15th International Conference on Geometry and Graphics

A Family of Butterfly Patterns Inspired by Escher

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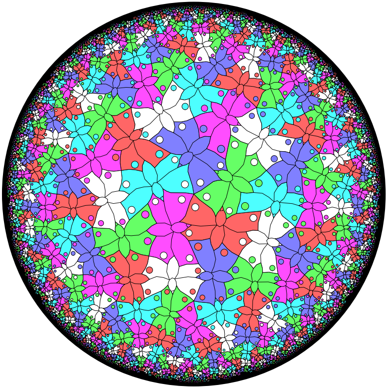


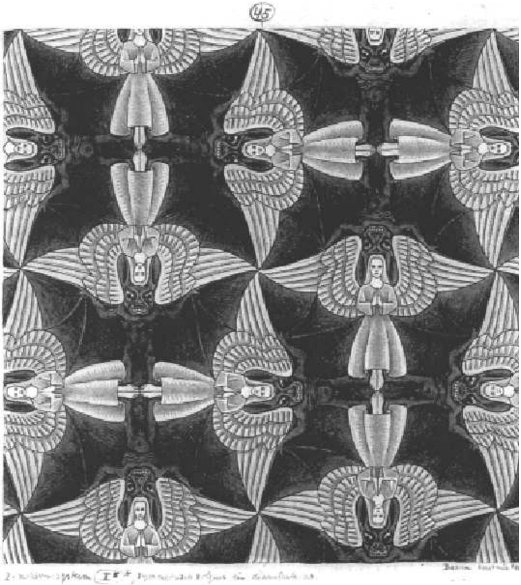
Outline

- Families of patterns - the basic idea
- The "Classical Geometries"
- Hyperbolic geometry
- Repeating patterns and regular tessellations
- Parameterized families of patterns
- The family of butterfly patterns
- Other families of patterns
- Future research

Families of Patterns

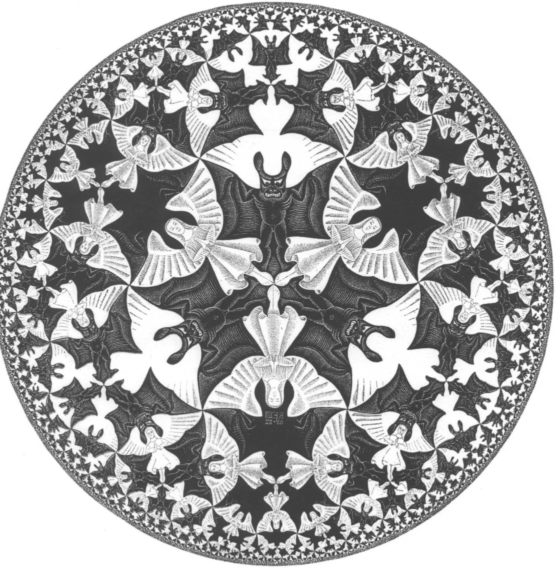
- Many artists have created related works.
- Example: the paintings of Picasso's "Blue Period"
- We will examine works that are related in a more precise mathematical way.
- We will consider patterns that can be classified by two integer parameters.
- M.C. Escher was most likely the first artist to create patterns related in this way.
- These patterns can exist in any of the three "Classical Geometries": the Euclidean plane, the sphere, and the hyperbolic plane.

Escher's (Euclidean) Regular Division Drawing 45



Escher's "Angles and Devils" carved sphere

Escher's (hyperbolic) Circle Limit IV



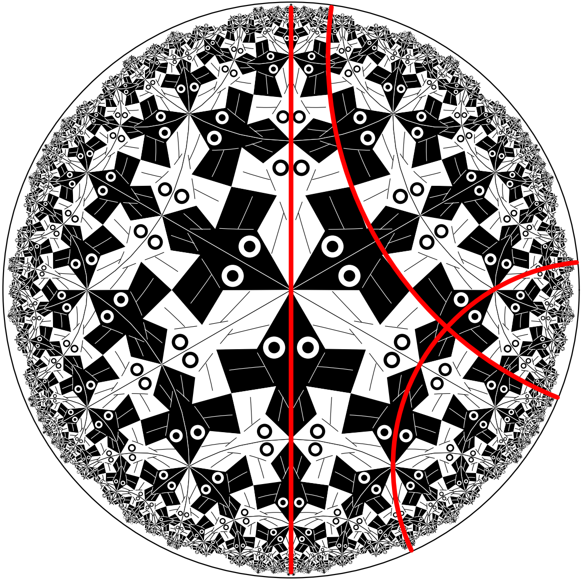
The Classical Geometries

- Euclidean geometry — zero curvature, i.e. flat
- Spherical geometry — constant positive curvature
- Hyperbolic geometry — constant negative curvature
- The hyperbolic plane cannot be smoothly embedded in Euclidean 3-space, unlike the sphere (proved in 1901 by David Hilbert).
- Therefore we must rely on **models** hyperbolic geometry — Euclidean constructs that can be interpreted as hyperbolic objects.

A Model of Hyperbolic Geometry

- M.C. Escher (and other artists) used the **Poincaré circle model** of hyperbolic geometry.
- Hyperbolic points are represented by Euclidean points within a (Euclidean) **bounding circle**.
- Hyperbolic lines are represented by (Euclidean) circular arcs that are orthogonal to the bounding circle (including diameters as special cases).
- Preferred by M.C. Escher and other artists since (1) it is contained in a finite area of the Euclidean plane, and (2) it is *conformal*, the hyperbolic measure of an angle is the same as its Euclidean measure, so that motifs retain approximately the same shape as they get smaller toward the bounding circle.

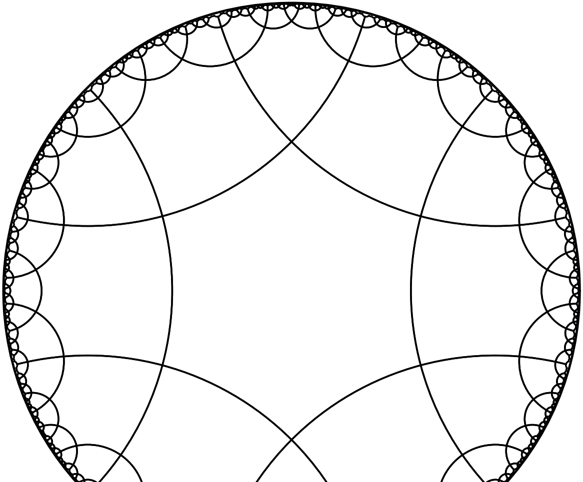
Escher's Circle Limit I Showing Hyperbolic Lines.



Repeating Patterns and Regular Tessellations

- A *repeating pattern* in any of the 3 "classical geometries" (Euclidean, spherical, and hyperbolic geometry) is composed of congruent copies of a basic subpattern or *motif*.
- For example if we ignore color, one butterfly is a motif for the butterfly pattern on the title page.
- The *regular tessellation*, $\{p, q\}$, is an important kind of repeating pattern composed of regular p-sided polygons meeting q at a vertex.
- If $(p-2)(q-2) < 4$, $\{p, q\}$ is a spherical tessellation (assuming $p > 2$ and $q > 2$ to avoid special cases).
- If $(p-2)(q-2) = 4$, $\{p, q\}$ is a Euclidean tessellation.
- If $(p-2)(q-2) > 4$, $\{p, q\}$ is a hyperbolic tessellation. The next slide shows the $\{6, 4\}$ tessellation.
- Escher based his 4 "Circle Limit" patterns, and many of his spherical and Euclidean patterns on regular tessellations.

The $\{6, 4\}$ tessellation.



A Table of the Regular Tessellations

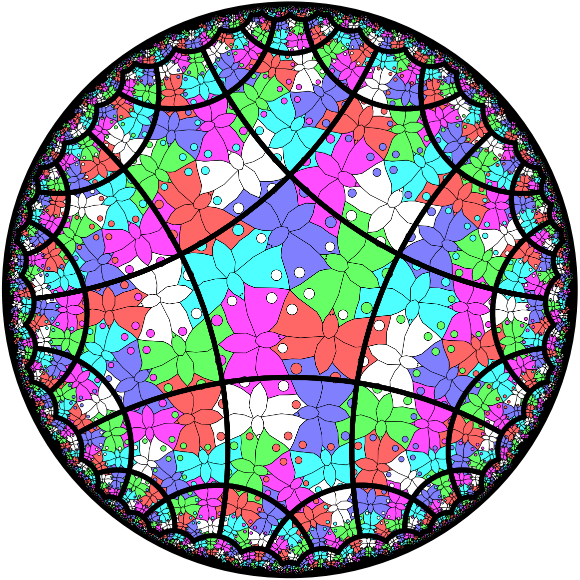
q \ p	3	4	5	6	7	8	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
8	*	*	*	*	*	*	$\cdots$
7	*	*	*	*	*	*	$\cdots$
6	□	*	*	*	*	*	$\cdots$
5	○	*	*	*	*	*	$\cdots$
4	○	□	*	*	*	*	$\cdots$
3	○	○	○	□	*	*	$\cdots$

□ - Euclidean tessellations

○ - spherical tessellations

* - hyperbolic tessellations

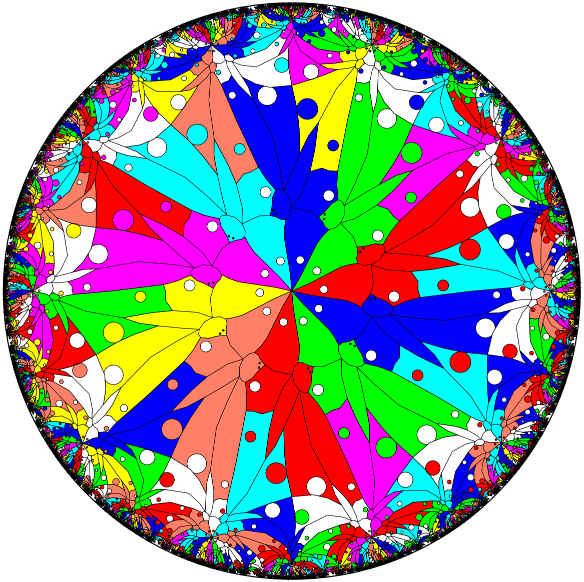
The $\{5, 4\}$ tessellation underlying the butterfly pattern



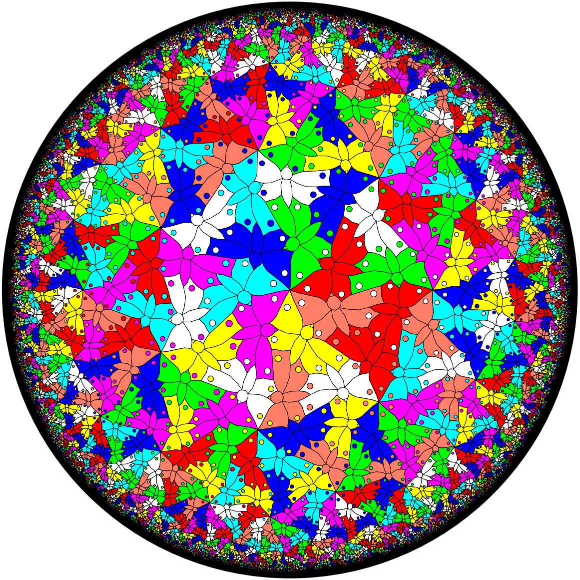
Parameterized Families of Patterns

- If a pattern is based on an underlying $\{p, q\}$ tessellation, we can conceive of other patterns with the same motif (actually slightly distorted) based on a different tessellation $\{p', q'\}$.
- This observation leads us to consider a whole *family* of such patterns indexed by p and q.
- We use (p, q) to denote the pattern of the family that is based on $\{p, q\}$.
- For example, the butterfly pattern above would be denoted $(5, 4)$.
- Unfortunately, large values of p or q or both usually do not produce aesthetically appealing patterns, since such values lead to distortion of the motif and/or push most of the pattern outward near the bounding circle.

A $(10, 4)$ butterfly pattern showing distortion



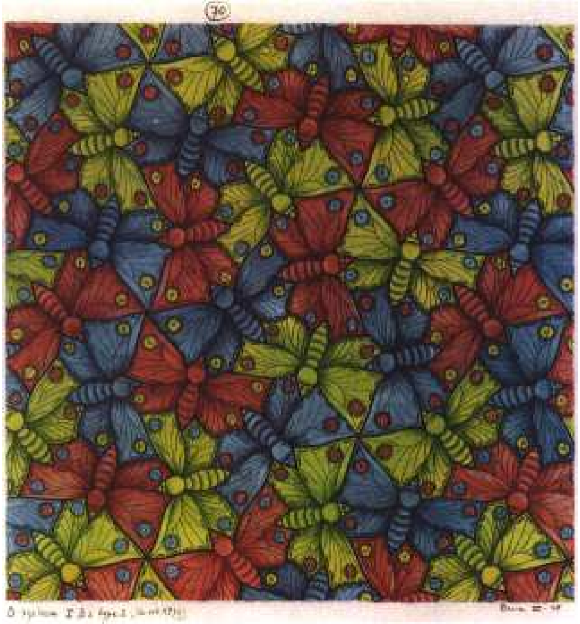
A $(7, 3)$ "error" butterfly pattern that has color symmetry, but does not adhere to the map-coloring principle.



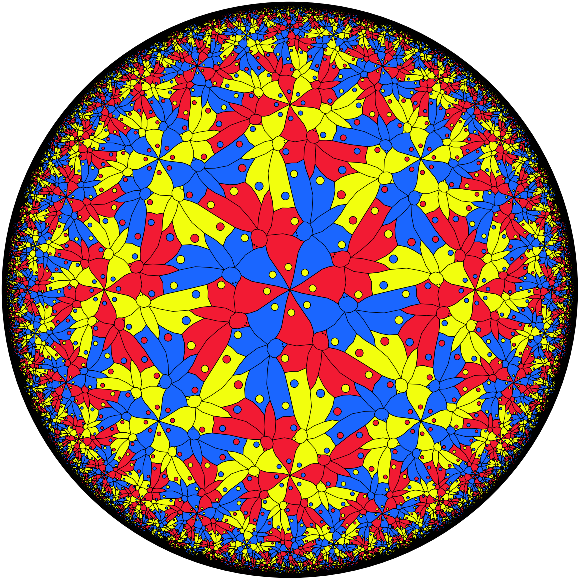
The Family of Butterfly Patterns

- Theoretically, we can create a butterfly pattern based on $\{p, q\}$ like the one above for any values of p and q provided $p \geq 3$ and $q \geq 3$.
- For these patterns, p butterflies meet at their left front wing tips and q butterflies meet at their right rear wings.
- Escher created only one member of this family of patterns, his Regular Division Drawing Number 70, based on the Euclidean hexagon tessellation $\{6, 3\}$. At least 3 colors are needed to satisfy the map-coloring principle at the meeting points of right rear wings.
- Following Escher, we add the restriction to our patterns that all circles on the butterfly wings around a p-fold meeting point of left wingtips be a different color from the butterflies meeting there.
- The hyperbolic butterfly pattern based on the $\{5, 4\}$ tessellation requires at least five colors for color symmetry since five is prime, and six colors if the circles on the wings are to be a different color.

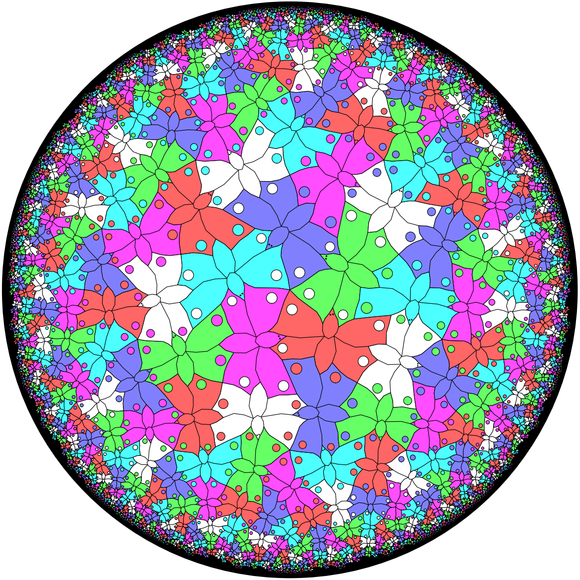
Escher's 3-colored butterfly pattern
Regular Division Drawing Number 70



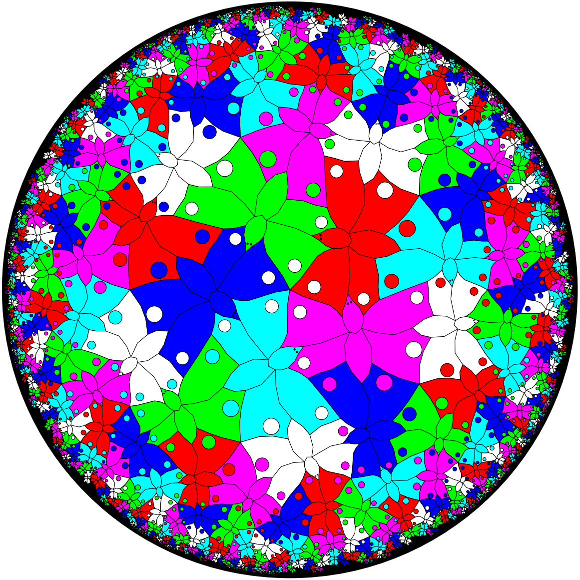
A 3-colored $(8, 3)$ butterfly pattern



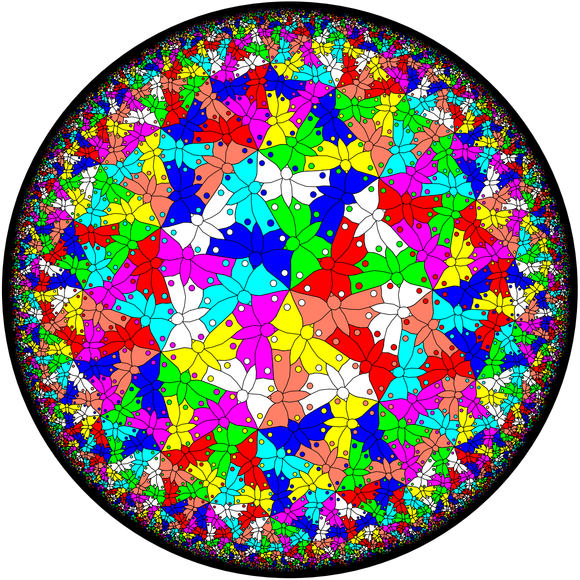
The $(5, 4)$ title pattern



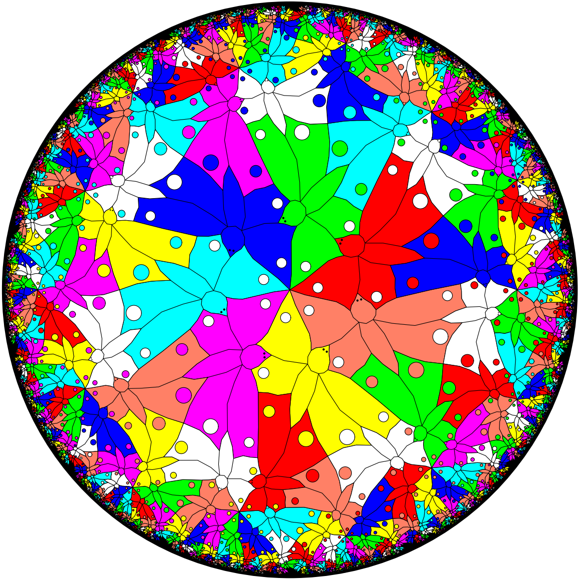
A 6-colored $(5,5)$ butterfly pattern



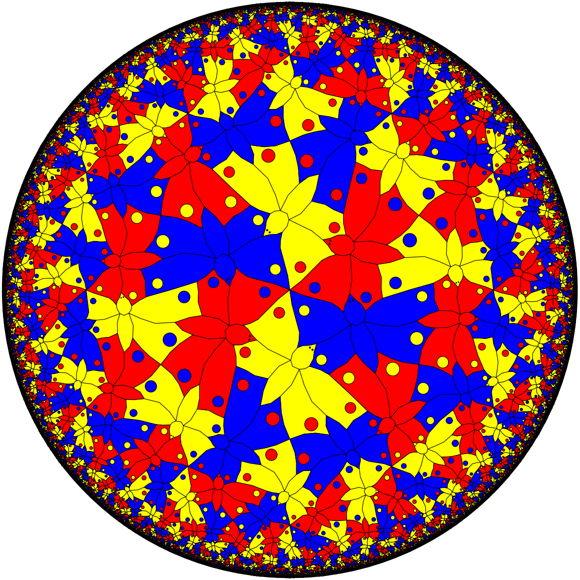
An 8-colored $(7,3)$ butterfly pattern



An 8-colored $(7,4)$ butterfly pattern



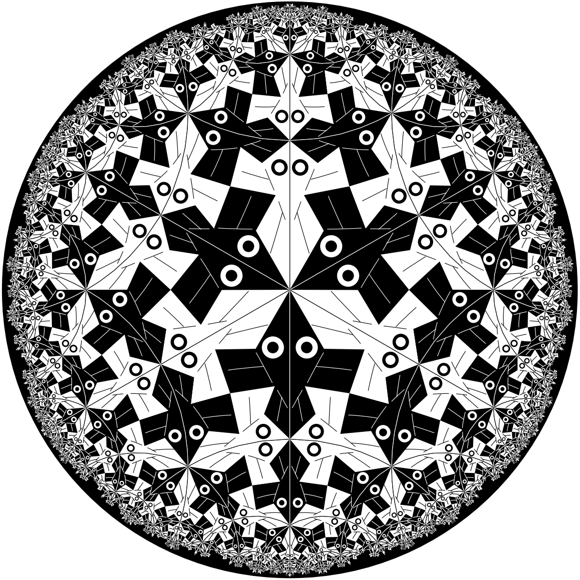
A 3-colored $(6, 4)$ butterfly pattern
that violates the color of circles convention



Other Families of Patterns

- We have seen Escher's three "Angles and Devils" patterns, all based on $\{p, q\}$ tessellations with p even:
  - Regular Division Drawing 45, which is based on the "square" $\{4, 4\}$ tessellation.
  - The carved sphere, based on the $\{4, 3\}$ tessellation.
  - The hyperbolic *Circle Limit IV*, based on the $\{6, 4\}$ tessellation.
- The fact that p must be even is a *divisibility condition*.
- Escher's *Circle Limit I* (shown below) is the only pattern he made in that family. These patterns are based on on $\{p, q\}$ tessellations with both p and q even, since the backbones of both the black and white fish are lines of reflection.

Escher's hyperbolic *Circle Limit I* pattern
Based on the $\{6, 4\}$ tessellation.



Future Work

- Investigate other families of Escher-like patterns, and draw such patterns, including 3-parameter families.
- Automatically generate the colors so that the pattern is symmetrically colored. Currently this must be done manually for each pattern in a family.
- Generate patterns of a family on a polyhedron of genus ≥ 2, since such polyhedra have the hyperbolic plane as their universal covering space.

Thank You

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