Seeing geometry through hyperbolic art

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Abstract

The hyperbolic plane is the least familiar of the three ”classical” geometries, which also include the Euclidean plane and the sphere. In this workshop, we will learn about hyperbolic geometry and more about geometry in general by studying artistic hyperbolic patterns, in particular, those of M.C. Escher.

Figure 1 below shows the Dutch artist M.C. Escher's *Circle Limit III* pattern, usually considered to be the most appealing of his four “Circle Limit” patterns. Figure 2 shows an Islamic-style pattern that is closely related to *Circle Limit III*, in that both patterns have one kind of 4-fold rotation point and the two kinds of 3-fold rotation points.

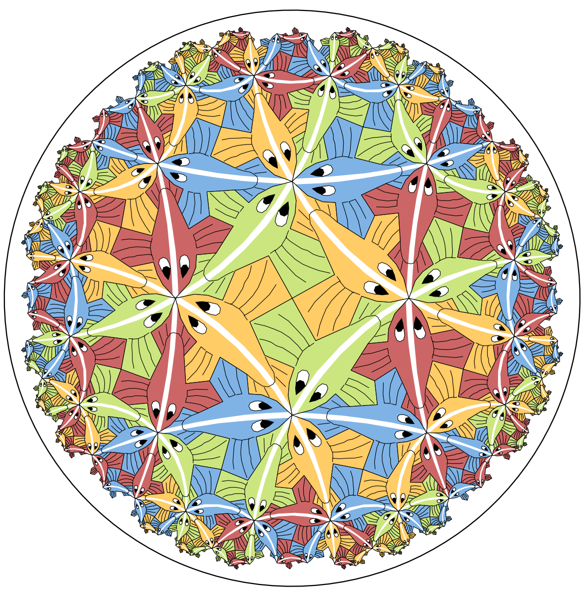

Figure 1: A computer rendition of Escher's *Circle Limit III* pattern.

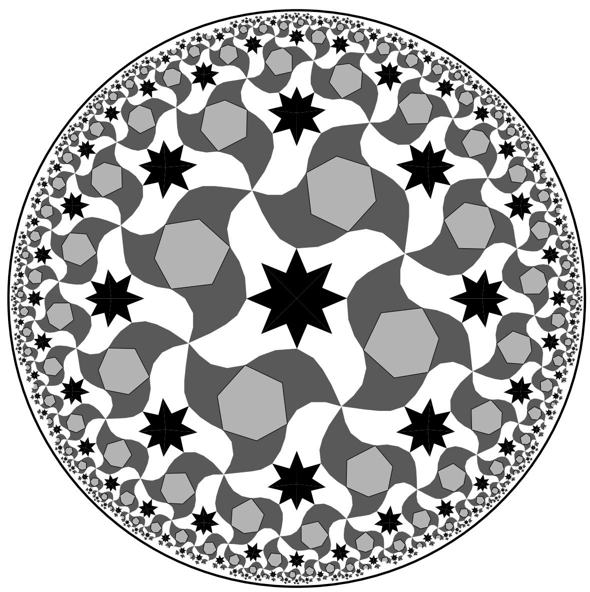

Figure 2: An Islamic-style hyperbolic pattern.

The three “classical” geometries are the sphere, the Euclidean plane, and the hyperbolic plane. They have constant positive, zero, and negative curvatures, respectively. The hyperbolic plane is the least familiar of the three geometries — probably because it cannot be smoothly embedded in ordinary 3-space, as can the other two. Thus, we must be content with Euclidean models of hyperbolic geometry. One of these models is the Poincaré circle model, shown in Figures 1 and 2. This model is the hyperbolic analog of a stereographic map of the sphere. We can gain insight as to how hyperbolic geometry works by examining patterns in the different models of it.

A *repeating pattern* in any of the three geometries consists of many, symmetrically placed copies of a basic subpattern called the *motif* for the pattern. For example, one of the fish serves as a motif in the *Circle Limit III* pattern, since hyperbolic distance in the circle model is measured in such a way that all the fish are the same size. Usually we require that the motifs do not overlap, and in Escher patterns there are no gaps either. It seems necessary to have a repeating pattern in a model in order to understand its hyperbolic nature.

One important kind of repeating pattern is the regular tessellation $\{p, q\}$ by regular p-sided polygons meeting q at a vertex. If $(p-2)(q-2) > 4$, one obtains a regular tessellation of the hypberbolic plane, otherwise the tessellation is Euclidean or spherical. The tessellation $\{4, 4\}$ is the familiar Euclidean tessellation of a grid of squares. Figure 3 below shows the regular hyperbolic tessellation $\{6, 4\}$. Many of Escher's patterns are based on regular tessellations. Figure 4 shows how *Circle Limit III* is based on the tessellation $\{8, 3\}$.

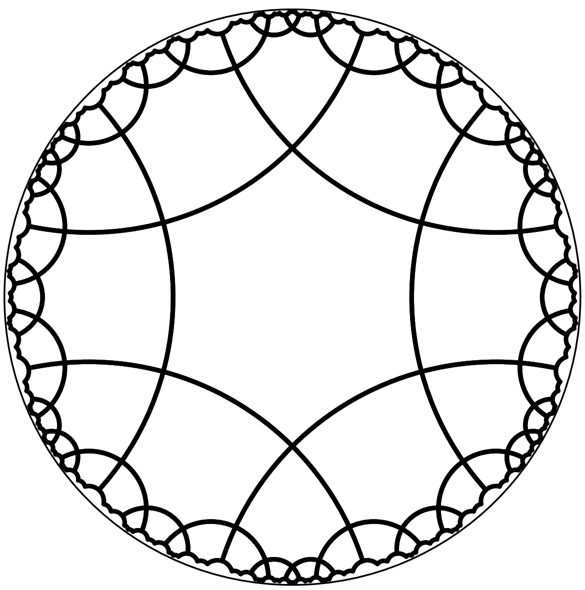

Figure 3: The regular hyperbolic tessellation $\{6, 4\}$.

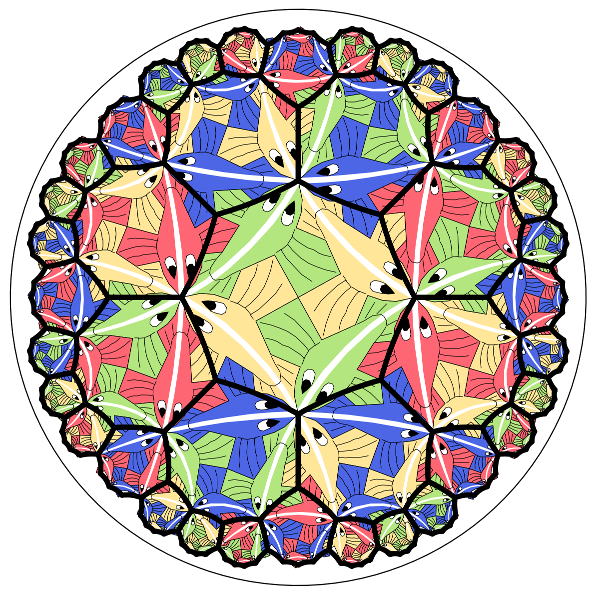

Figure 4: The tessellation $\{8, 3\}$ superimposed on the *Circle Limit III* pattern.

In this workshop, we will analyze repeating patterns in each of the classical geometries. We will discover the underlying tessellations and the transformations used to create the pattern. There are other interesting geometric details that bear investigation, such as color symmetry and the white backbone lines in the *Circle Limit III* pattern.

The following link describes the origin of the poster pattern for Mathematics Awareness Month 2003: `http://www.mathaware.org/mam/03/essay1.html` (which is on the MAM 2003 web site: `http://www.mathaware.org/mam/03/`).