattern will be Euclidean.
- If $1/p + 1/q + 1/r < 1$, the pattern will be hyperbolic.

Escher's Euclidean $(3, 3, 3)$ Regular Division Drawing 85

A Rhombic Dodecahedron $(2, 2, 2)$ Pattern (1952)
A Spherical $(2, 2, 2)$ Pattern (1963)

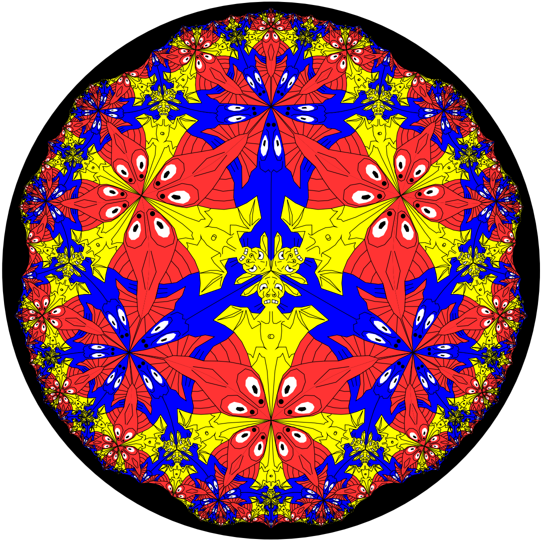


A Study for the $(2, 2, 2)$ Ivory Netsuke Carving

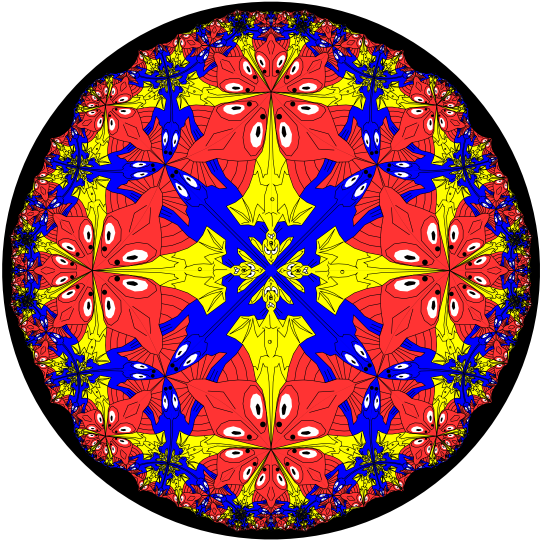


Masatoshi's Netsuke Carving (1963)

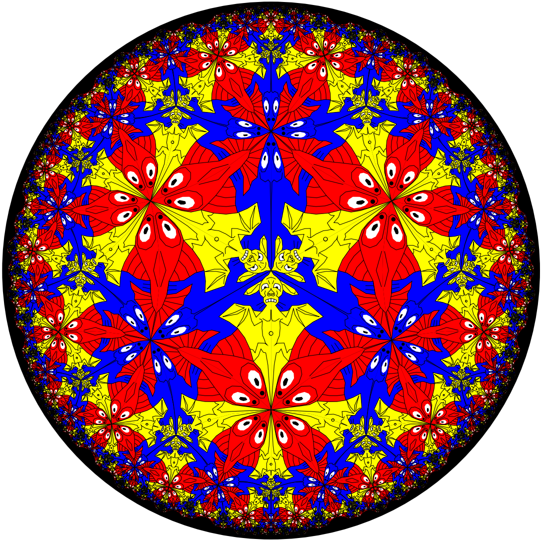
A $(4, 5, 3)$ "Three Element" Pattern



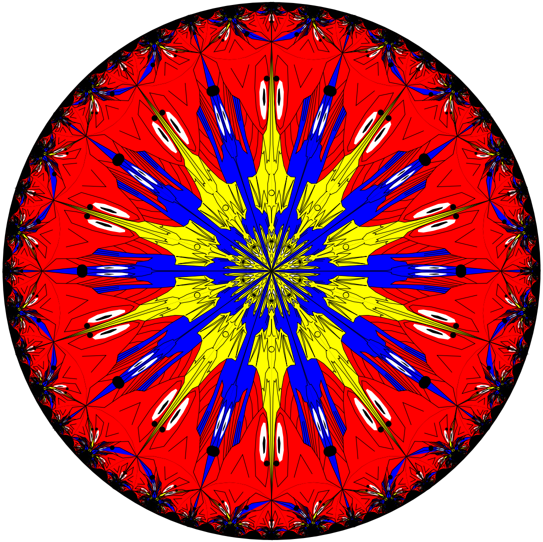
A $(5, 3, 4)$ “Three Element” Pattern



A $(4, 4, 3)$ "Three Element" Pattern



A $(4, 4, 10)$ "Three Element" Pattern



Future Work

- Extend the repeating pattern program so that it can also draw Euclidean and spherical patterns.
- Investigate other families of patterns by Escher and other artists.
- Create more patterns!

Thank you!

To all who worked on ICGG 2010