Artistic Patterns: from Randomness to Symmetry

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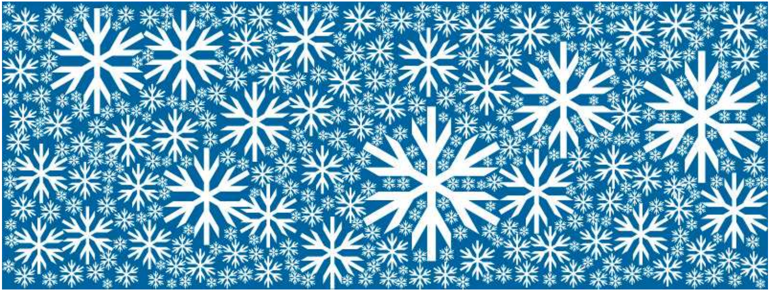


Outline

- Background and the “Area Rule”
- The algorithm
- A conjecture
- Dependence on parameters c and N
- Another conjecture
- Circles, squares, and triangles
- Other sample patterns
- Conclusions and future work
- Contact information

Background

Our original goal was to create patterns by randomly filling a region R with successively smaller copies of a motif, creating a fractal pattern.

This goal can be achieved if the motifs follow an "area rule" which we describe in the next slide.

The resulting algorithm is quite robust in that it has been found to work for hundreds of patterns in (combinations of) the following situations:

- The region R is connected or not.
- The region R has holes — i.e. is not simply connected.
- The motif is not connected or simply connected.
- The motifs have multiple (even random) orientations.
- The pattern has multiple (even all different) motifs.
- If R is the fundamental region for one of the 17 plane crystallographic (or "wallpaper") groups, that region can be replicated using isometries from the group to tile the plane. The code is different and more complicated in this case.

The Area Rule

If we wish to fill a region R of area A with successively smaller copies of a motif (or motifs), it has been found experimentally that this can be done for $i = 0, 1, 2, \ldots$, with the area A_i of the i-th motif obeying an inverse power law:

$$A_i = \frac{A}{\zeta(c, N)(N + i)^c}$$

where where $c > 1$ and $N > 0$ are parameters, and $\zeta(c, N)$ is the Hurwitz zeta function: $\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(q+k)^s}$ (and thus $\sum_{k=0}^{\infty} A_i = A$).

We call this the **Area Rule**

The Algorithm

The algorithm works by successively placing copies m_i of the motif at locations inside the bounding region R.

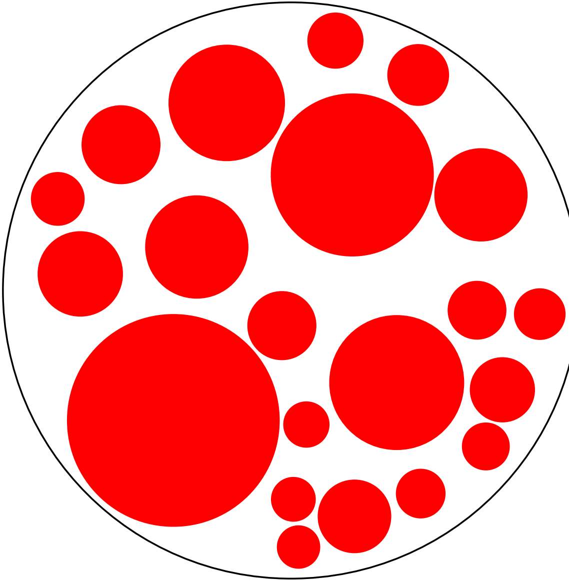
This is done by repeatedly picking a random **trial** location (x, y) inside R until the motif m_i placed at that location doesn't intersect any previously placed motifs.

We call such a successful location a **placement**. We store that location in an array so that we can find successful locations for subsequent motifs.

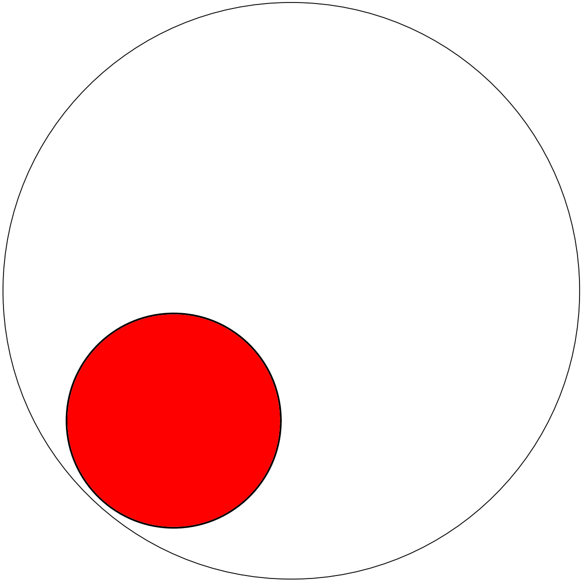
We show an example of how this works in the following slides.

A pattern of 21 circles partly filling a circle

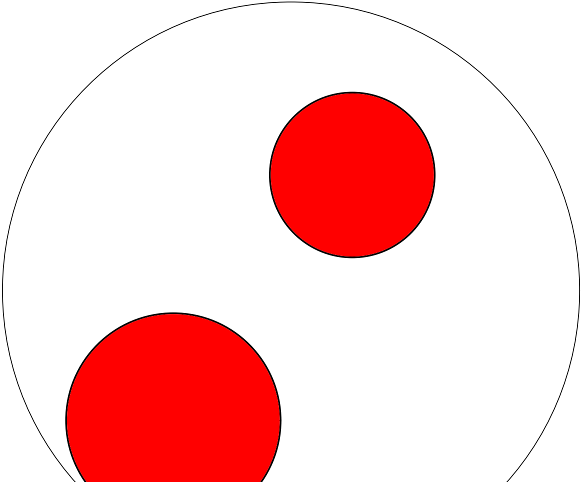
(Note: $c = 1.30$ and $N = 2$ in this example)



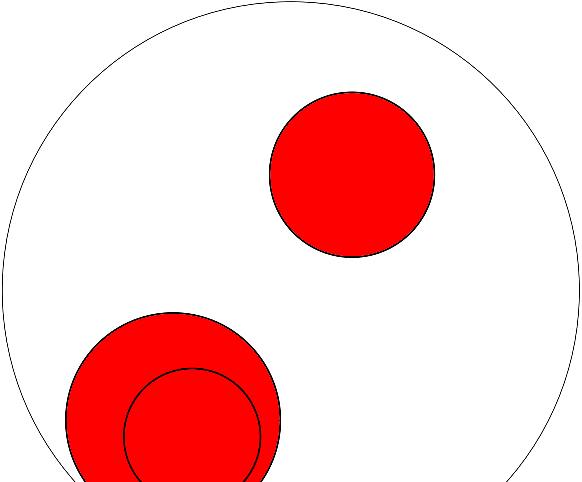
Placement of the first motif



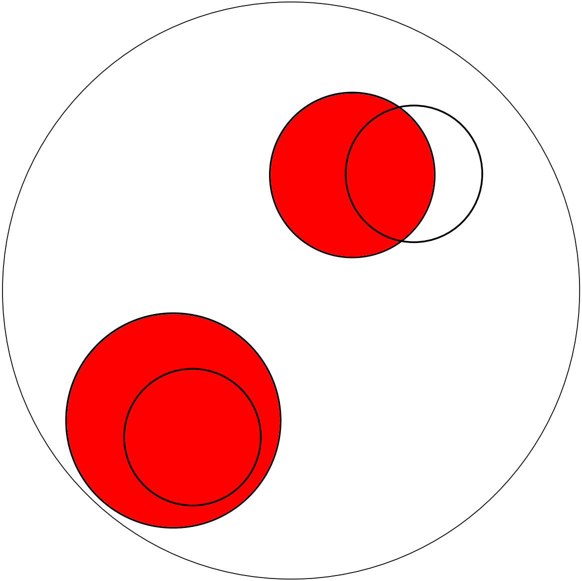
Placement of the second motif



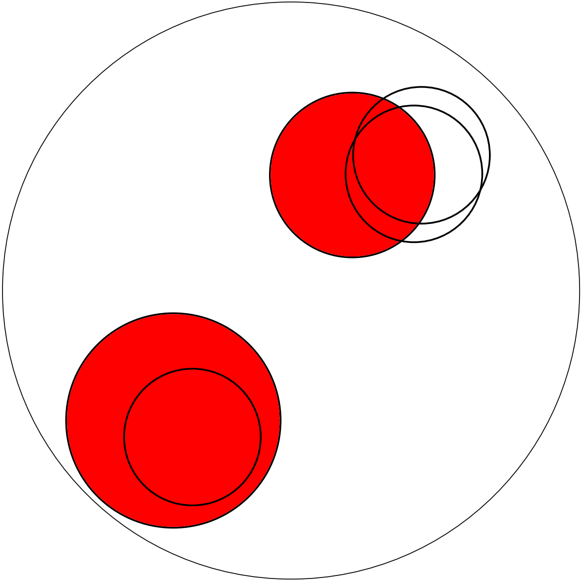
First trial for the third motif



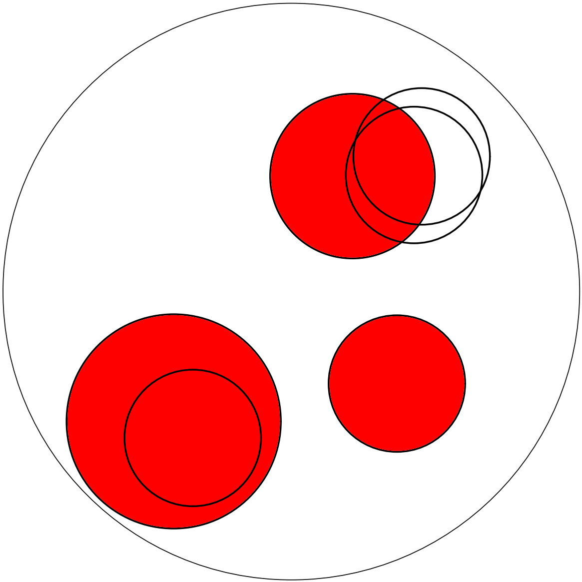
Second trial for the third motif



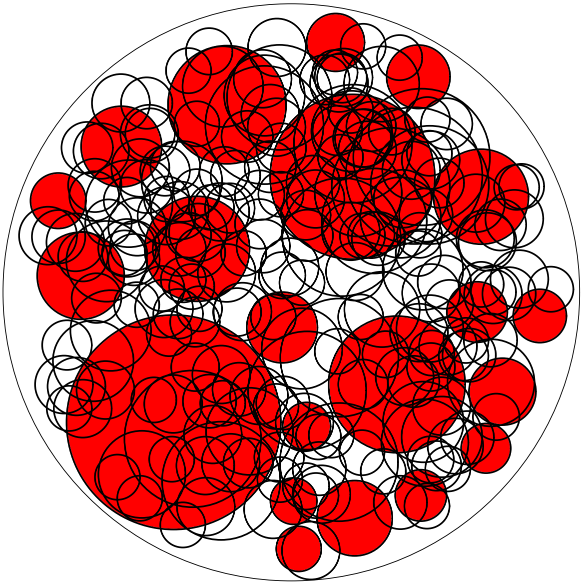
Third trial for the third motif

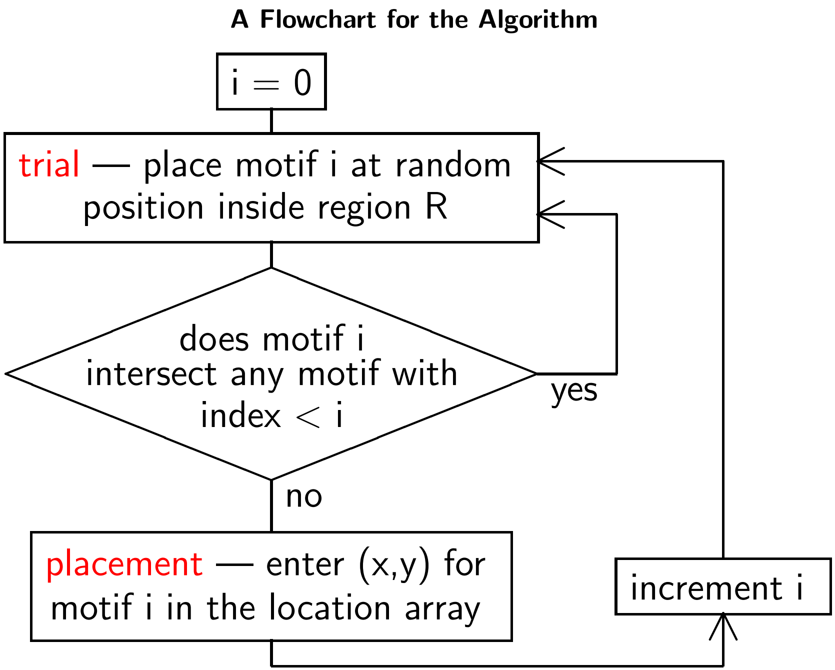


Successful placement of the third motif



All 245 trials for placement of the 21 circles




A Flowchart for the Algorithm
i = 0
trial — place motif i at random position inside region R
does motif i intersect any motif with index < i
yes
no
placement — enter (x,y) for motif i in the location array
increment i

A Conjecture

Conjecture: The algorithm will randomly fill *any* reasonably defined region R with *any* reasonably defined motif(s), and it will not halt for $1 < c < c\_max$ and $N > N\_min > 0$, for appropriate values of c_max and N_min (which depend on the shapes of R and the motifs).

Typically values of c_max seem to be somewhat less than 1.5; often the values of N that were used were 2 or greater (not necessarily integer).

This algorithm has been implemented in dimensions 1, 2, 3, and 4, though we note that 1D patterns are not very interesting, and the "front" motifs in 3D and 4D obscure the motifs behind them.

In 1D, in which the motifs are line segments, it has been proved that the algorithm never halts for any c with $1 < c < 2$.

Also, the fractal dimensions of the patterns (not the unused portion of R) can be calculated to be $1/c, 2/c$, and $3/c$ in the 1D, 2D, and 3D cases respectively, which leads to the conjecture that the fractal dimension is d/c in d-dimensional space.

Dependence of patterns on **c** and **N**

By examining the formula that gives the Area Rule:

$$A_i = \frac{A}{\zeta(c, N)(N+i)^c}$$

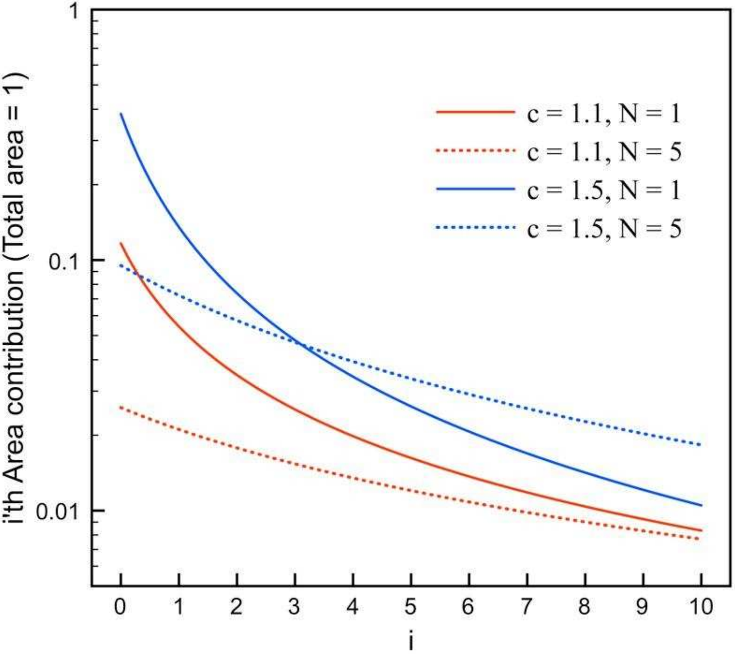
one can see that as c increases or N decreases, there is a larger difference in the sizes of the first few motifs.

Conversely, as c decreases or N increases, the first few motifs are closer in size.

The next slide shows a graph of how the sizes of the i-th motif decrease for different values of c and N.

Following that, we show how patterns depend on c (as far as symmetry goes there is much less dependence on N).

Graph of areas A_i for different values of c and N



Another Conjecture

Another Conjecture: As the parameter c increases from 1 to c_max, the patterns start out as quite random, then become more symmetric.

Note that in the lower limit as c_max decreases to 1, the motifs decrease in size, becoming dots, so that in the limit, the pattern is just a random selection of points.

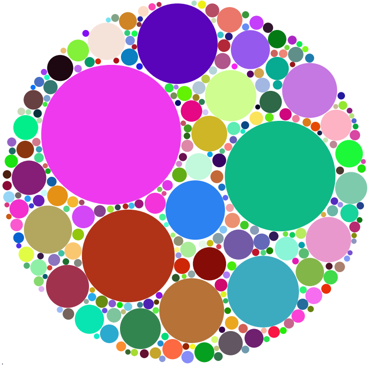
In the examples below, the patterns start with low c values, and then become more symmetric as c increases toward c_max.

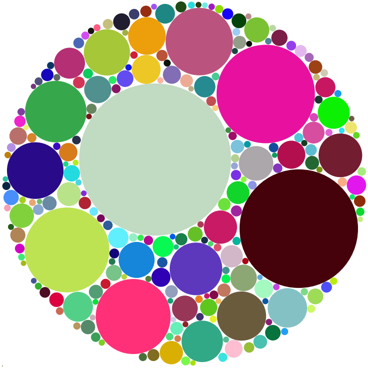
Circle patterns with c = 1.24 and 1.32

The value of N is 2.5 for each of these patterns.

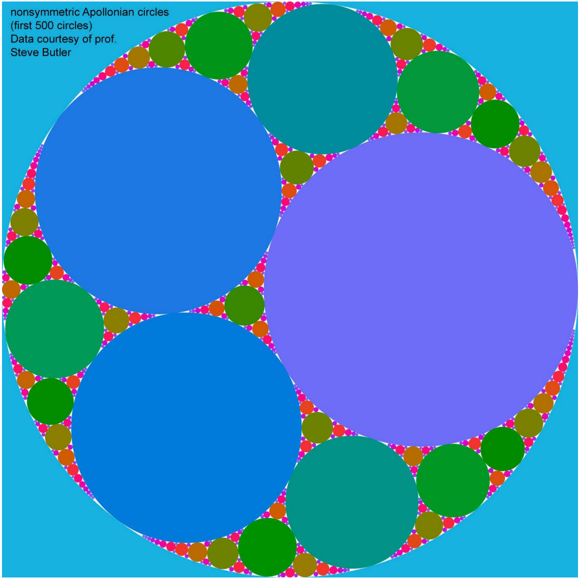
Circle patterns with c = 1.40 and 1.48

The value of N is 2.5 for each of these patterns.



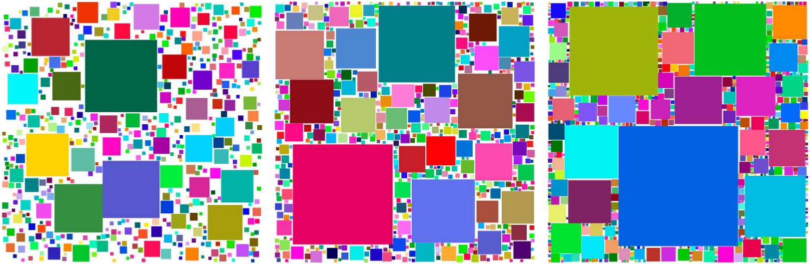


Apollonian Circles



Squares

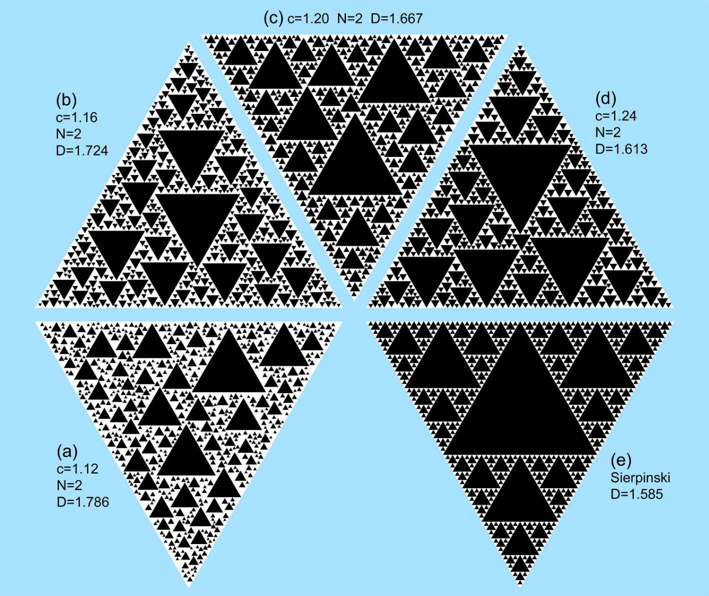
In these patterns, $N = 2$, and c takes the values 1.16 (a), 1.32 (b) and 1.48 (c).



(a) (b) (c)

Triangles

In these patterns, $N = 2$, and $c = 1.12$ (a), 1.16 (b), 1.20 (c), and 1.24 (d). The fractal dimensions are 1.786, 1.724, 1.667, 1.613, and 1.585.



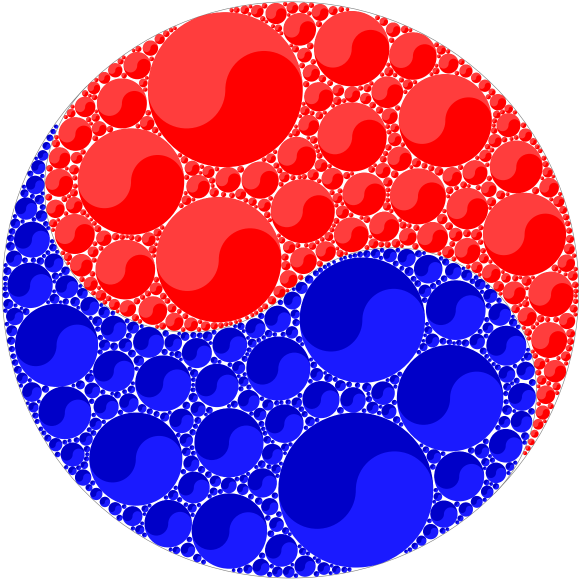
Sample Patterns

In the following slides, we exhibit the robustness of the algorithm by showing combinations of:

- Connectivity of the bounding region R.
- Non simply connected regions R.
- Non connected or non simply connected motifs.
- The motifs with multiple or even random orientations.
- Multiple, even all different, motifs.
- Periodicity — “wallpaper” groups.

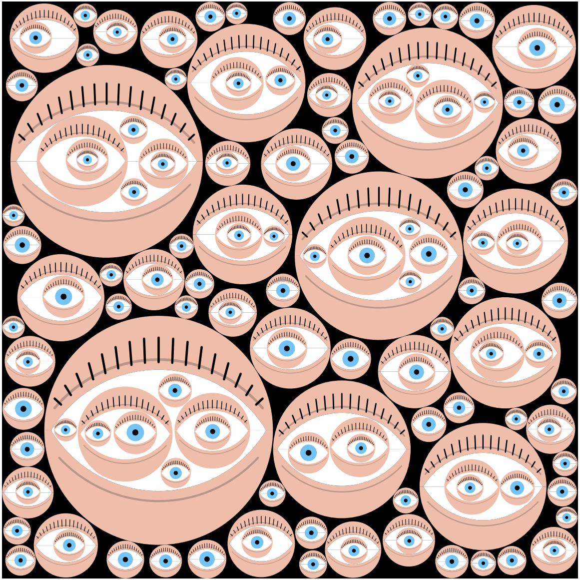
Two regions forming a yin and yang

In this pattern, $c = 1.47$ and $N = 3$, with 92% fill;
it has 180° rotational color symmetry.



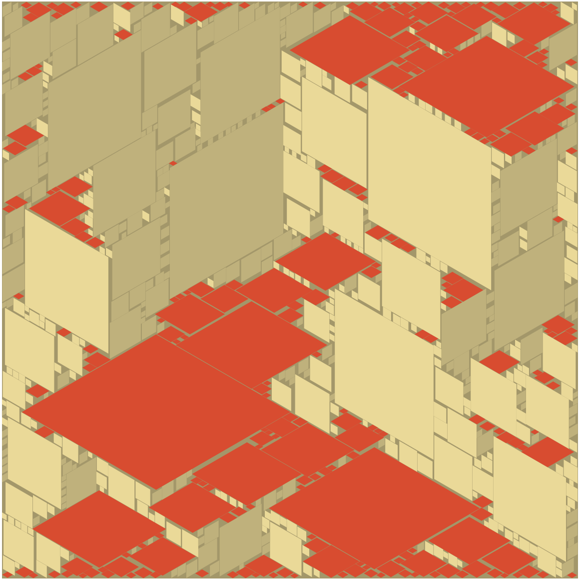
A pattern non-simply connected eye motifs

In this pattern, $c = 1.20$ and $N = 3$, with 56% fill; only eyes with no contained eyes have pupils.



Rhombi in three orientations and colors

In this pattern, $c = 1.52$ and $N = 8$ with 91% fill.

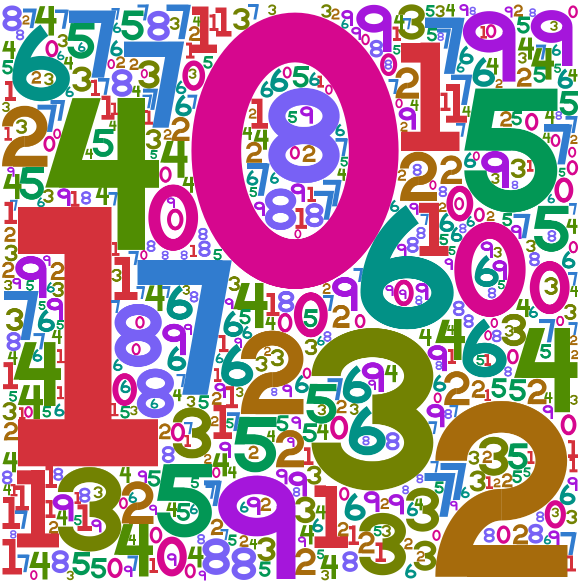


A periodic pattern of randomly oriented peppers

In this pattern, $c = 1.26$ and $N = 3$ with 80% fill.

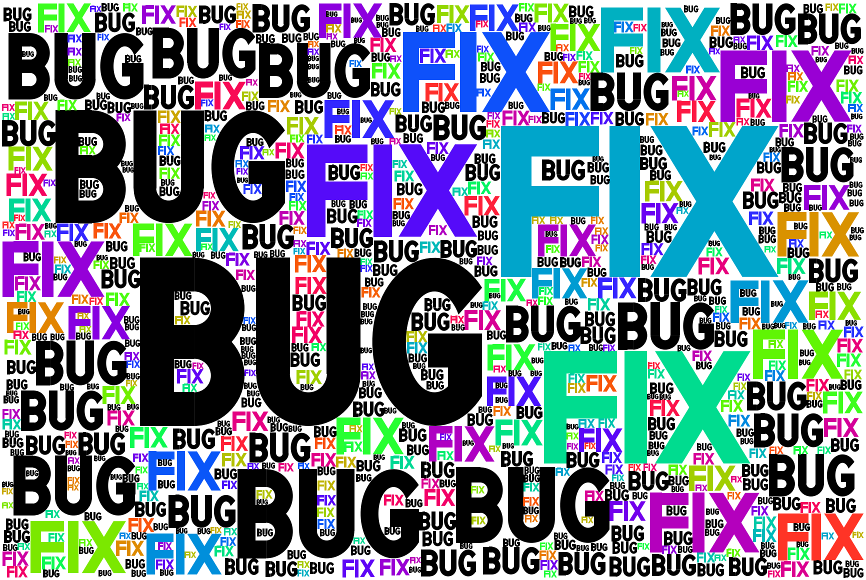
A pattern of the 10 digit motifs

In this pattern, $c = 1.19$ and $N = 2$ with 68% fill.

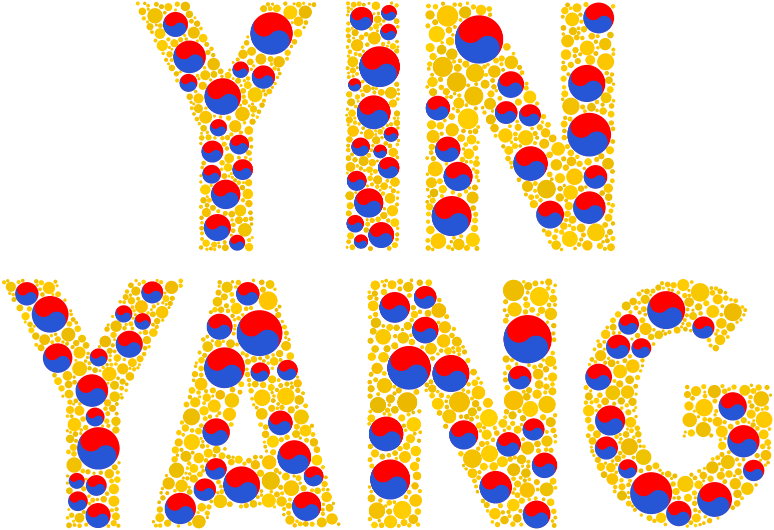


A pattern with the words BUG and FIX as motifs

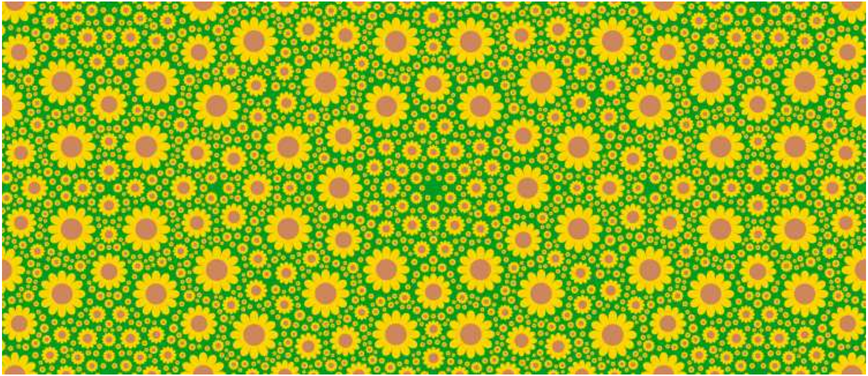
In this pattern, $c = 1.155$ and $N = 2$ with 62% fill.



YIN YANG latin letters filled with two motifs

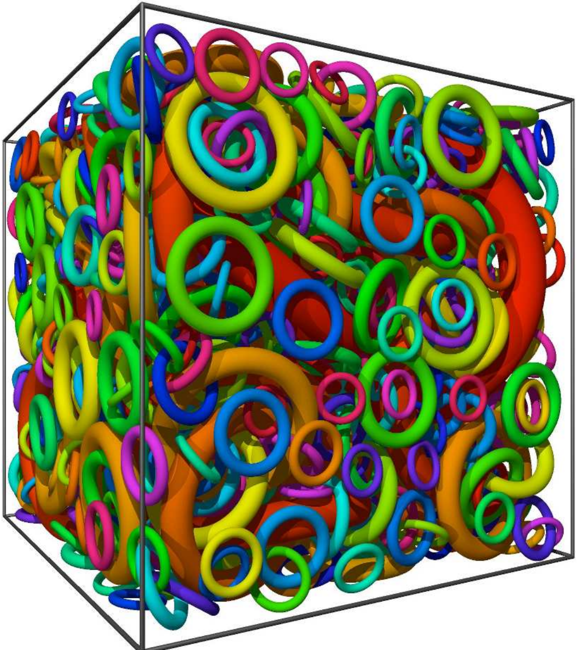


A flower pattern with $p6mm$ symmetrey



A 3D pattern of tori by Paul Bourke

Note that some tori are linked.



Future Work

- We would like to obtain more fractal, but regular patterns that we could approximate with our algorithm and see if the high c values approach the fractal dimension of the regular pattern.
- We have shown a pattern that is periodic with $p6mm$ symmetry, and thus can tile the plane. It would also seem possible to create locally fractal patterns having global symmetries of the other plane symmetry groups using our techniques.
- There are a few things that can be proved mathematically about these patterns, but there are a number of conjectures that have yet to be proved.

Acknowledgements and Contact

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