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A FAMILY OF BUTTERFLY PATTERNS INSPIRED BY ESCHER

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ABSTRACT: M.C. Escher is noted for his repeating patterns, usually with animal motifs. For a few motifs he created more than one pattern with different combinatorial characteristics, even in different geometries, leading to the concept of families of patterns with the same motif but different combinatorics. Hyperbolic geometry is useful in that it provides an infinite number of combinatorial possibilities. This paper investigates one of those families, based on a butterfly motif.

Keywords: M.C. Escher, hyperbolic geometry, mathematical art

1. INTRODUCTION

In 1948, the Dutch artist M.C. Escher created his Regular Division Drawing number 70, a repeating Euclidean pattern of butterflies. We show a closely related hyperbolic pattern in Figure 1 below.

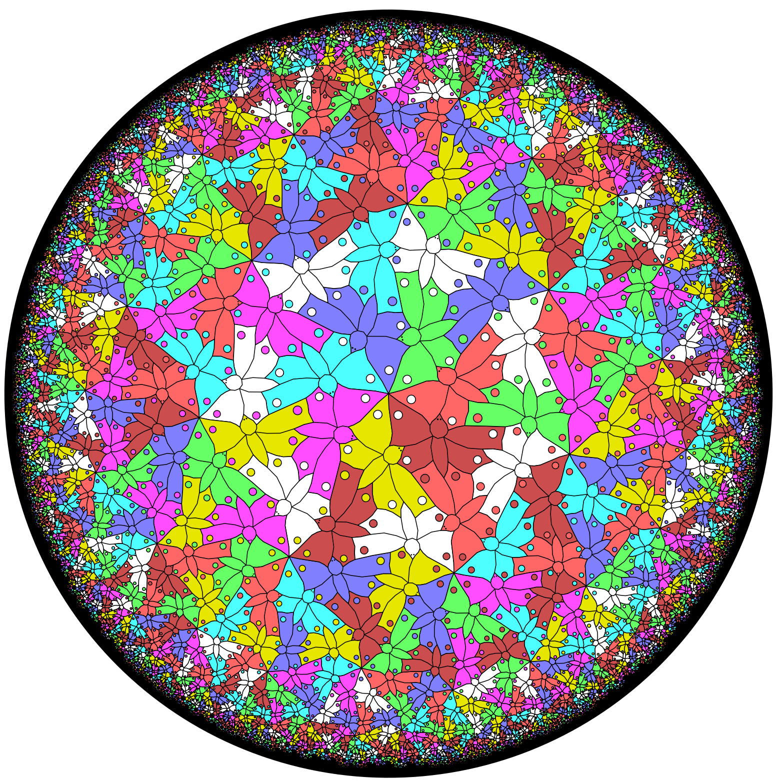

Figure 1: A pattern of butterflies based on the $\{7, 3\}$ tessellation.

This paper is organized as follows. First we present some background on Escher and his butterfly patterns. Then we discuss hyperbolic geometry, useful for interpreting the patterns we show, and regular tessellations. Next we define what we mean by a family of patterns, showing several related butterfly patterns. Finally, we offer conclusions and indicate directions of future work.

2. ESCHER AND BUTTERFLY PATTERNS

Escher was well known for his repeating patterns, which had two characteristics: they filled the plane without gaps or overlaps, and they exhibited color symmetry. One of his patterns that exhibited these properties was his Regular Division Drawing number 70 of butterflies (page 172 of [2]). He later used the butterfly motif in his wood engraving "Butterflies" (cat. no. 369, page 305 of [2]), a water color circular design of butterflies decreasing to a point limit, and Regular Division Drawing 79 (page 179 of 2]). These latter two works clearly exhibit the pattern of interlocking circles upon which they are based — not so evident in Regular Division Drawing 70. However the pattern of interlocking circles of butterflies of each color is evident in the related hyperbolic pattern shown in Figure 1.

Escher mapped some of his "Regular Division" drawings onto spheres, regular polyhedra, and the hyperbolic plane, but not Regular Division Drawing 70. However the Canadian mathematician H.S.M. Coxeter described how 18 of Escher's butterflies could be placed on a torus (pages 24–27 of [1]). Doris Schattschneider and Wallace Walker placed 60 of those butterflies on a regular icosahedron [3]. And my students and I have created several hyperbolic patterns based on those butterflies, including Figure 1.

3. HYPERBOLIC GEOMETRY AND REGULAR TESSELLATIONS

The patterns that I have created can be interpreted as patterns in the hyperbolic plane, and specifically in the *Poincaré disk* model of hyperbolic geometry. The hyperbolic points in this model are represented by Euclidean points within a bounding circle. Hyperbolic lines are represented by (Euclidean) circular arcs orthogonal to the bounding circle (including diameters). The hyperbolic measure of an angle is the same as its Euclidean measure in the disk model (we say such a model is *conformal*), but equal hyperbolic distances correspond to ever-smaller Euclidean distances as figures approach the edge of the disk. Figure 2 shows some lines superimposed on a computer rendition of Escher's hyperbolic pattern *Circle Limit I*. Hyperbolic reflections across lines are represented by inversions in the circular arcs representing those lines (including Euclidean reflections across diameters). Other hyperbolic transformations can be built up as products (of at most three) reflections. For example, successive reflections across two intersecting lines produces a hyperbolic rotation about their intersection point by twice the angle of intersection (as in the Euclidean case).

Many of Escher's Regular Division drawings are based on the *regular tessellations* $\{p, q\}$ formed by regular p-sided polygons meeting q at each vertex. If $(p - 2)(q - 2) = 4$, the tessellation is Euclidean and there are three solutions: the tilings by square, equilateral triangles, and regular hexagons, denoted $\{4, 4\}$, $\{3, 6\}$, and $\{6, 3\}$ respectively. If $(p - 2)(q - 2) < 4$, the tessellation is spherical and there are five solutions: $\{3, 3\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 3\}$, and $\{5, 3\}$. These correspond to "blown up" versions of the Platonic solids, the tetrahedron, octahedron, icosahedron, cube, and dodecahedron, respectively. If $(p - 2)(q - 2) > 4$, the tessellation is hyperbolic and there are an infinite number of solutions. Figure 3 shows the $\{6, 4\}$ tessellation superimposed on a computer rendition of Escher's hyperbolic pattern *Circle Limit I*.

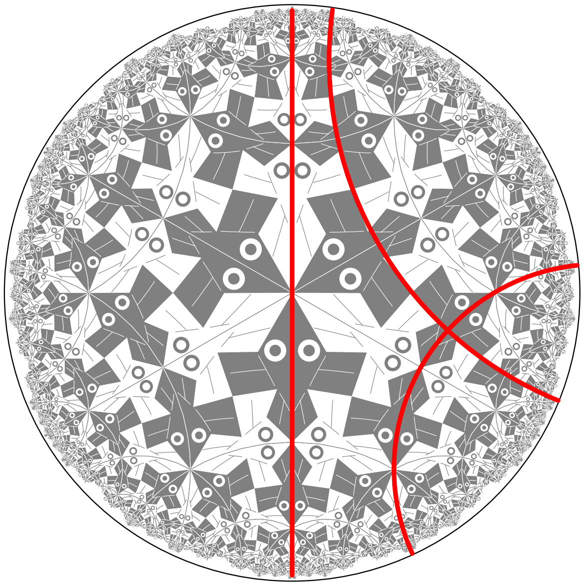

Figure 2: A computer rendition of *Circle Limit I* showing three hyperbolic lines on it.

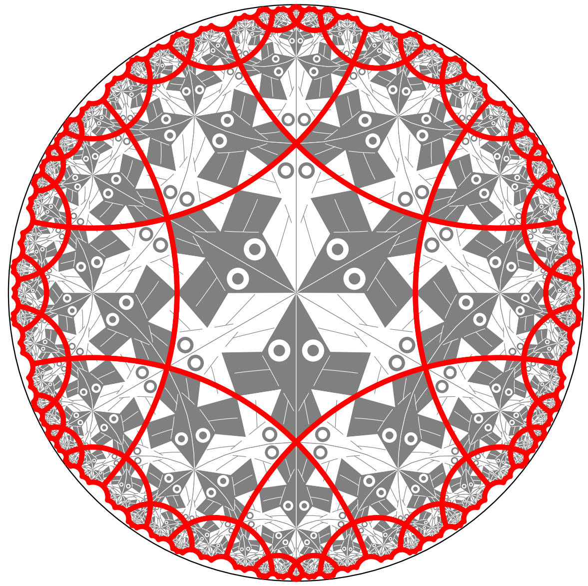

Figure 3: A computer rendition of *Circle Limit I* with the $\{6, 4\}$ tessellation superimposed.

The table below shows the possible spherical, Euclidean, and hyperbolic regular tessellations.

Table 1: A Table of the Regular Tessellations

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
	8	*	*	*	*	*	*	…
	7	*	*	*	*	*	*	…
q	6	□	*	*	*	*	*	…
	5	○	*	*	*	*	*	…
	4	○	□	*	*	*	*	…
	3	○	○	○	□	*	*	…
		3	4	5	6	7	8	…
					p			

□ - Euclidean tessellations
○ - spherical tessellations
* - hyperbolic tessellations

In particular, it indicates that there are infinitely many hyperbolic tessellations, but only a finite number of Euclidean and spherical tessellations, thus there are many more possible patterns based on hyperbolic tessellations. Escher created only four hyperbolic patterns, his *Circle Limit* prints. Undoubtedly the reason he did not make more was the fact that creating such patterns by hand is a laborious and time-consuming process. My students and I were able to avoid this problem by using computer technology.

4. FAMILIES OF BUTTERFLY PATTERNS

Regular Division Drawing 70 and Figure 1 above are based on the $\{6,3\}$ and $\{7,3\}$ tessellations respectively. Schattschneider and Walker's butterfly decoration of the icosahedron mentioned above is based on the $\{3,5\}$ tessellation. This leads us to define a 2-parameter *family* of butterfly patterns (p,q) for all $p \geq 3$ and $q \geq 3$, where the pattern (p,q) is based on the tessellation $\{p,q\}$. In (p,q) p butterflies meet at left front wing tips and q of them meet at right rear wing tips. In the past I have built tetrahedral, octahedral, and icosahedral models decorated with butterflies. Those pattern would be denoted $(3,3)$, $(3,4)$, and $(3,5)$ respectively.

All the patterns discussed above have *color symmetry*: any symmetry operation on the pattern permutes the colors. One can see that a rotation by $2\pi/7$ about the center of Figure 1 cyclicly permutes the colors except for white, which remains fixed. We also require the group of color permutations to be transitive on the colors, so that all the colors get "mixed up". Following Escher, for these butterfly patterns we also want the circles on the wings to be a different color than all the butterflies that meet at a left front wing tip; this "circle" color is white for the butterflies at the center of Figure 1 above. We call this the "color of circles convention". We also require that the pattern obey the map-coloring principle that butterflies sharing an edge must be different colors. In the subfamily $(p,3)$, if p is even, then three colors suffice, as in the case of Regular Division Drawing 70 and Figure 4 below which shows $(8,3)$. If p is odd, more colors are required, as can

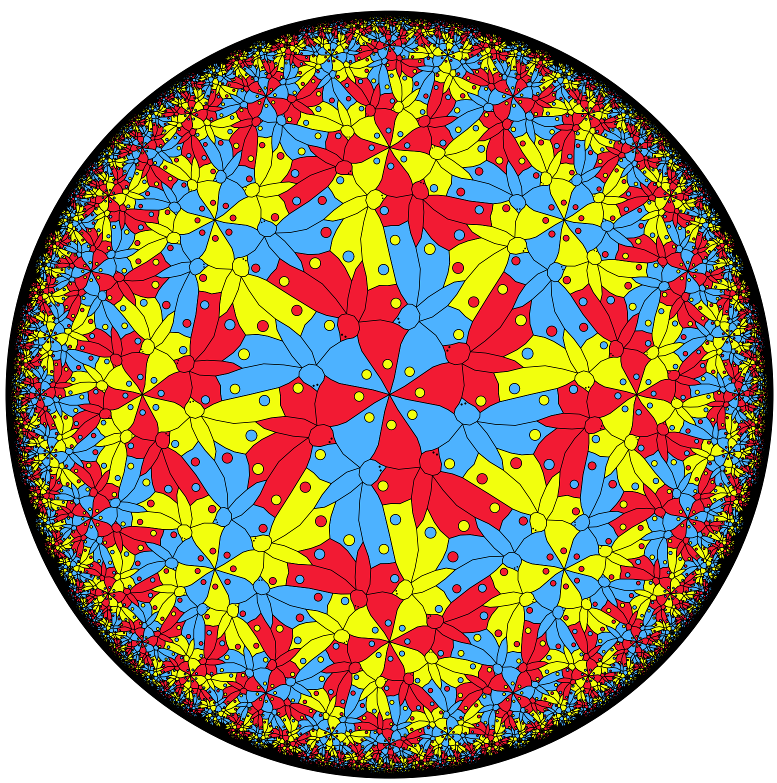

Figure 4: A pattern of butterflies based on the $\{8,3\}$ tessellation.

be seen in Figure 1, and in the figures below.

Figures 5 and 6 show 6-colored butterfly patterns based on the $\{5, 4\}$ and $\{5, 5\}$ tessellations respectively.

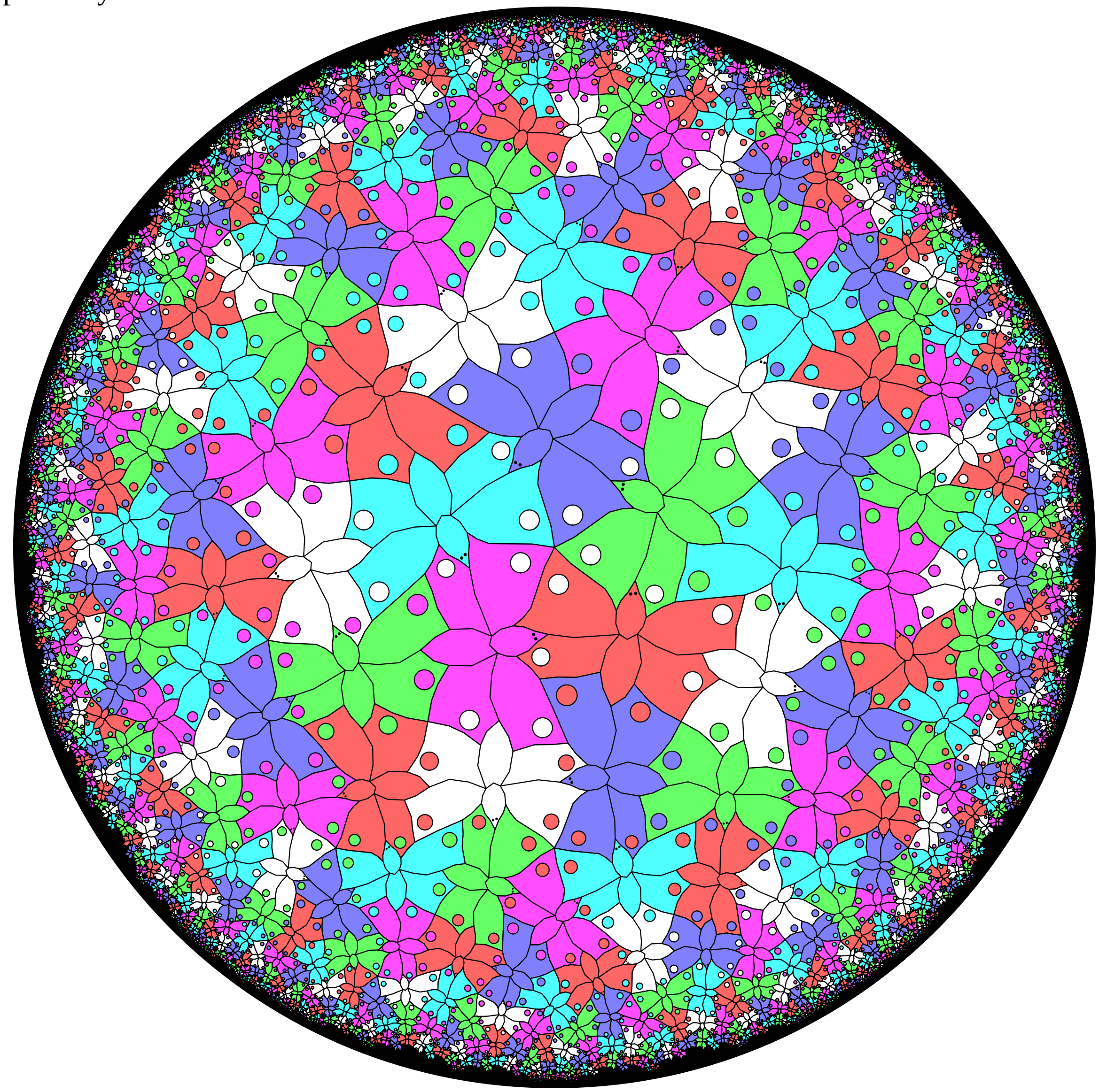

Figure 5: A pattern of butterflies based on the $\{5, 4\}$ tessellation.

Figures 7 and 8 show two butterfly patterns based on the $\{6, 4\}$ tessellation. Figure 7 satisfies the “colors of the circles convention” in that the circular dots on the wings at a meeting point of left front wing tips is a different color than any of those wings — this requires four colors. But the pattern of Figure 8 with only three colors violates that convention.

Figure 9 shows an 8-colored butterfly pattern based on the $\{7, 4\}$ tessellation.

Though it would seem that we could just keep generating butterfly patterns with different values of p and q, there are aesthetic limitations in that large values of p or q or both tend distort the motifs to the point where they can’t be recognized. We start to see this in the pattern of Figure 9, but it is even more obvious in Figure 10, which is based on the $\{10, 4\}$ tessellation.

This brings us to the end of our current investigation of this family of butterfly patterns.

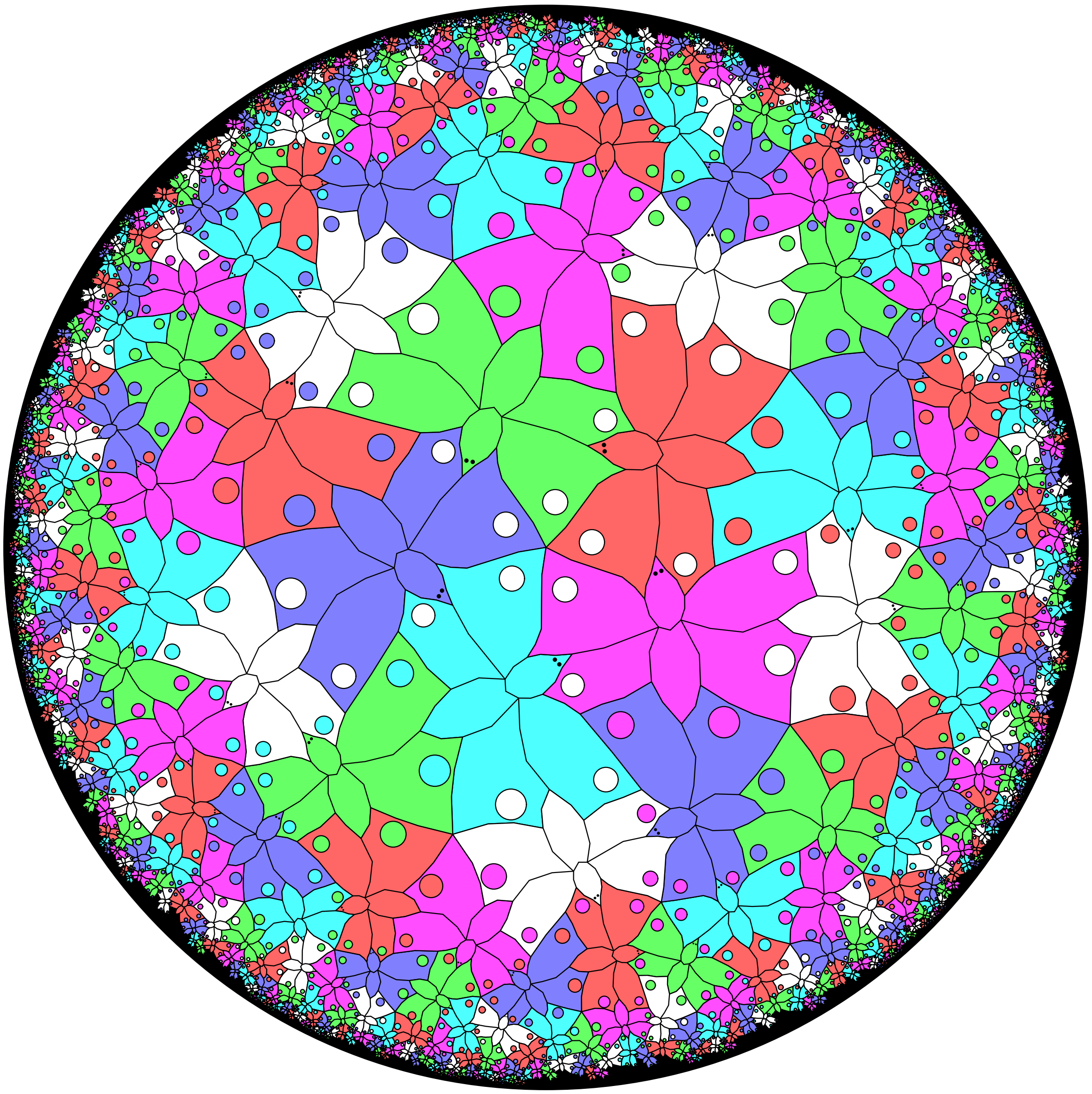

Figure 6: A pattern of butterflies based on the $\{5, 5\}$ tessellation.

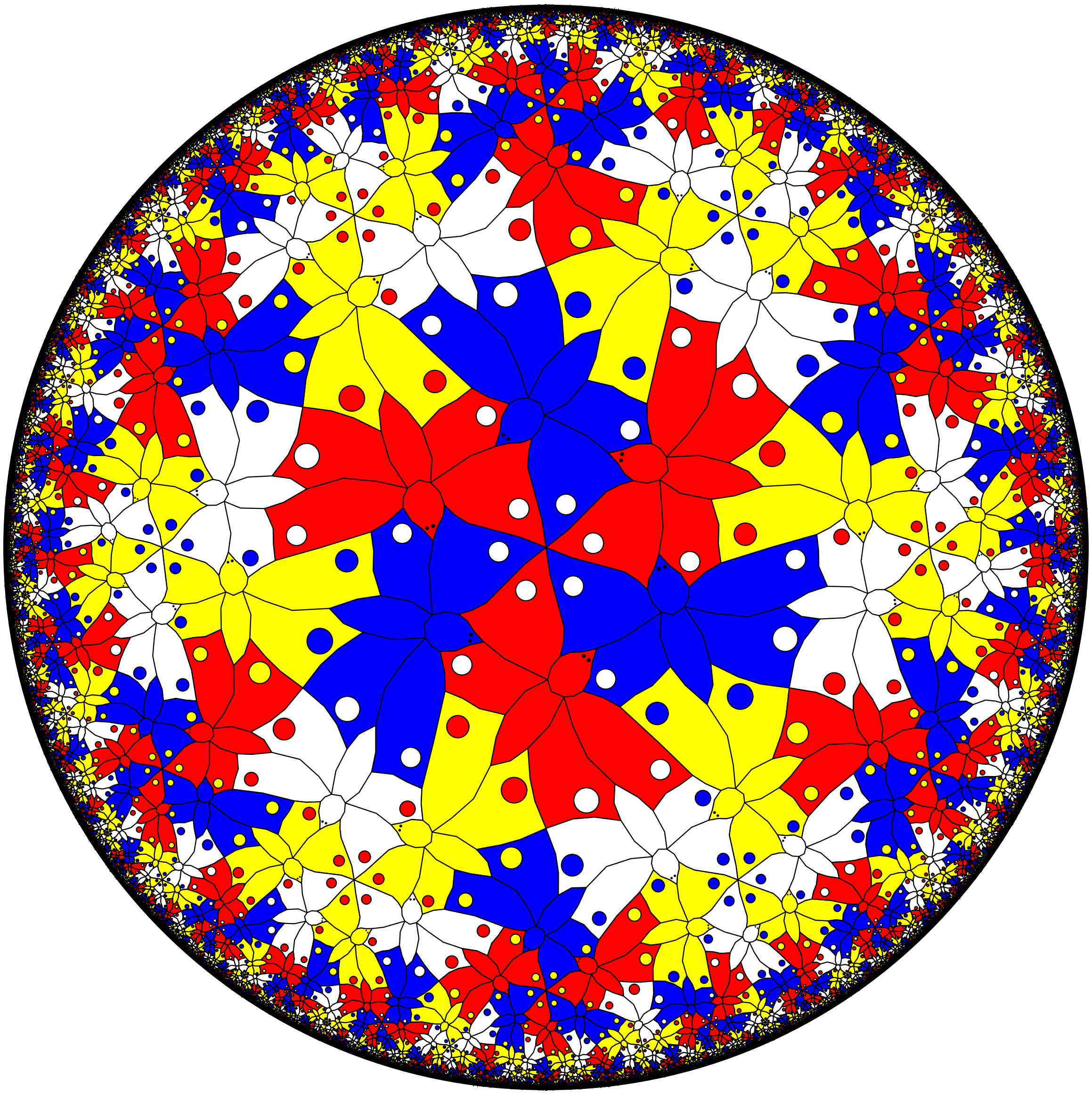

Figure 7: A pattern of butterflies based on the $\{6, 4\}$ tessellation.

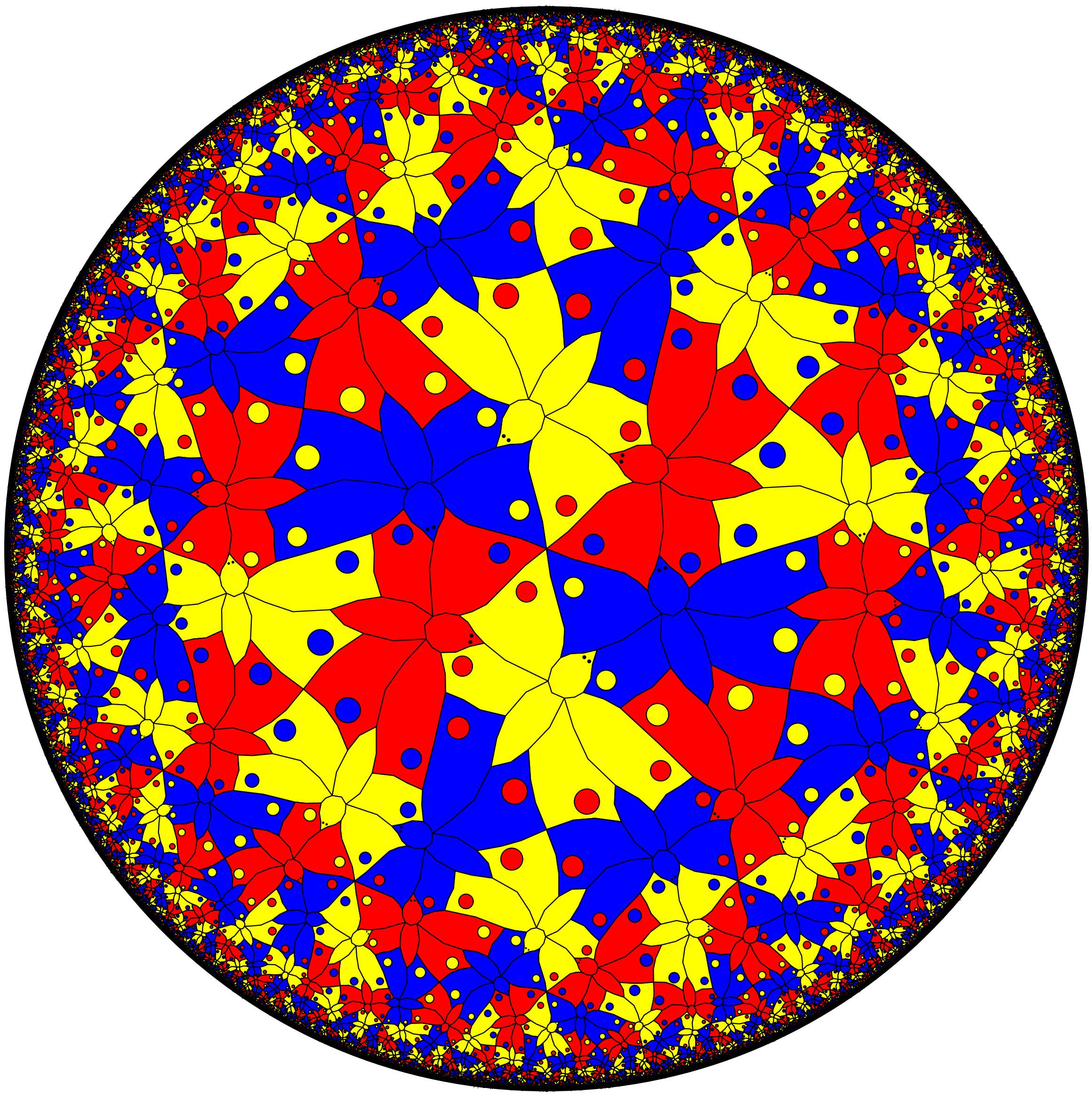

Figure 8: A pattern of butterflies based on the $\{6, 4\}$ tessellation that violates the "color of circles convention".

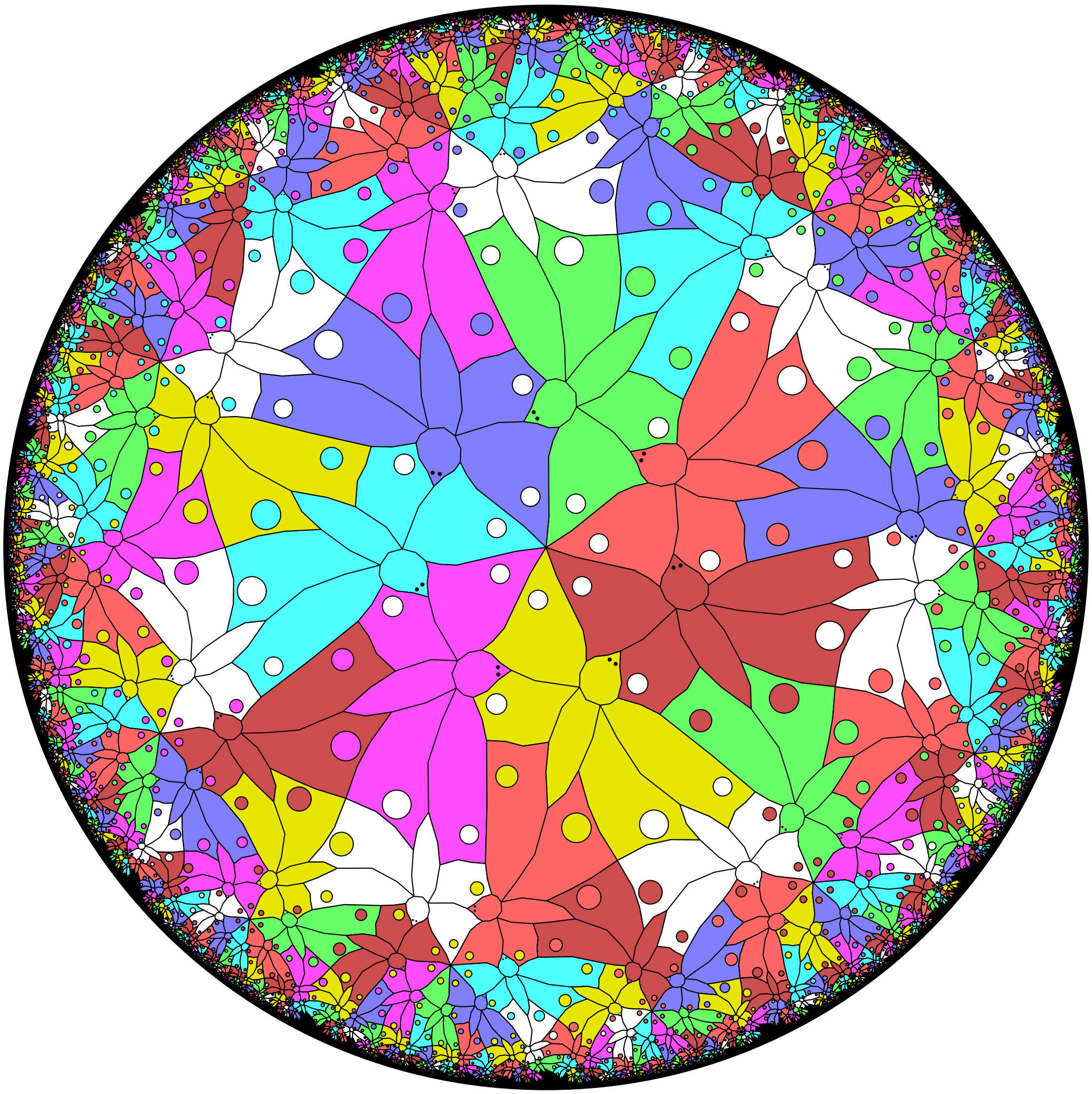

Figure 9: A pattern of butterflies based on the $\{7, 4\}$ tessellation.

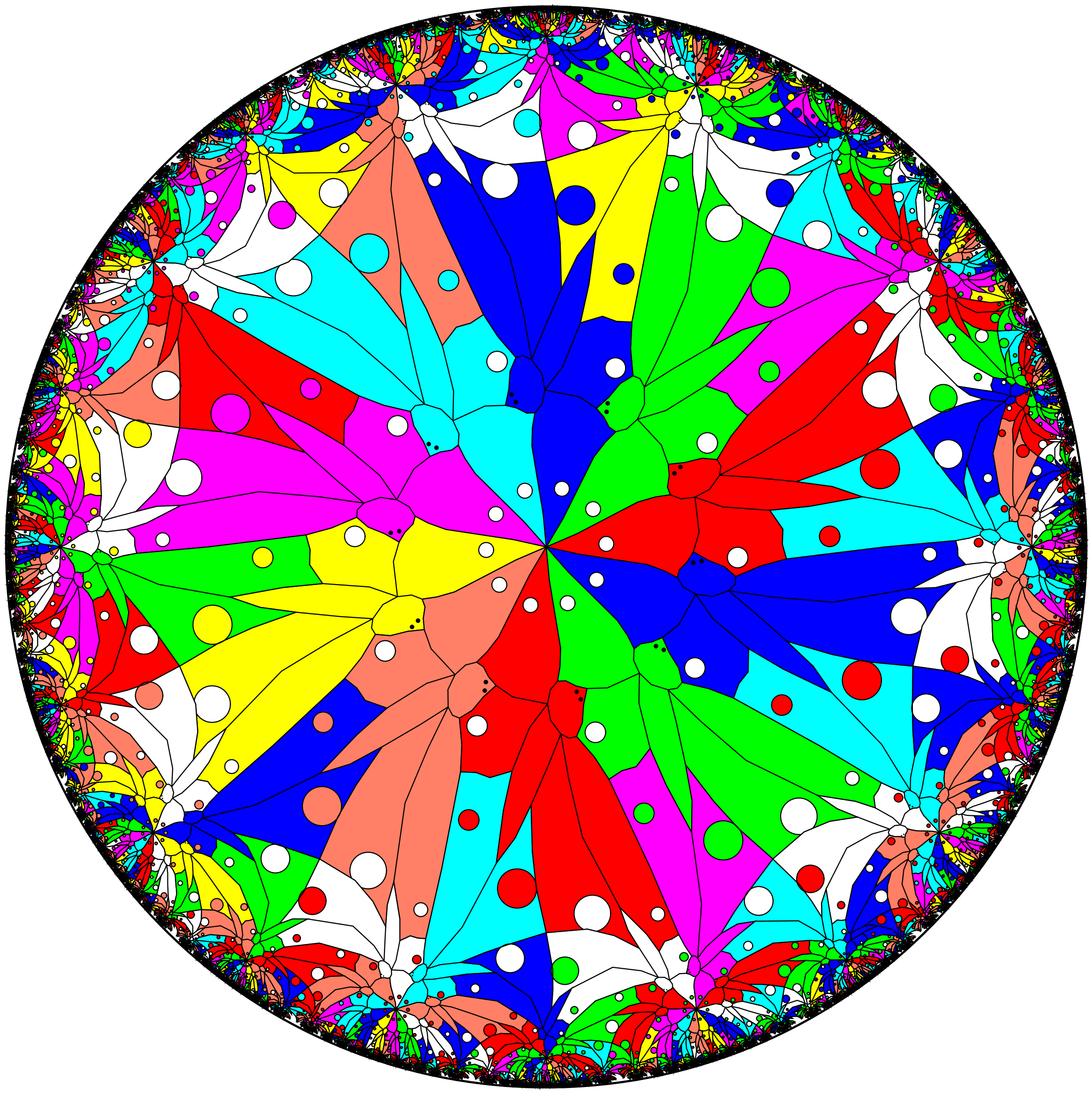

Figure 10: A pattern of distorted butterflies based on the $\{10, 4\}$ tessellation.

CONCLUSIONS

We have showed that some repeating patterns do not just exist in isolation, but are related to many other patterns with similar motifs but different combinatorical properties. We have also seen that most of these patterns exist in the hyperbolic plane. All the related patterns can be thought of as belonging to a single family of patterns, as we have shown with the family of butterfly patterns. The same techniques can be used to investigate other families of patterns — notably those related to Escher's Regular Division Drawings, many of which are based on regular tessellations $\{p, q\}$. The colorings chosen for the butterflies in the patterns we have presented we determined "by hand", and as we have seen, different values of the parameters p and q lead to quite different numbers of colors. It seems to be a difficult task, for future research, to automate the color assignments to have both color symmetry and satisfy the map-coloring principle.

ACKNOWLEDGMENTS

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