How to Create Repeating Hyperbolic Patterns

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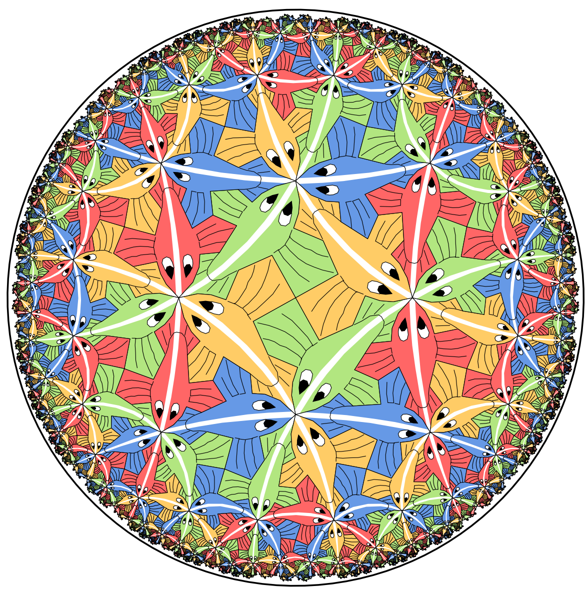


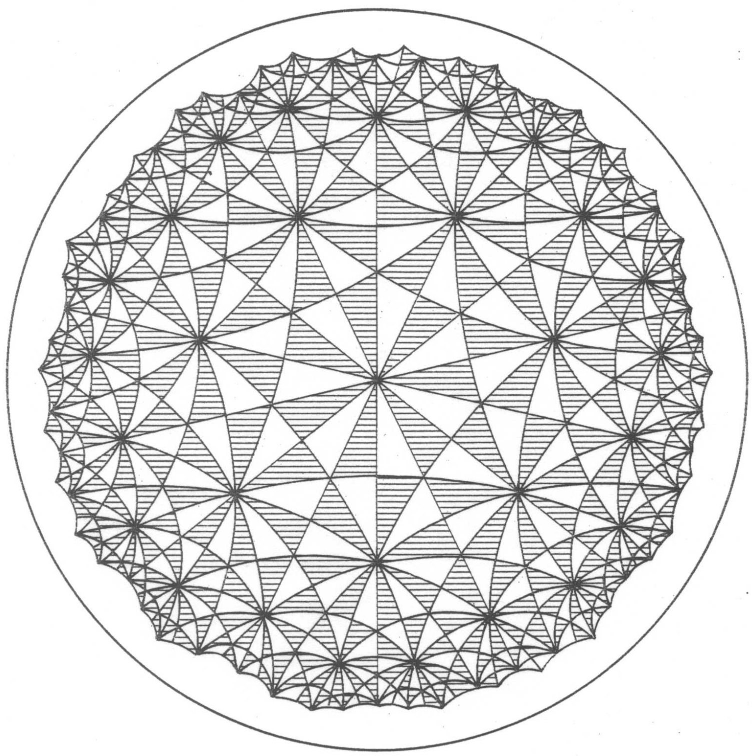
Outline

1. History
2. Computer generation of repeating hyperbolic patterns
3. Some new patterns
4. Future work

1. History

1. **Pre-Escher**
2. **Escher's patterns**
3. **Post-Escher = Dunham**

Triangle group (7,3,2) tessellation
Originally in *Vorlesungen über die Theorie der elliptischen Modulfunctionen*
F. Klein and R. Fricke, 1890.



H.S.M. Coseter's Figure 7
in *Crystal Symmetry and Its Generalizations*
Trans. Royal Soc. of Canada, 1957.

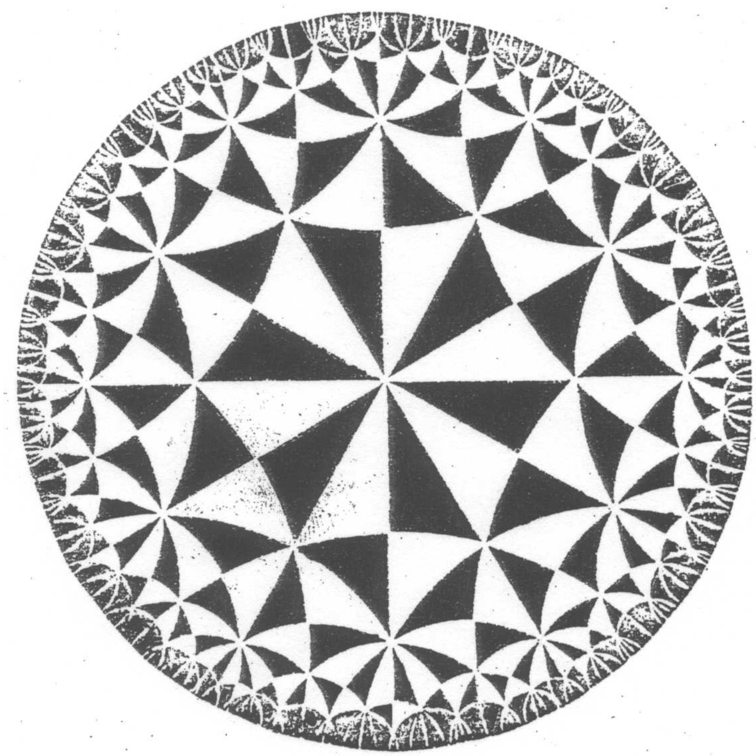
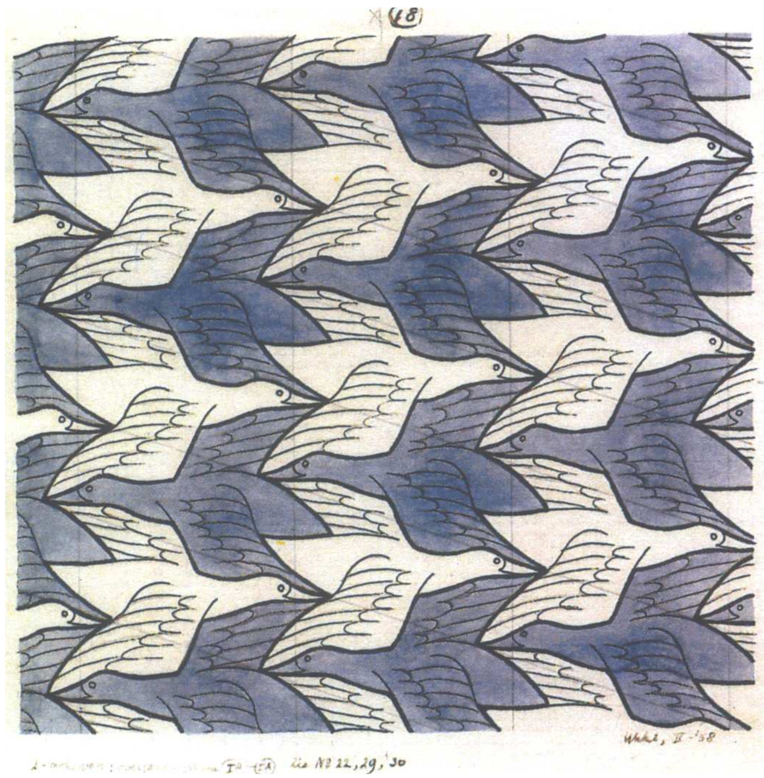
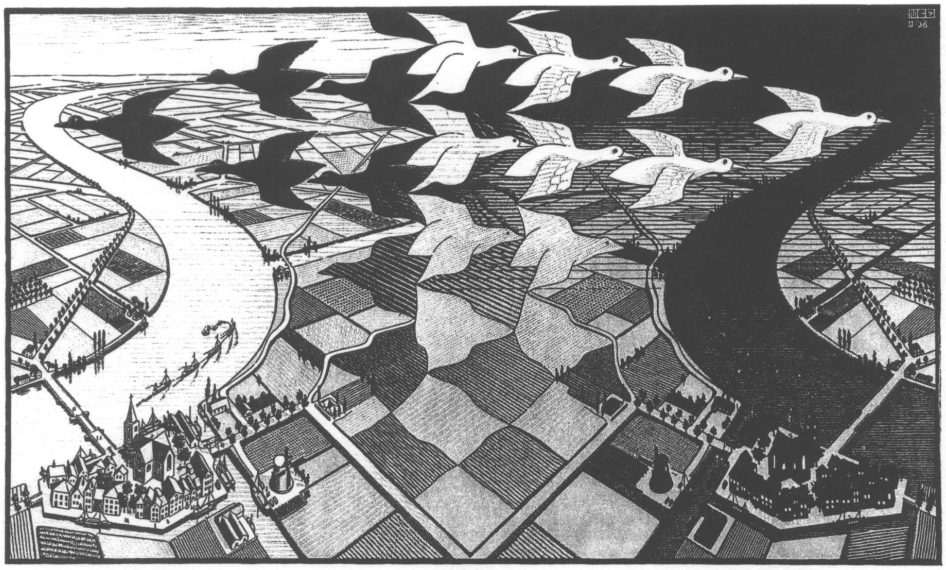


FIGURE 7

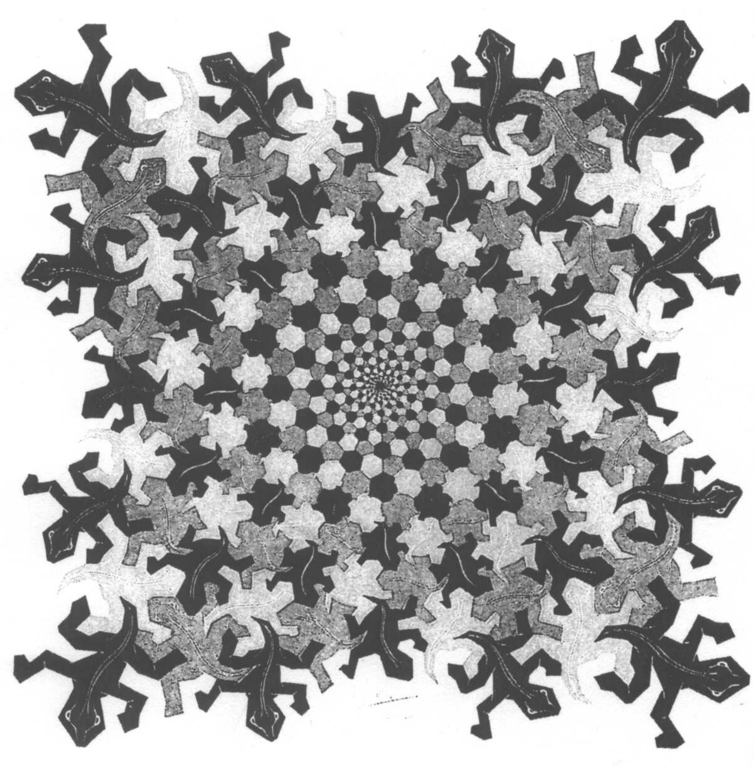
Escher's Notebook Drawing 18



Escher's *Day and Night*



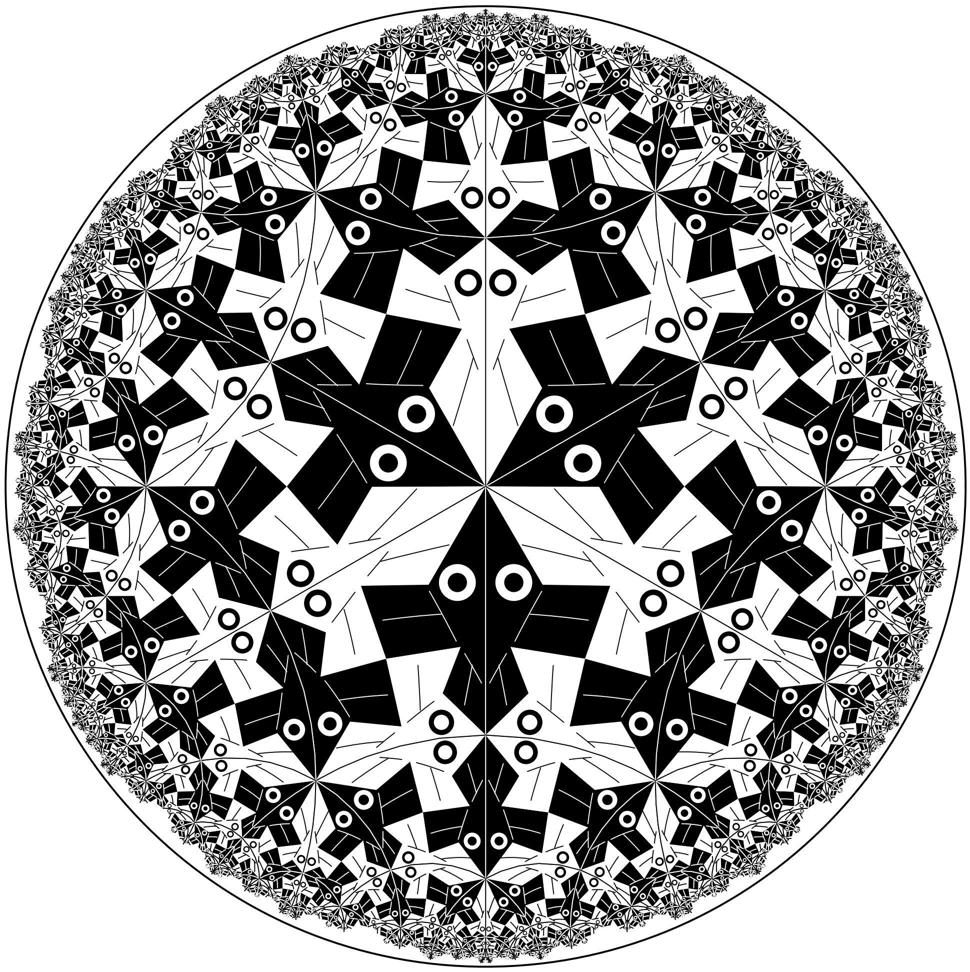
Escher's *Development II* (point limit)



Escher's *Square Limit* (line limits)

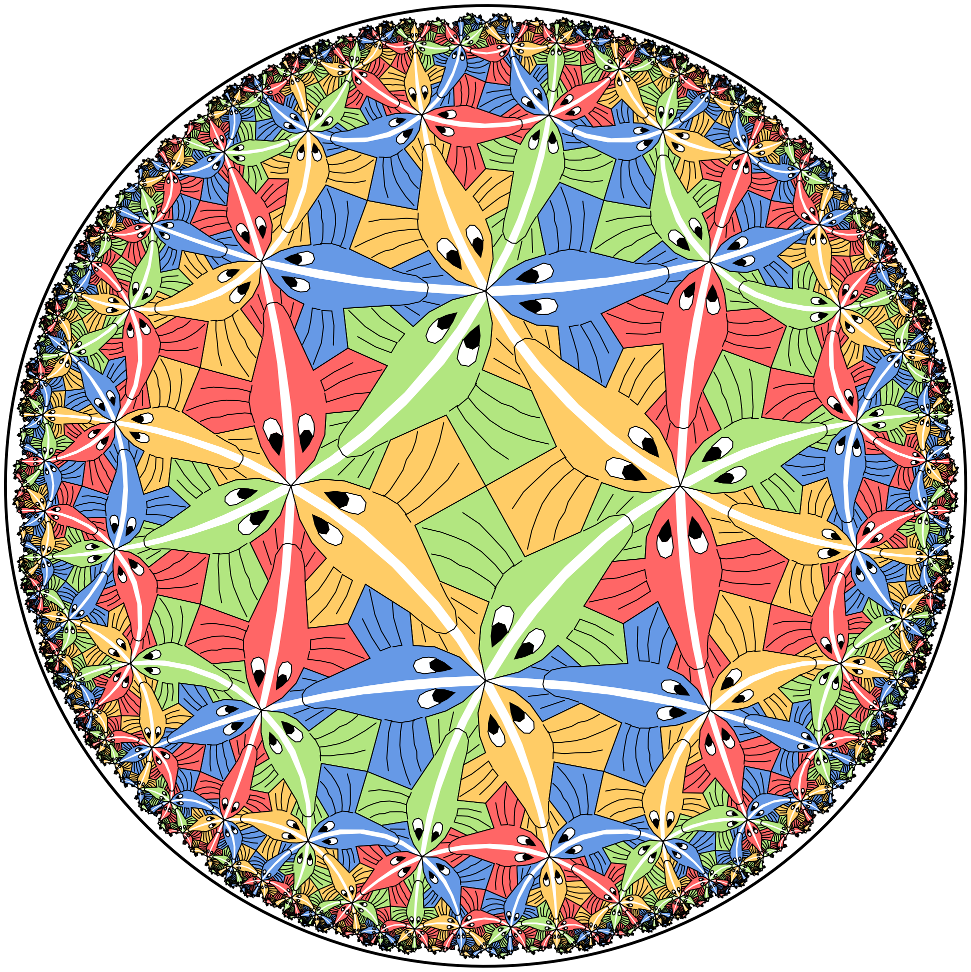
M.C. Escher's "Circle Limit" Patterns

Circle Limit I

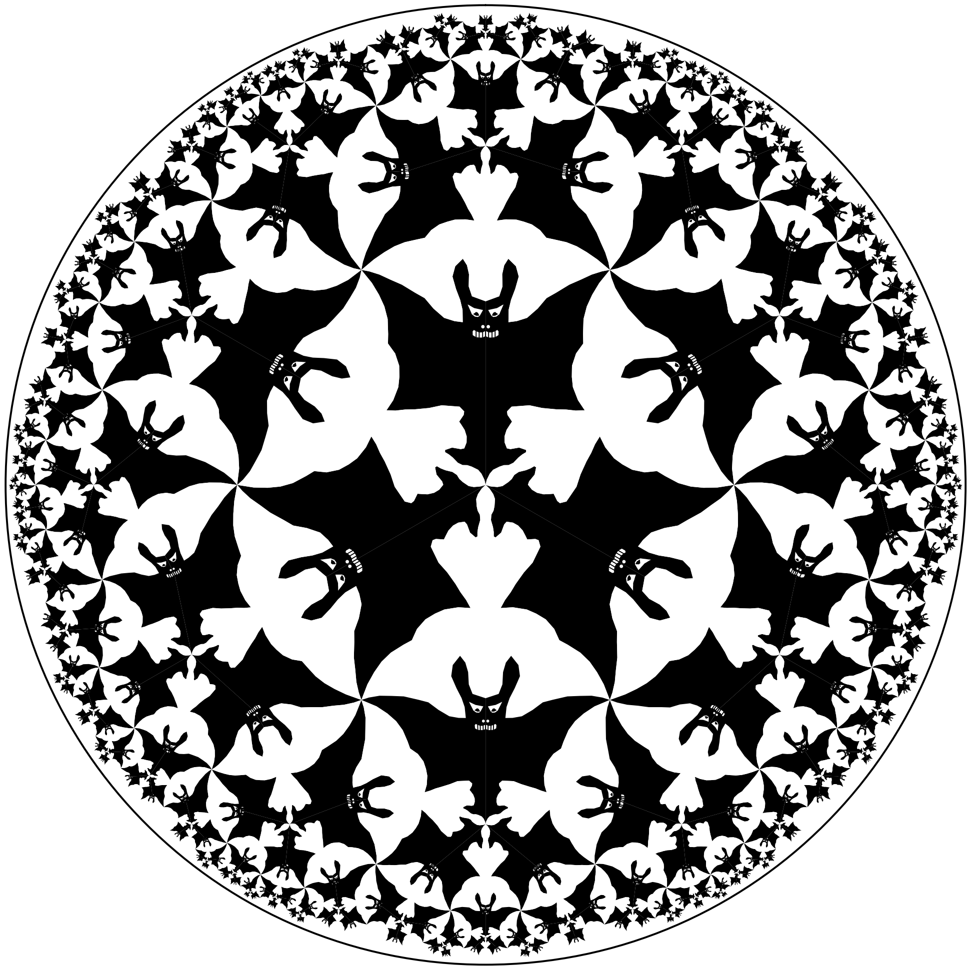


Circle Limit II

Circle Limit III



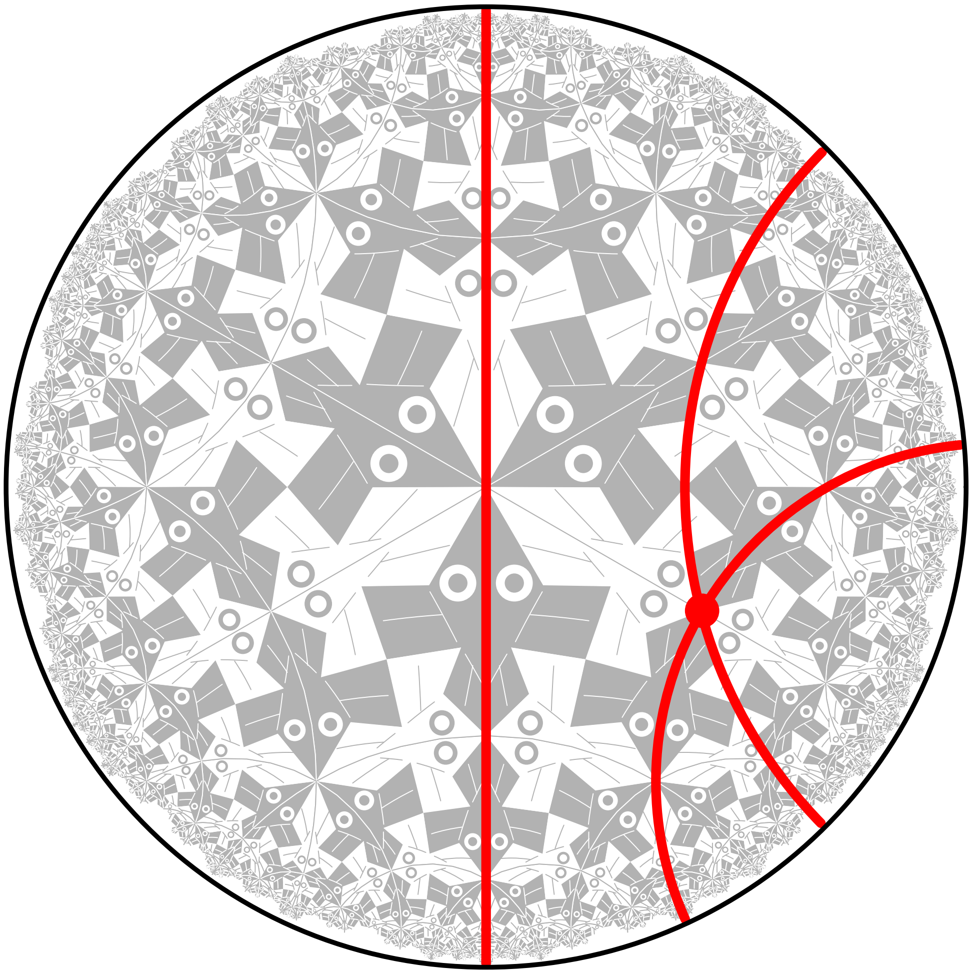
Circle Limit IV



2. Generation of Repeating Hyperbolic Patterns

Following Escher, we use the Poincaré disk model of hyperbolic geometry.

Poincaré Disk Model of Hyperbolic Geometry



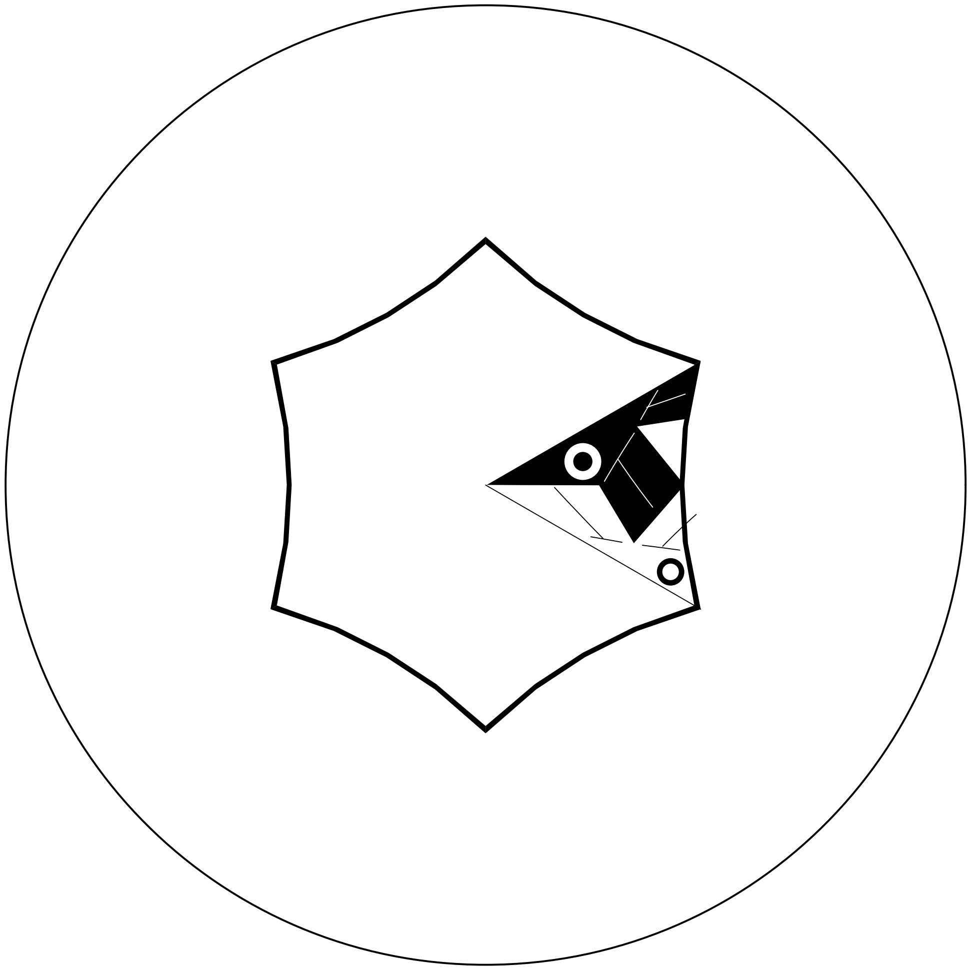
The Pattern Generation Process

Consists of two steps:

1. **Design the basic subpattern or *motif* — done by a hyperbolic drawing program.**
2. **Transform copies of the motif about the hyperbolic plane: *replication***

Repeating Patterns

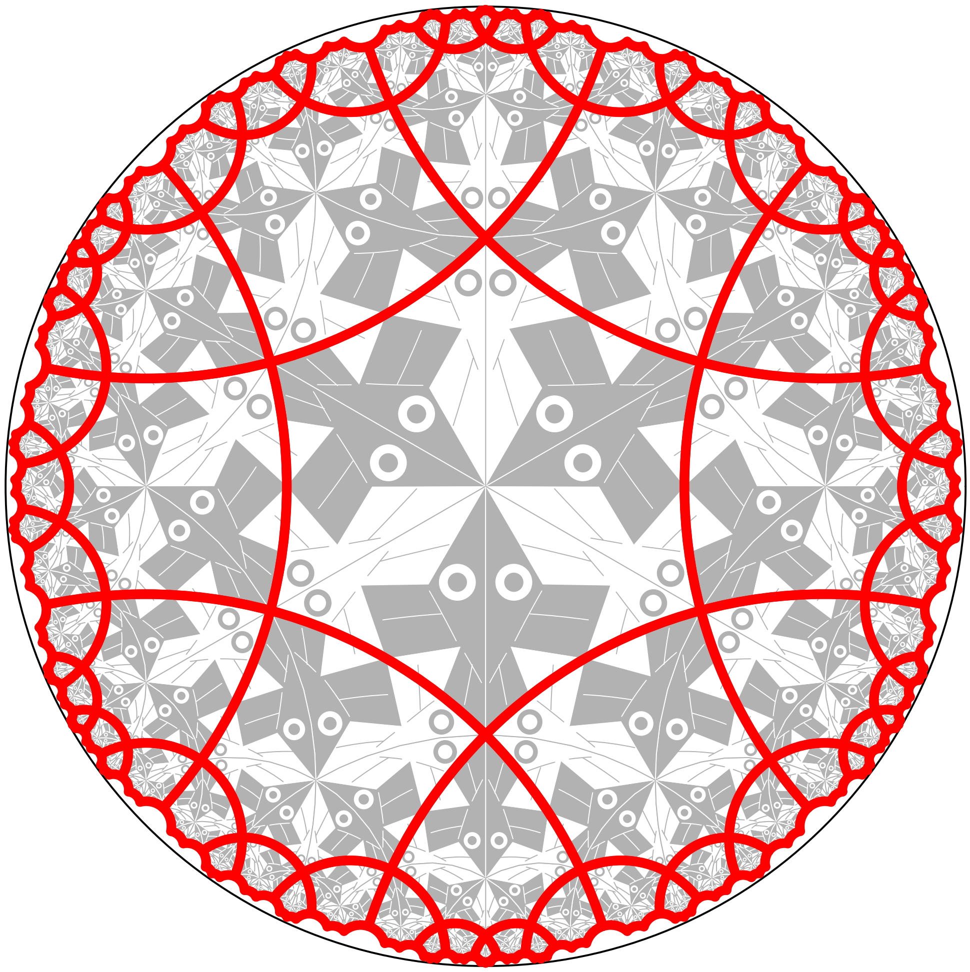
A *repeating pattern* is composed of congruent copies of the motif. A motif for *Circle Limit I*.



The Regular Tessellations $\{p, q\}$

- **Escher based his four "Circle Limit" patterns (and many of his Euclidean and spherical patterns) on regular tessellations.**
- **The *regular tessellation* $\{p, q\}$ is a tiling composed of regular p-sided polygons, or *p-gons* meeting q at each vertex.**
- **It is necessary that $(p-2)(q-2) > 4$ for the tessellation to be hyperbolic.**
- **If $(p-2)(q-2) = 4$ or $(p-2)(q-2) < 4$ the tessellation is Euclidean or spherical respectively.**

The Regular Tessellation $\{6, 4\}$



A Table of the Regular Tessellations

	⋮			⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
	11			*	*	*	*	*	*	*	*	*	...
	10			*	*	*	*	*	*	*	*	*	...
	9			*	*	*	*	*	*	*	*	*	...
	8			*	*	*	*	*	*	*	*	*	...
	7			*	*	*	*	*	*	*	*	*	...
q	6			□	*	*	*	*	*	*	*	*	...
	5			○	*	*	*	*	*	*	*	*	...
	4			○	□	*	*	*	*	*	*	*	...
	3			○	○	○	□	*	*	*	*	*	...
	2												
	1												
		1	2	3	4	5	6	7	8	9	10	11	...
							p						

□ - Euclidean tessellations

○ - spherical tessellations

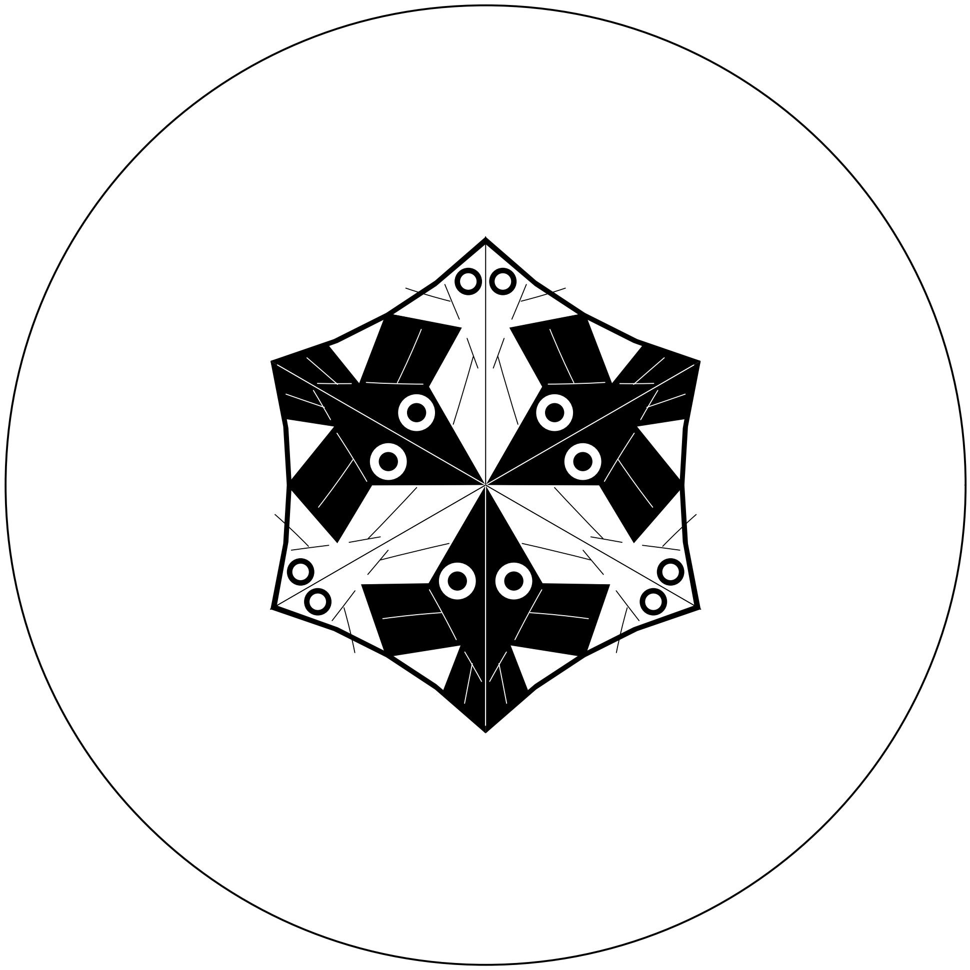
* - hyperbolic tessellations

Replicating the Pattern

In order to reduce the number of transformations and to simplify the replication process, we form the *p-gon pattern* from all the copies of the motif touching the center of the bounding circle.

- **Thus to replicate the pattern, we need only apply transformations to the p-gon pattern rather than to each individual motif.**
- **Some parts of the p-gon pattern may protrude from the enclosing p-gon, as long as there are corresponding indentations, so that the final pattern will fit together like a jigsaw puzzle.**
- **The p-gon pattern is often called the *translation unit* for repeating Euclidean patterns.**

The p-gon pattern for *Circle Limit I*



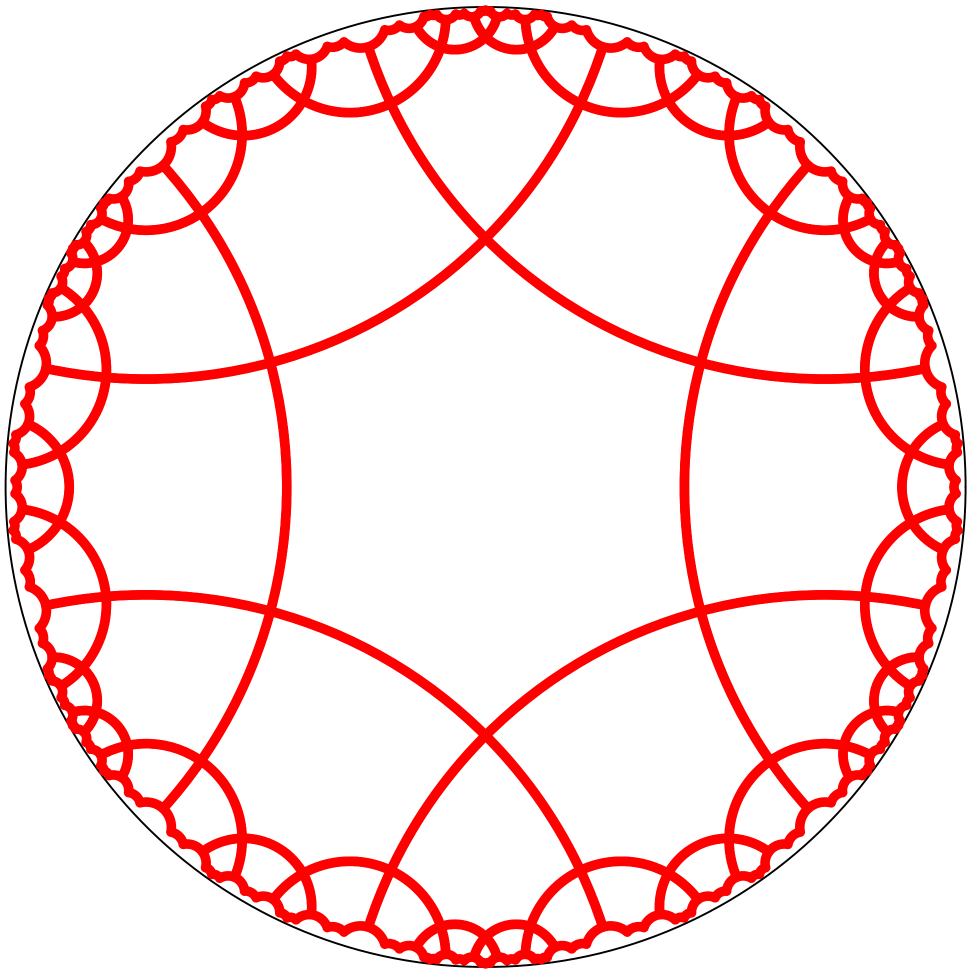
Layers of p-gons

We note that the p-gons of a $\{p, q\}$ tessellation are arranged in *layers* as follows:

- **The first layer is just the central p-gon.**
- **The $k + 1^{st}$ layer consists of all p-gons sharing and edge or a vertex with a p-gon in the k^{th} layer (and no previous layers).**
- **Theoretically a repeating hyperbolic pattern has an infinite number of layers, however if we only replicate a small number of layers, this is usually enough to appear to fill the bounding circle to our Euclidean eyes.**

The Regular Tessellation $\{6, 4\}$ — Revisited

To show the layer structure and exposure of p-gons.



Exposure of a p-gon

We also define the exposure of a p-gon in terms of the number of edges it has in common with the next layer (and thus the fewest edges in common with the previous layer.

- **A p-gon has *maximum exposure* if it has the most edges in common with the next layer, and thus only shares a vertex with the previous layer.**
- **A p-gon has *minimum exposure* if it has the least edges in common with the next layer, and thus shares an edge with the previous layer.**
- **We abbreviate these values as MAX_EXP and MIN_EXP respectively.**

The Replication Algorithm

The replication algorithm consists of two parts:

1. **A top-level "driver" routine** `replicate()` **that draws the first layer, and calls a second routine,** `recursiveRep()`**, to draw the rest of the layers.**
2. **A routine** `recursiveRep()` **that recursively draws the rest of the desired number of layers.**

A tiling pattern is determined by how the p-gon pattern is transformed across p-gon edges. These transformations are in the array `edgeTran[]`

The Top-level Routine `replicate()`

```
Replicate ( motif ) {

  drawPgon ( motif, IDENT ) ;  // Draw central p-gon

  for ( i = 1 to p ) {    // Iterate over each vertex

    qTran = edgeTran[i-1]

    for ( j = 1 to q-2 ) {  // Iterate about a vertex

      exposure = (j == 1) ? MIN_EXP : MAX_EXP ;
      recursiveRep ( motif, qTran, 2, exposure ) ;
      qTran = addToTran ( qTran, -1 ) ;

    }
  }
}
```

The function `addToTran()` is described next.

The Function `addToTran()`

Transformations contain a matrix, the orientation, and an index, `pPosition`, of the edge across which the last transformation was made (`edgeTran[i].pPosition` is the edge matched with edge `i` in the tiling). Here is `addToTran()`

```
addToTran ( tran, shift ) {
  if ( shift % p == 0 ) return tran ;
  else return computeTran ( tran, shift ) ;
}
```

where `computeTran()` is:

```
computeTran ( tran, shift ) {
  newEdge = (tran.pPosition +
             tran.orientation * shift) % p ;
  return tranMult(tran, edgeTran[newEdge]) ;
}
```

and where `tranMult ( t1, t2 )` multiplies the matrices and orientations, sets the `pPosition` to `t2.pPosition`, and returns the result.

The Routine `recursiveRep()`

```
recursiveRep ( motif, initialTran, layer, exposure ) {

   drawPgon ( motif, initialTran ) ;  // Draw p-gon pattern

   if ( layer < maxLayer ) {          // If any more layers
      pShift = ( exposure == MIN_EXP ) ? 1 : 0 ;
      verticesToDo = ( exposure == MIN_EXP ) ? p-3 : p-2 ;

      for ( i = 1 to verticesToDo ) {  // Iterate over vertices
         pTran = computeTran ( initialTran, pShift ) ;
         qSkip = ( i == 1 ) ? -1 : 0 ;
         qTran = addToTran ( pTran, qSkip ) ;
         pgonsToDo = ( i == 1 ) ? q-3 : q-2 ;

         for ( j = 1 to pgonsToDo ) { // Iterate about a vertex
            newExposure = ( i == 1 ) ? MIN_EXP : MAX_EXP ;
            recursiveRep(motif, qTran, layer+1, newExposure);
            qTran = addToTran ( qTran, -1 ) ;
         }
         pShift = (pShift + 1) % p ;  // Advance to next vertex
      }
   }
}
```

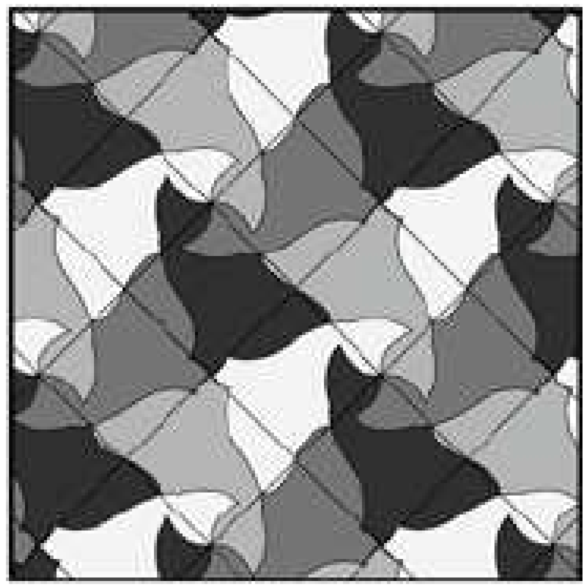
Special Cases

The algorithm above works for $p > 3$ and $q > 3$.

If $p = 3$ or $q = 3$, the same algorithm works, but with different values of `pShift, verticesToDo, qSkip,` **etc.**

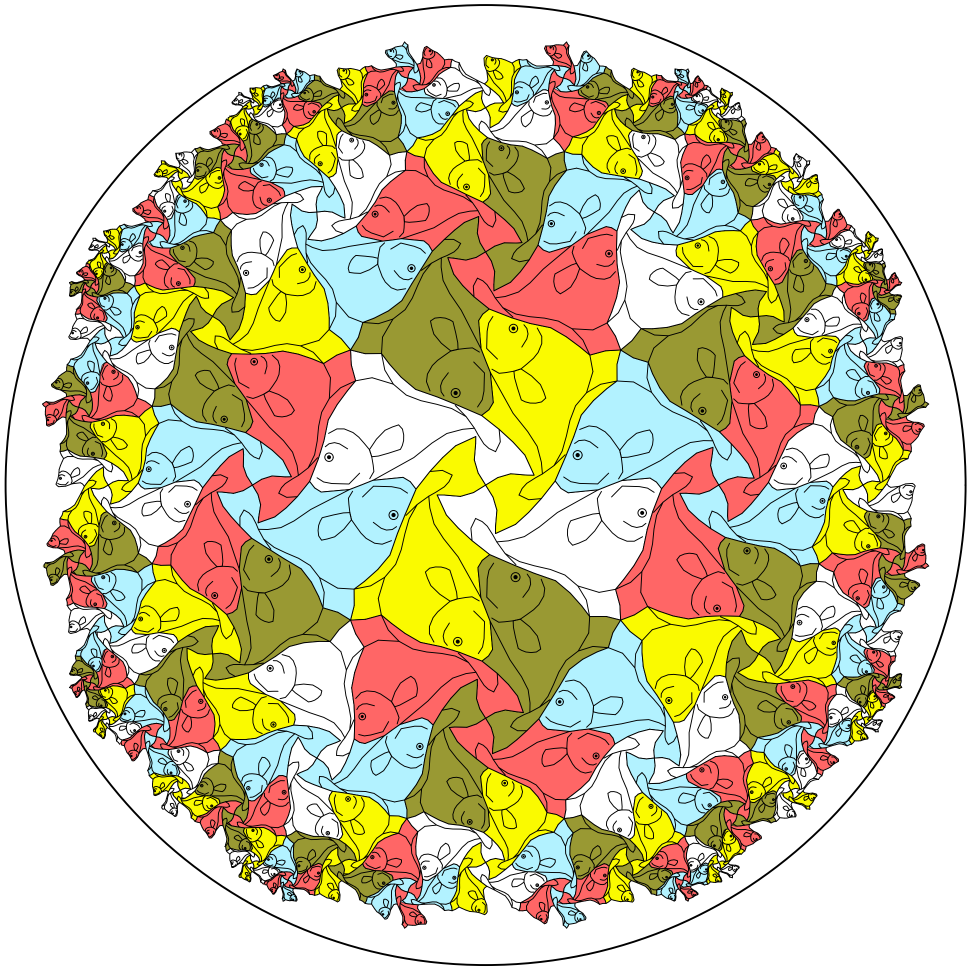
3. Some New Hyperbolic Patterns

Escher's Euclidean Notebook Drawing 20, based on the $\{4, 4\}$ tessellation.

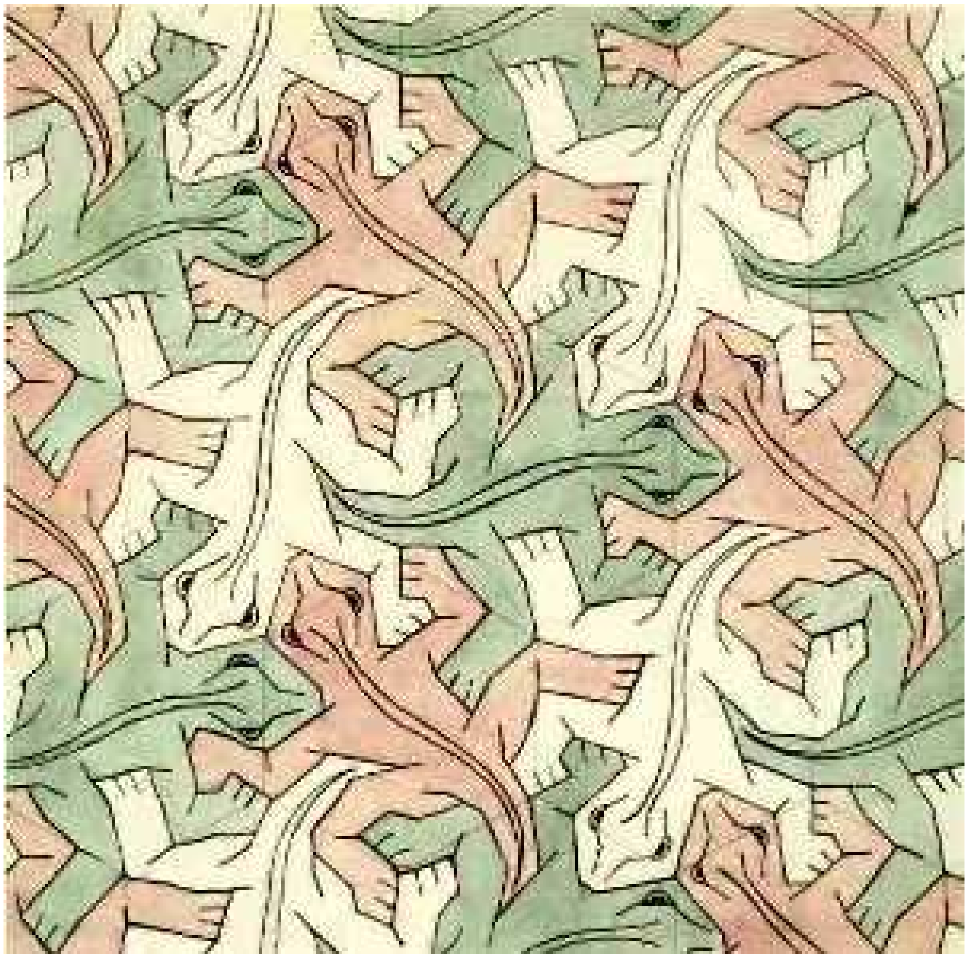


Escher's Spherical Fish Pattern Based on $\{4, 3\}$

A Hyperbolic Fish Pattern Based on $\{4, 5\}$

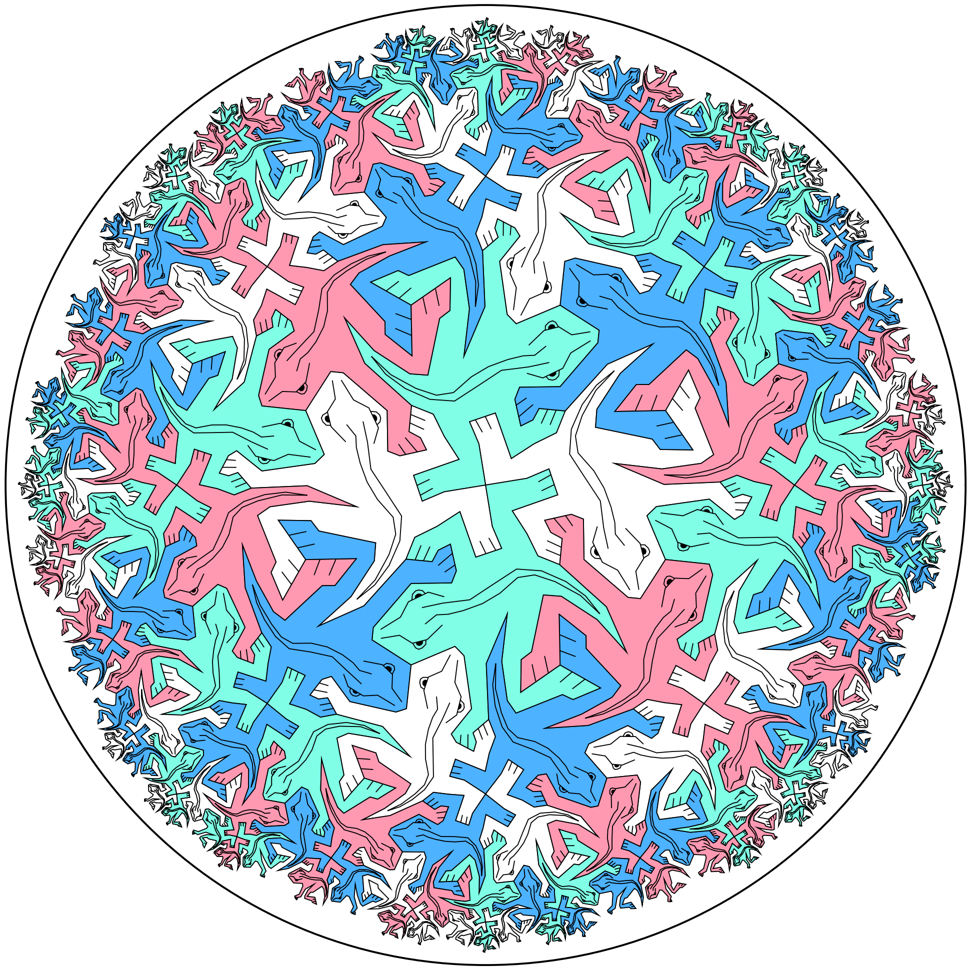


Escher's Euclidean Notebook Drawing 25, based on the $\{6, 3\}$ tessellation.



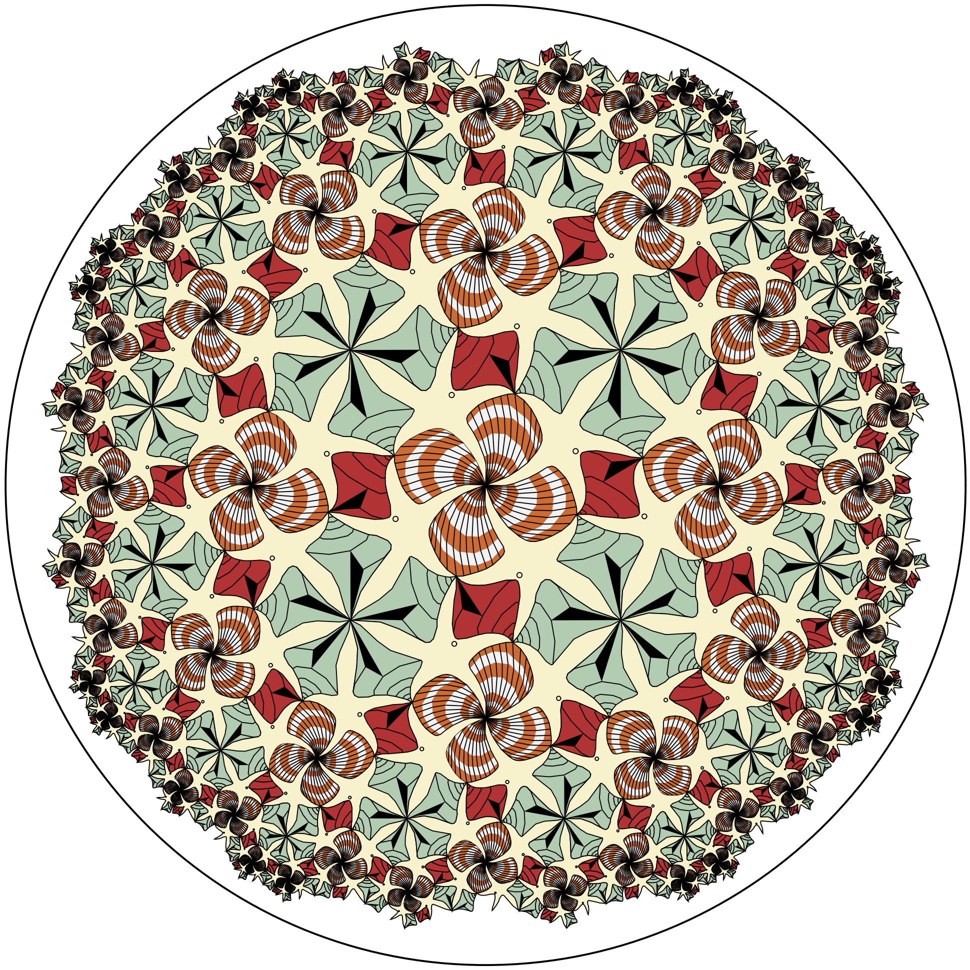
Escher's Print *Reptiles* based on Notebook Drawing 25

A Hyperbolic Lizard Pattern Based on $\{8, 3\}$

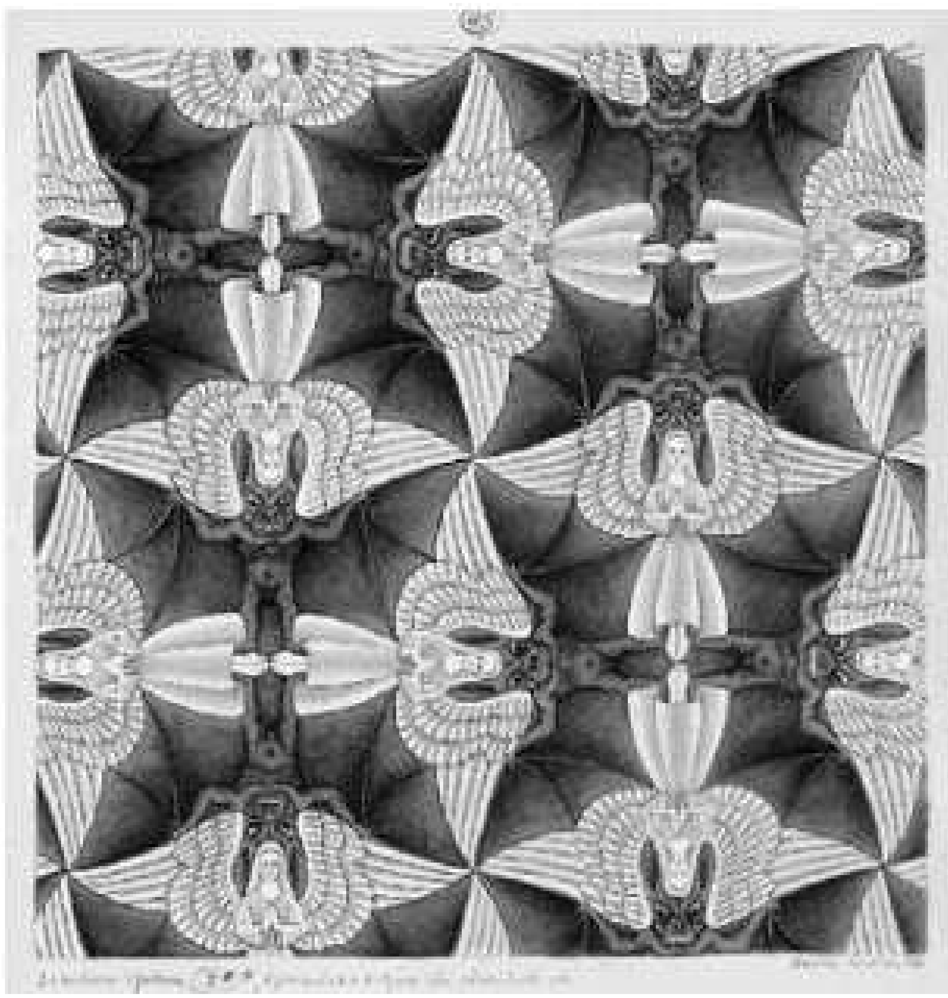


Escher's Euclidean Notebook Drawing 42, based on the $\{4,4\}$ tessellation.

A Hyperbolic Shell Pattern Based on $\{4, 5\}$

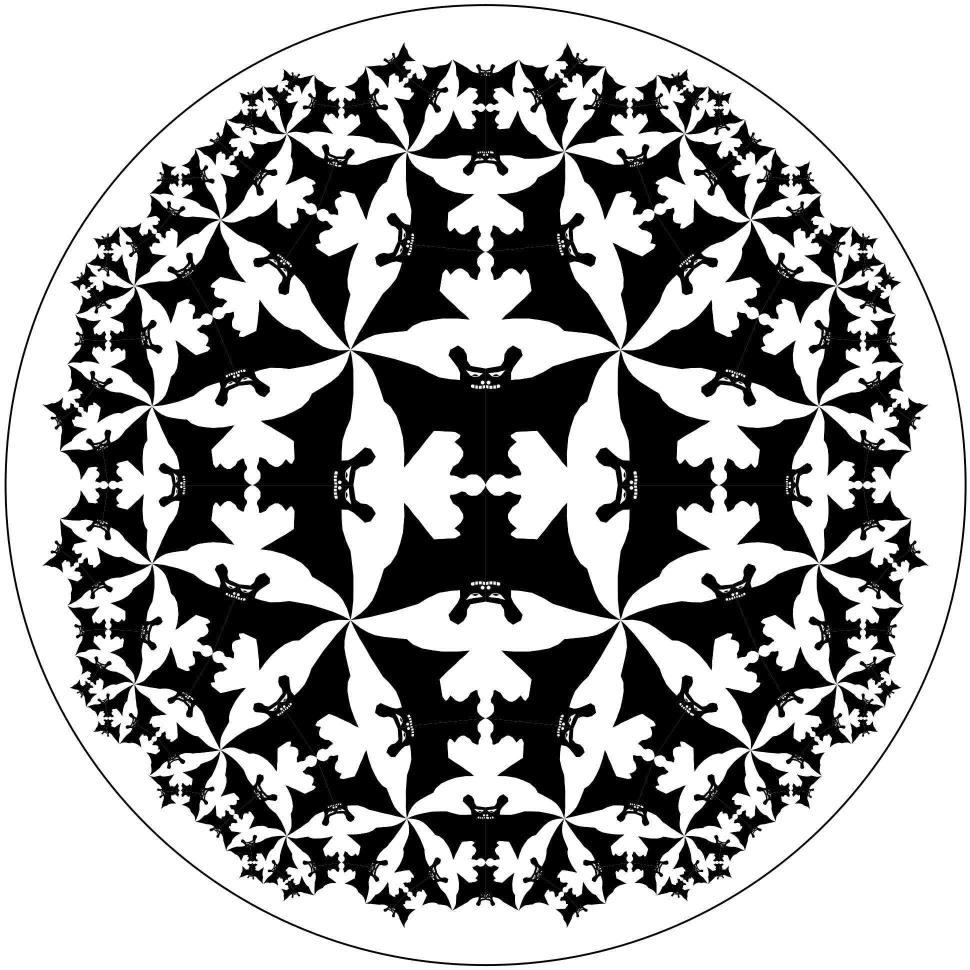


Escher's Euclidean Notebook Drawing 45, based on the $\{4,4\}$ tessellation.

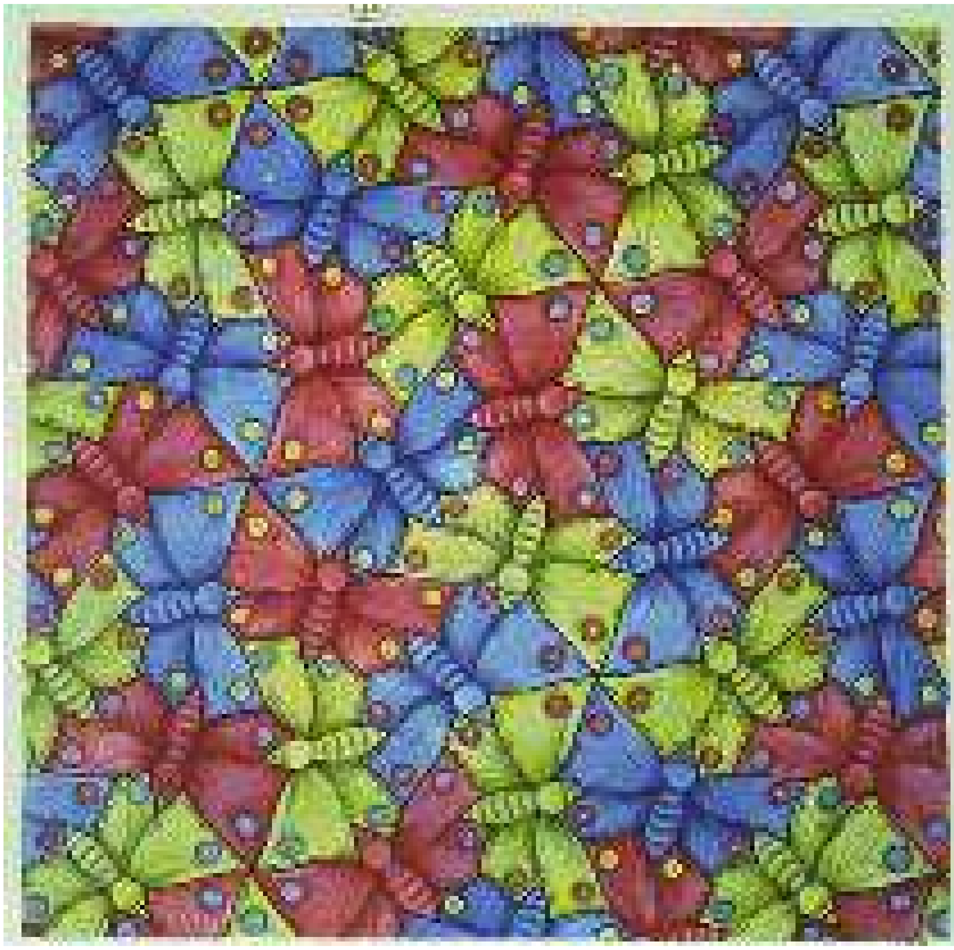


Escher's Spherical "Heaven and Hell" Based on $\{4, 3\}$

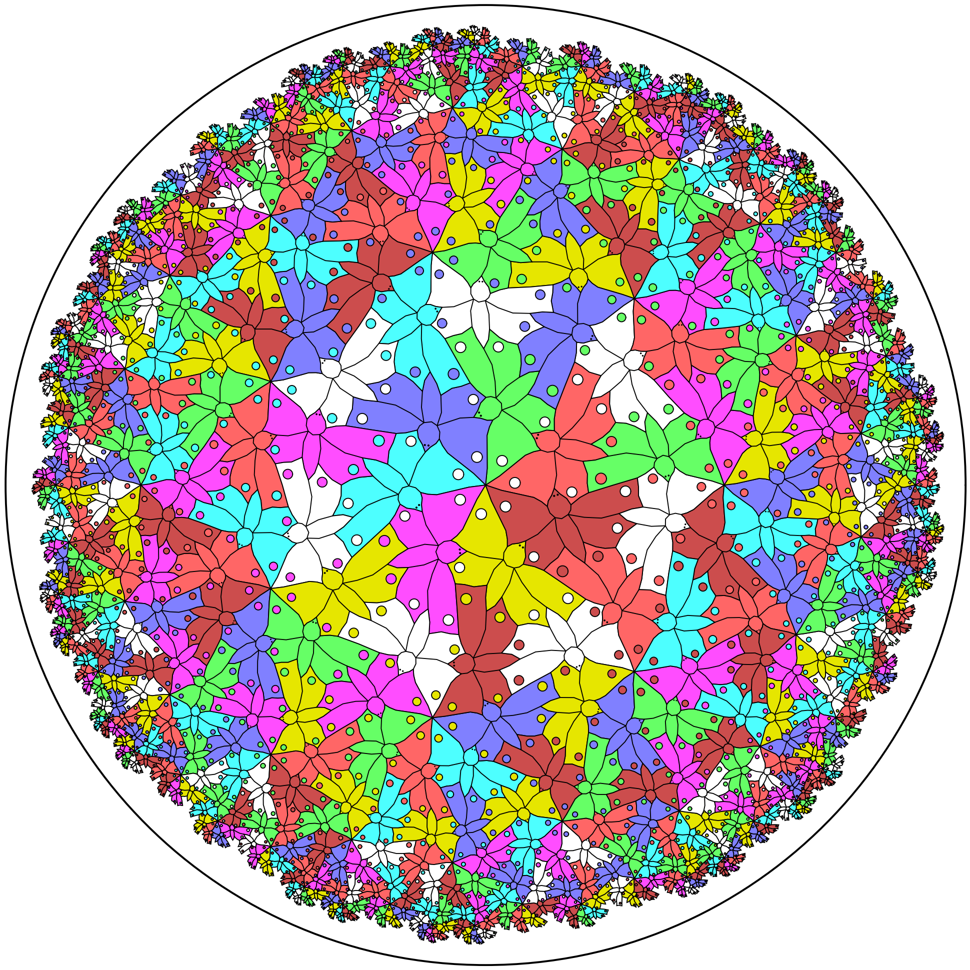
A Hyperbolic "Heaven and Hell" Pattern Based on $\{4, 5\}$



Escher's Euclidean Notebook Drawing 70, based on the $\{6, 3\}$ tessellation.



A Hyperbolic Butterfly Pattern Based on $\{4, 3\}$



4. Future Work

- **Extend the algorithm to handle tilings by non-regular polygons.**
- **Extend the algorithm to the cases of infinite regular polygons:** $\{p, \infty\}$ **composed of infinite** p**-sided polygons, or** $\{\infty, q\}$ **composed of infinite-sided polygons meeting** q **at a vertex.**
- **Automatically generate patterns with color symmetry.**

The End

I hope not!