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A PROPERTY OF AREA AND PERIMETER

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Background

Our original goal was to randomly fill a **region** R of area A with successively smaller copies of a **fill-shape** (called a *m*otif in previous work), creating a fractal pattern.

We achieved that goal with our original algorithm, which we describe below. For that algorithm, the size of the fill-shapes was specified by an inverse power law.

Recently we used a modified version of the algorithm in which the size of the fill-shapes is determined using an estimate of the amount of "room" left after each placement.

We have evidence that the size of the fill-shapes in the new algorithm also obeys an inverse power law.

The Original Algorithm

We found experimentally that we could fill the region R if for $i = 0, 1, 2, \ldots$, the area A_i of i-th fill-shape obeyed an inverse power law:

$$A_i = \frac{A}{\zeta(c, N)(N + i)^c}$$

where where $c > 1$ and $N > 0$ are parameters, and $\zeta(c, N)$ is the Hurwitz zeta function: $\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(q+k)^s}$ (and thus $\sum_{k=0}^{\infty} A_i = A$).

We called this the Area Rule.

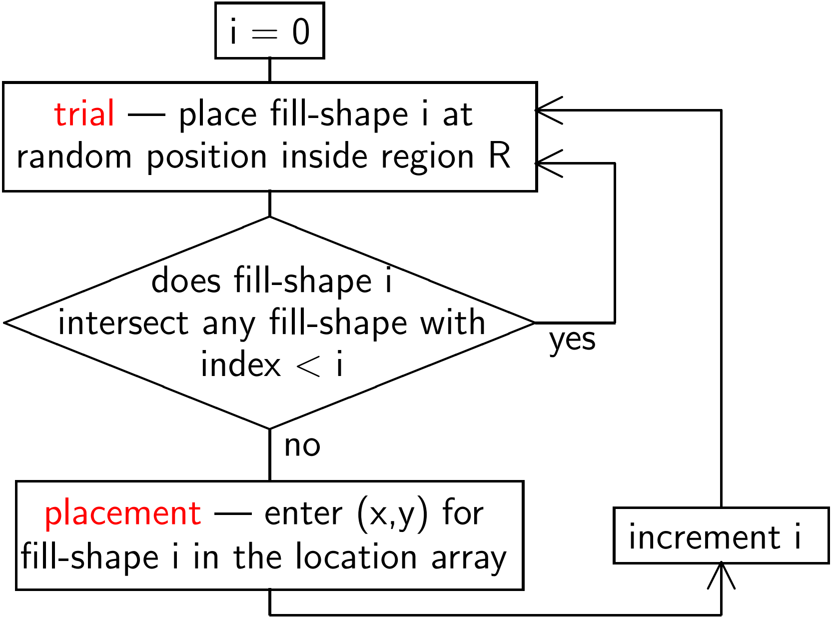
Algorithm Details

The algorithm works by successively placing copies m_i of the fill-shapes at locations inside the bounding region R.

This is done by repeatedly picking a random **trial** location (x, y) inside R until the fill-shape m_i placed at that location doesn't intersect any previously placed fill-shapes.

We call such a successful location a **placement**. We store that location in an array so that we can find successful locations for subsequent fill-shapes.

A Flowchart for the Algorithm



A Sample Pattern of Peppers

The New Algorithm

The new algorithm is the same as the original algorithm except for the determination of the size of the next fill-shape. Rather than specifying the size by the area rule, we use information about the part of the region R not yet filled to give the size of the next fill-shape.

Now, for discussion, we will restrict both R the fill-shapes to be disks, Although the algorithm works in the general case too.

We use the term **gasket** for the unfilled part of R after placing the first i fill-shapes. So the gasket looks like Swiss cheese when the fill-shapes are disks. Now we let A_i be the area of the gasket and P_i be its perimeter, then the radius r_{i+1} of the next disk is given by:

$$r_{i+1} = \gamma \frac{A_i}{P_i} \tag{1}$$

where $\gamma > 0$ is a dimensionless parameter which can be as large as 3 (but if $\gamma > 2$ the algorithm must be modified as discussed below).

Notes and Motivation

1. In the formula above, the size of the next fill-shape is certainly proportional to A_i since that is the remaining area available. Also P_i measures the boundary of that area that the next fill-shape can "run into" (it also measures the "fragmentation" of that area), hence the inverse relationship.

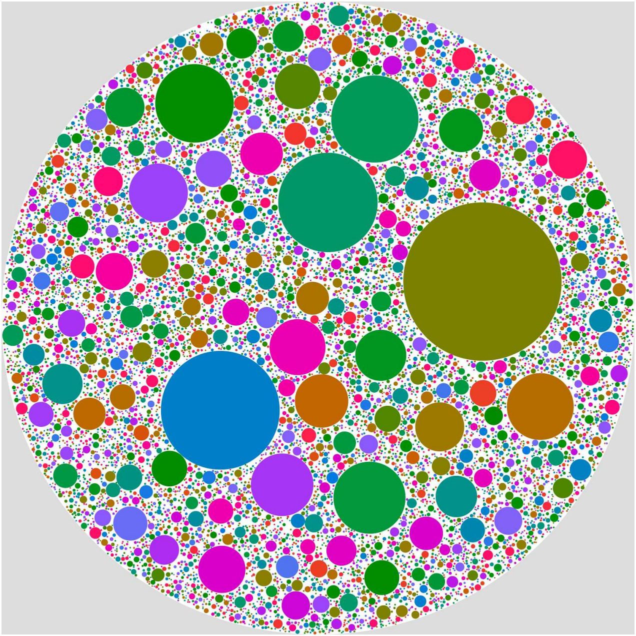
2. To get started, the area and perimeter of the bounding circle are πR^2 and $2\pi R$ respectively, so the initial area/perimeter ratio is $R/2$. Thus if $\gamma > 2$ the first disk doesn't fit within the bounding circle.

3. The motivation for formula for r_{i+1} comes from the **Dimensionless Gasket Width**:

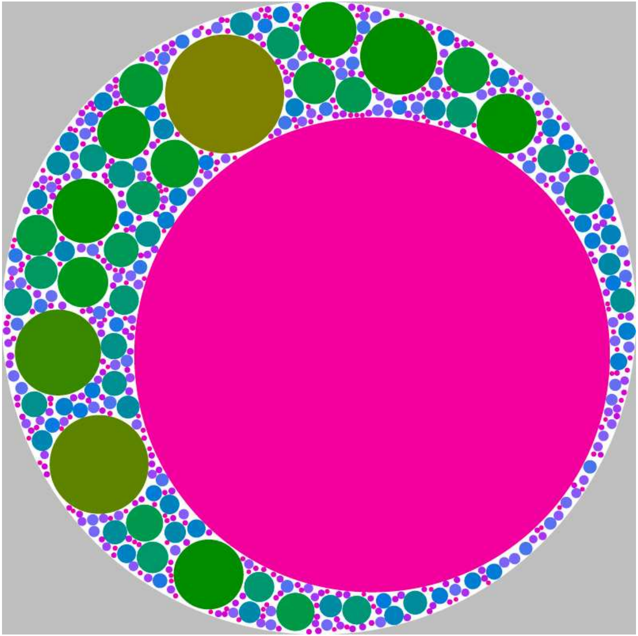
$$DGW = A_i/(r_{i+1}P_i)$$

which was used to analyze the halting probability of the original algorithm and was found to converge to a limit L in non-halting cases. Solving for r_{i+1} indicates how it should depend on A_i and P_i: $r_{i+1} = (1/L)A_i/P_i$. So we define γ by $\gamma = 1/L$.

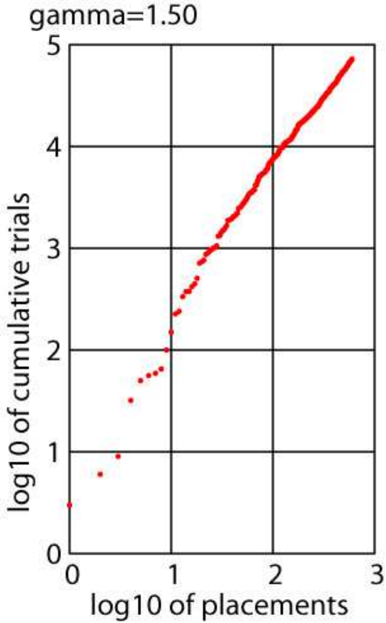
A Sample Pattern ($\gamma = 1/2$)



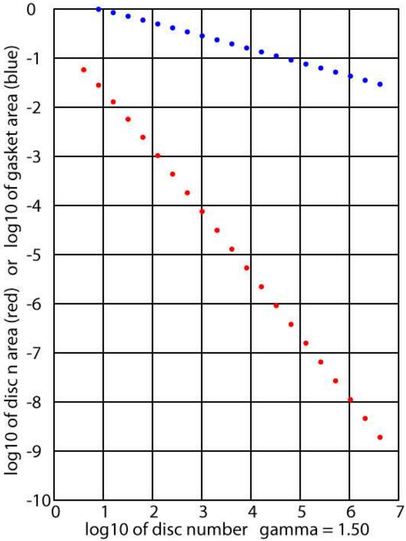
Another Pattern ($\gamma = 3/2$)



Log-log plot of trials versus placements



Log-log plot of disk areas (red) and the gasket area (blue)



The Property of Area and Perimeter

1. The plot above was again for $\gamma = 3/2$. The plot of the logarithms of the areas appears to be close to linear, indicating a power relationship. The slope of the last pair is -1.272727, which appears to be the beginning of a repeating decimal, and thus hints at a rational value for the exponent.

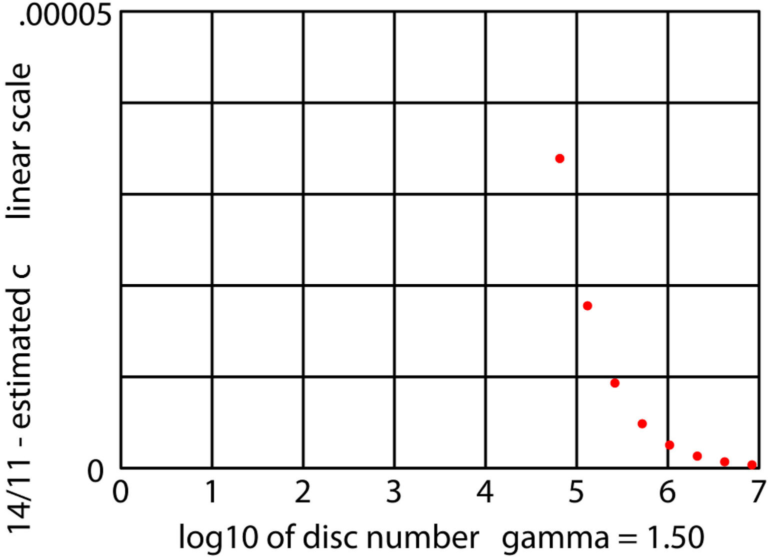
2. By running the algorithm with different rational values of γ, it was found that the following exponent was an exact fit to computational accuracy:
$$c = -\frac{4 + 2\gamma}{4 + \gamma}$$
The fact that the area, πr_i^2, of the i-th disc is proportional to $1/i^c$. is the **Property of Area and Perimeter** of the gasket refered to in the title.

3. **A Challenge:** derive the formula for c above from the recursion of Equation (1).

Convergence to the exponent c

The next plot shows how fast the estimated values of c (from successive disk area values at $i =$ a power of 2) converge to the value $14/11$ given by the formula above when $\gamma = 3/2$.

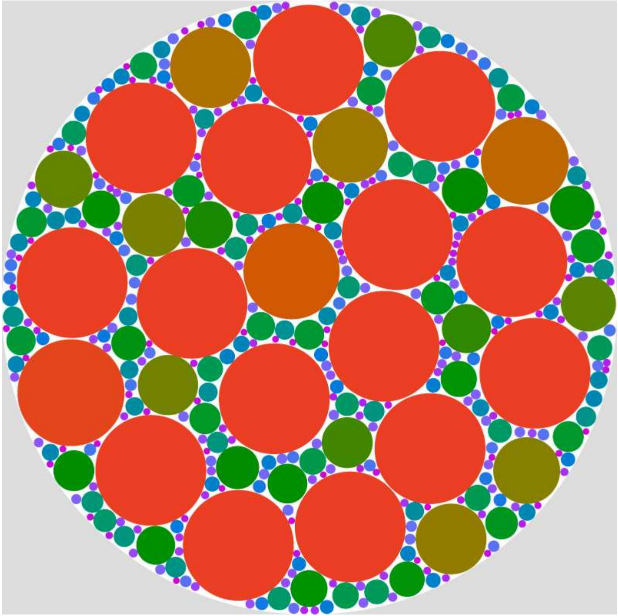
Convergence of c to $14/11$



Variation 1 — Larger γ Values

- We have noted that the new algorithm as given will not work if $\gamma > 2$. But we can start with disks of a fixed size (smaller than the bounding circle of course) and keep placing them until the disks given by the radius formula are smaller, then switch to the new algorithm. The next slide shows a sample pattern when $\gamma = 5/2$.
- It can be shown that the fractal dimension of the disk pattern is given by $D = 2/c$ in both the new algorithm and this variant.

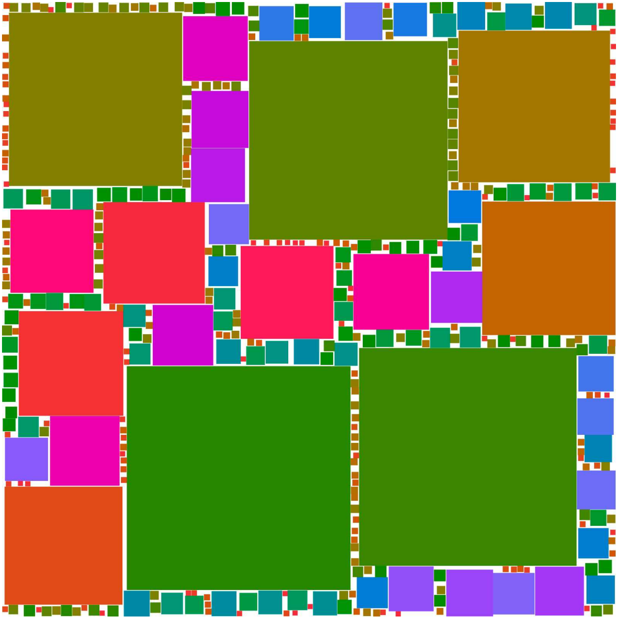
A Pattern using the modified algorithm



Variation 2 — Other Shapes

- We started investigating our new algorithm with discs as fill-shapes within a circular region R since the test for overlap of discs is simple: if the distance between their centers is larger than the sum of their radii, they don't overlap.
- However, as with the original algorithm, we conjecture that the new algorithm proceeds without halting for "reasonable" combinations of R and the fill-shape, and for positive values of γ within a small interval.
- Perhaps the next simplest combination for R and the fill-shapes are squares. And, in fact that combination works as shown in the next slide. For squares γ can be twice as big and is equal to 2.8 in that slide. There are 400 squares.

A Pattern of squares using the new algorithm



Variation 3 — Higher Dimensions

- Another variation is to consider n-dimensional balls as fill-shapes within a bounding sphere in n dimensions. Here the next radius is given by the formula
$$r_{i+1} = \gamma \frac{V_i}{A_i}$$
where V_i is the n-dimensional gasket volume and A_i is the $(n-1)$-dimensional gasket surface area.
- By running several samples with different (rational) γ values, and for $n = 3$ and $n = 4$ we discovered that the ball volumes seemed to follow a power law whose exponent $-c$ is given by the table on the next slide.
- For $n = 1$ we have directly verified the c value in the table.

A Table of c values for Spheres

Dimension	c-value
1	$1+\gamma$
2	$(4+2\gamma)/(4+\gamma)$
3	$(9+3\gamma)/(9+2\gamma)$
4	$(16+4\gamma)/(16+3\gamma)$
n	$(n^2+n\gamma)/(n^2+(n-1)\gamma)$

Table: A table of c values for balls within a sphere in n-dimensions.

Cubes within a Cube in other Dimensions

- We also considered *n*-dimensional cubes as fill-shapes within a bounding *n*-cube for dimensions other than 2. Here the next side length is given by the formula

$$s_{i+1} = \gamma \frac{V_i}{A_i}$$

  where V_i is the *n*-dimensional gasket volume and A_i is the $(n-1)$-dimensional gasket surface area as before.
- Again, by running several samples with different (rational) γ values, and for $n = 3$ and $n = 4$ we discovered that the ball volumes seemed to follow a power law whose exponent $-c$ is given by the table on the next slide.
- Again for $n = 1$ we have directly verified the c value in the table.

A Table of c values for Cubes

Dimension	c-value
1	$(2+\gamma)/2$
2	$(8+2\gamma)/(8+\gamma)$
3	$(18+3\gamma)/(18+2\gamma)$
4	$(32+4\gamma)/(32+3\gamma)$
n	$(2n^2+n\gamma)/(2n^2+(n-1)\gamma)$

Table: A table of c values for filled n-cubes within an n-cube in n-dimensions.

Future Work

- Prove that the new algorithm always converges for all γ less than some minimum value.
- Prove that the areas of the disks, or spheres in 3D, decrease according to a power law, and similarly for squares and cubes.
- Prove that the power laws are given by the formulas above.

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