An Algorithm to Generate Repeating Hyperbolic Patterns

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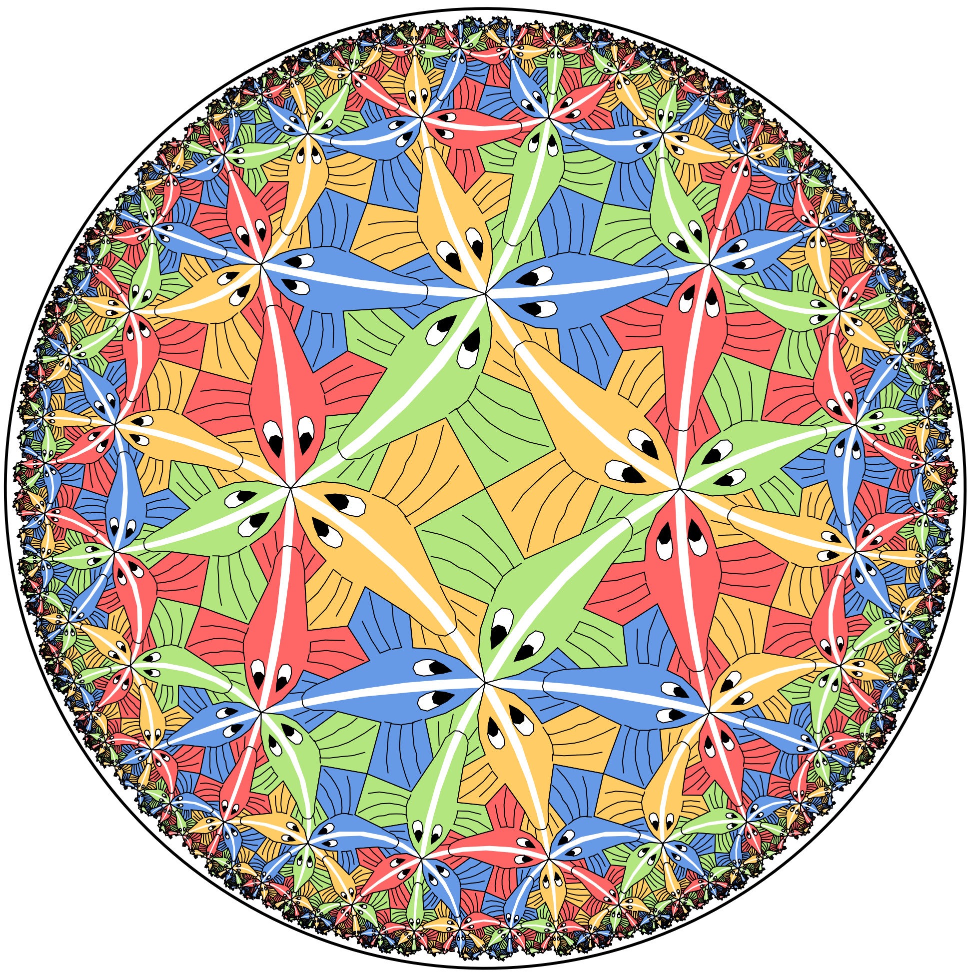
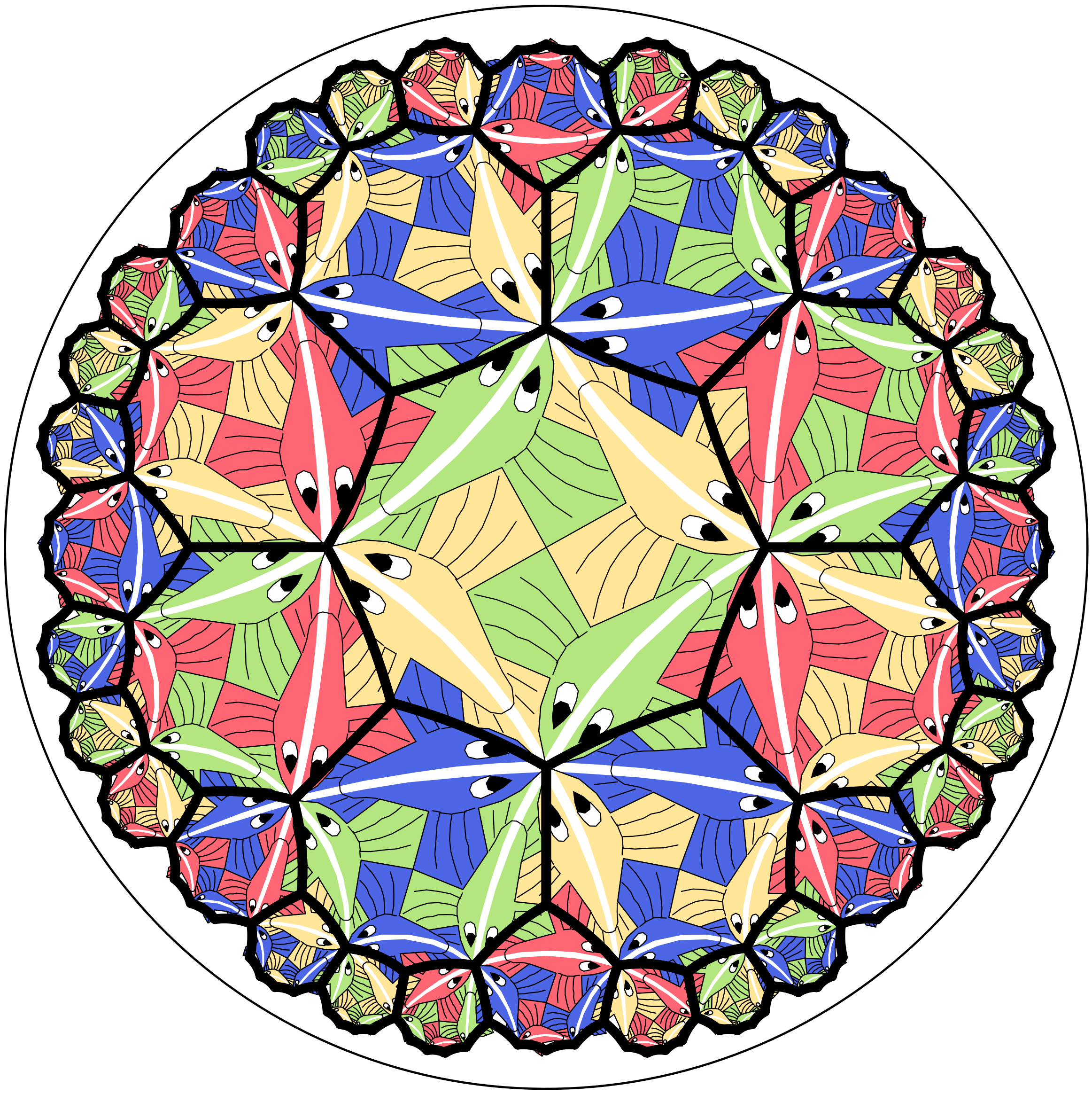
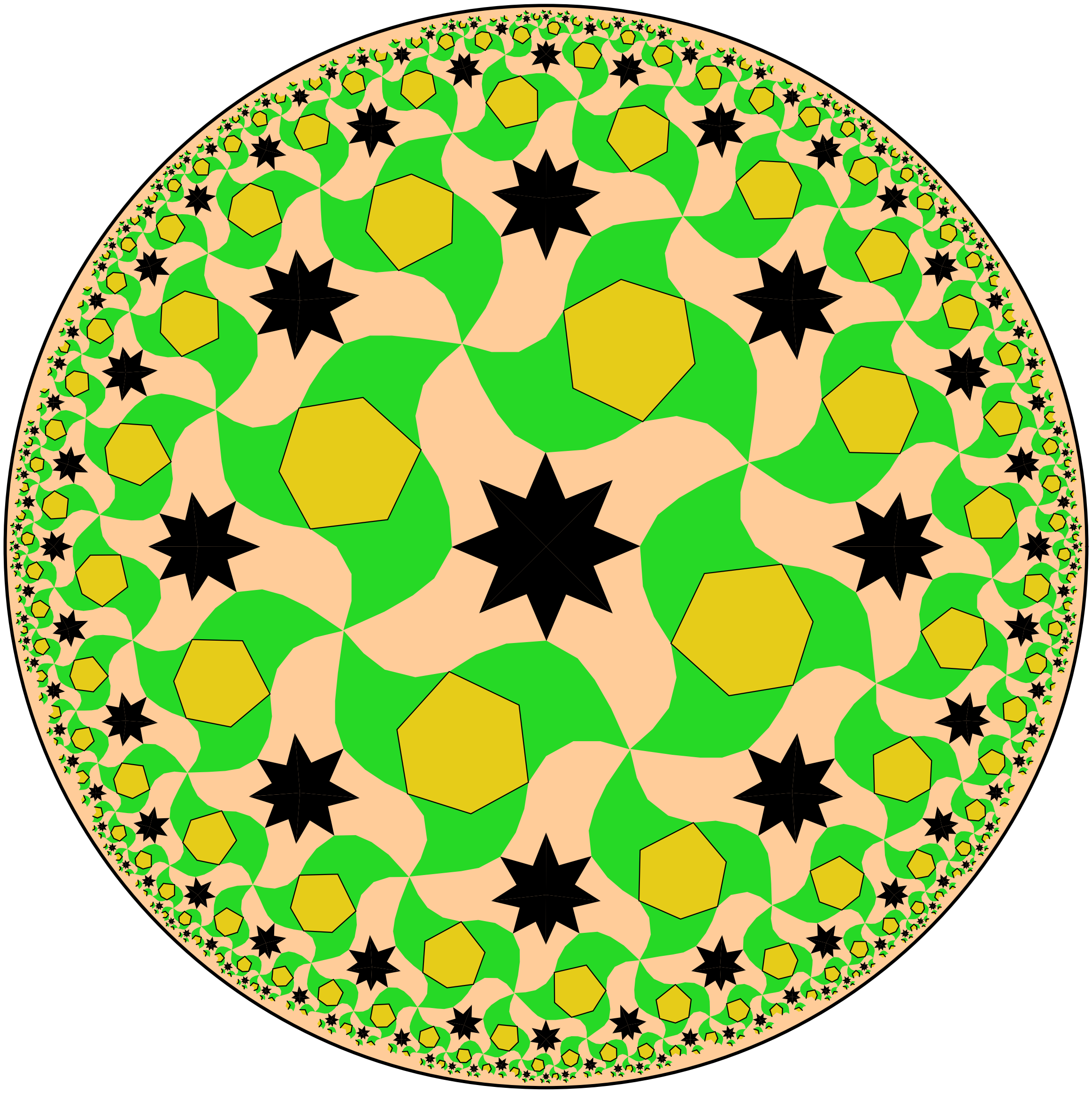


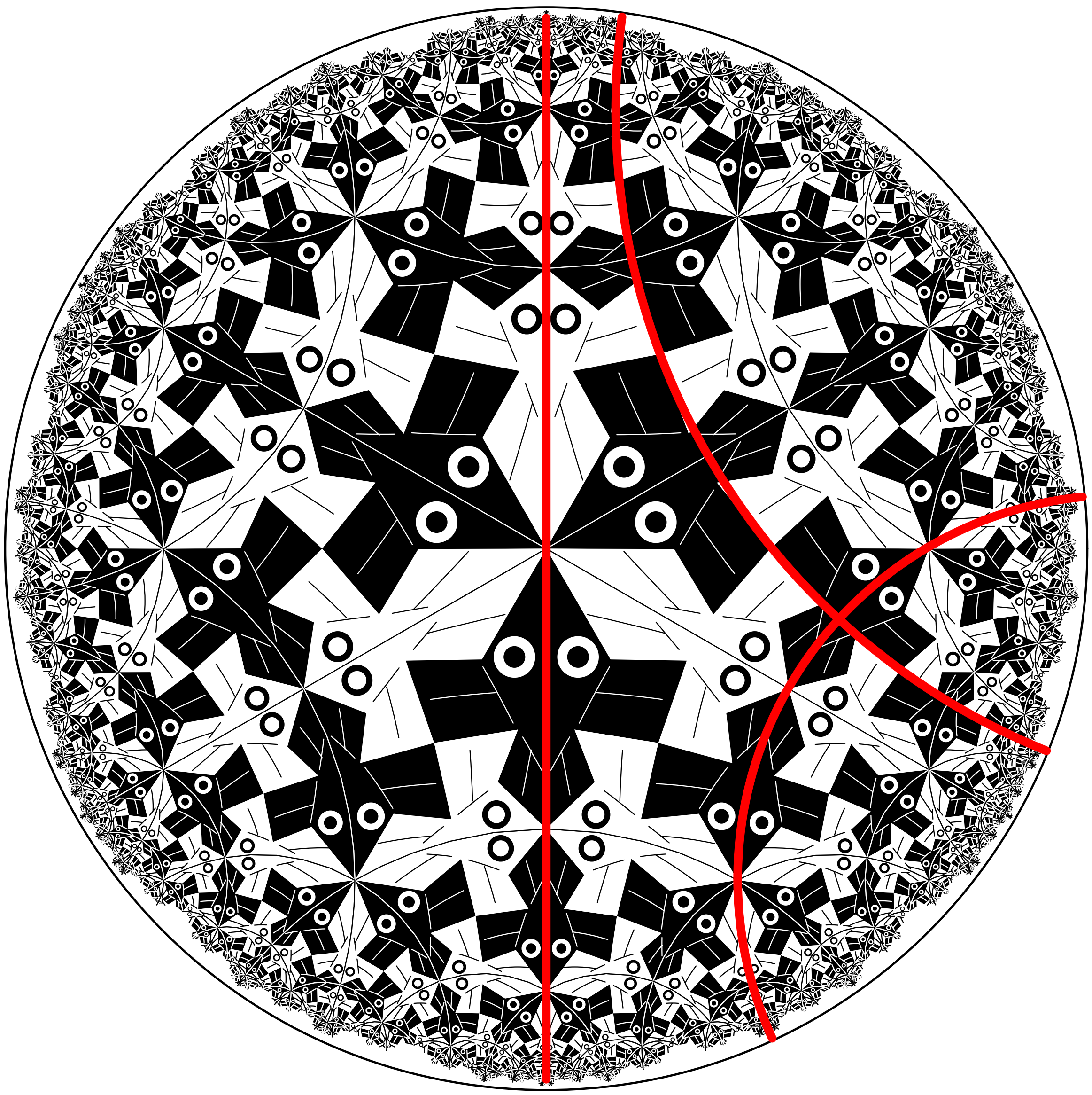
The $\{8,3\}$ tessellation on *Circle Limit III*



An Islamic pattern based on the {8,3} tessellation



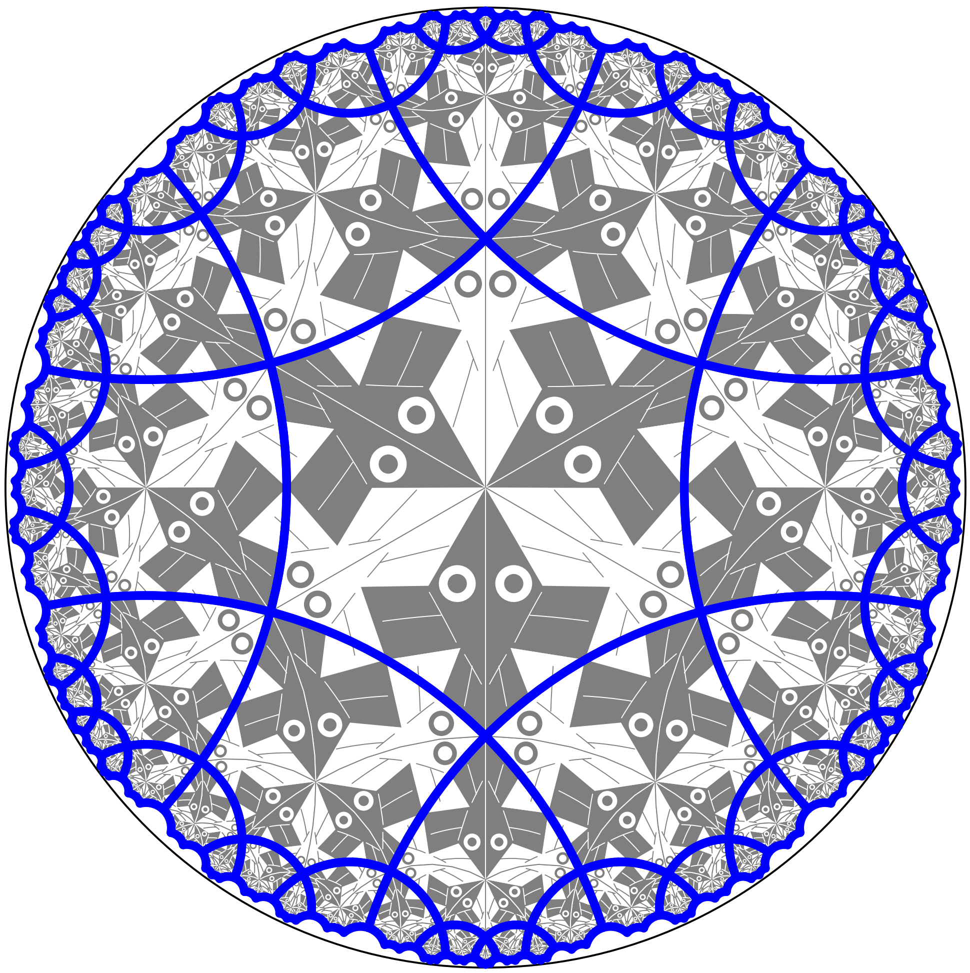
Poincaré Circle Model of Hyperbolic Geometry



- **Points:** points within the **bounding circle**
- **Lines:** circular arcs perpendicular to the bounding circle (including diameters as a special case)

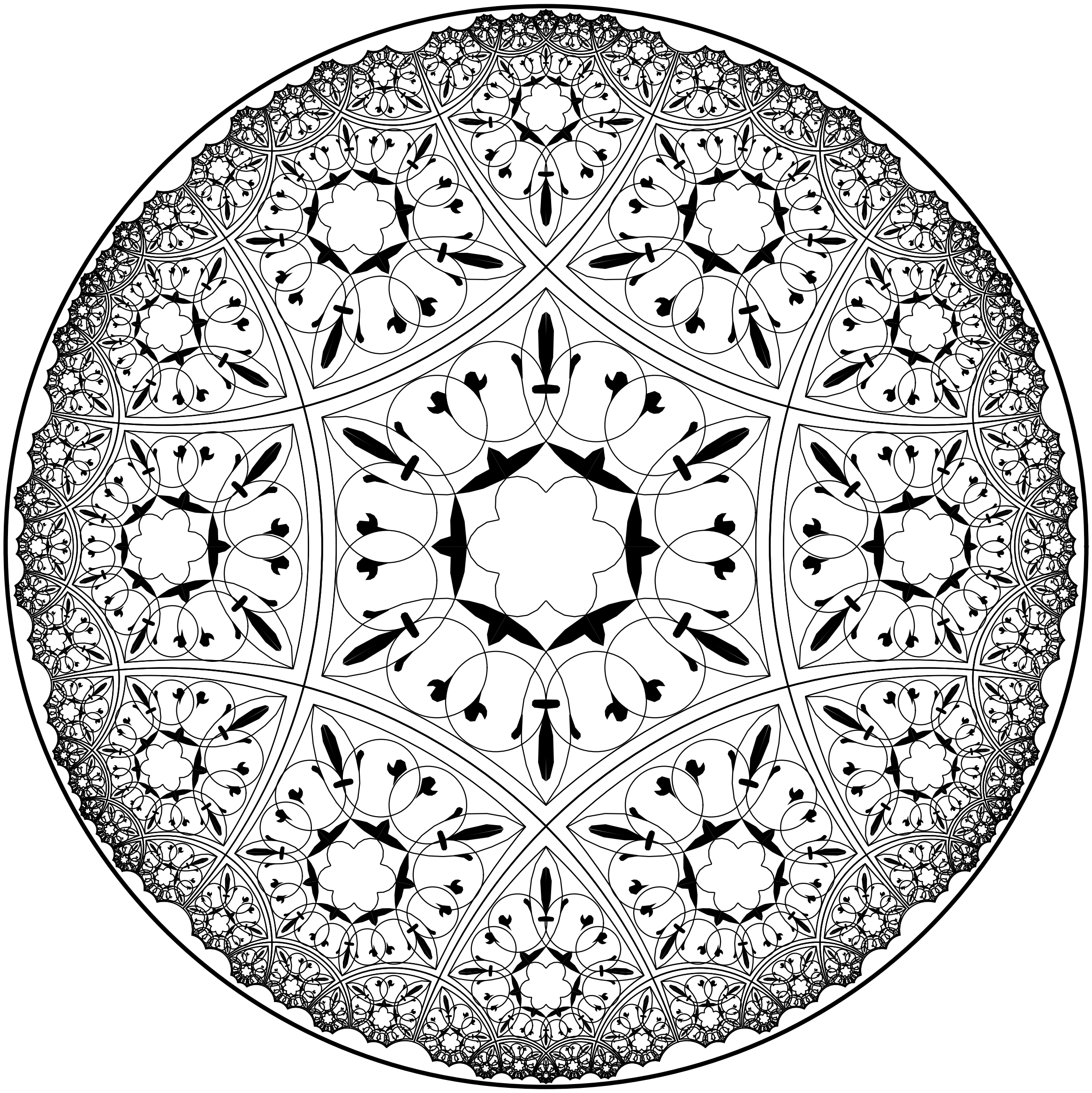
The Regular Tessellations $\{p,q\}$

There is a *regular tessellation,* $\{p,q\}$ of the hyperbolic plane by regular *p*-sided polygons meeting *q* at a vertex provided
$(p-2)(q-2) > 4$



The tessellation $\{6,4\}$ superimposed on the *Circle Limit I* pattern.

An arabesque pattern based on the $\{6,4\}$ tessellation



The General Replication Algorithm

A *motif* is a basic sub-pattern, of which the entire repeating pattern is comprised.

***Replication* is the process of transforming copies of the motif about the hyperbolic plane in order to create the whole repeating pattern.**

A *fundamental region* for the symmetry group of a pattern is a closed topological disk such that copies of it cover the plane without gaps or overlaps.

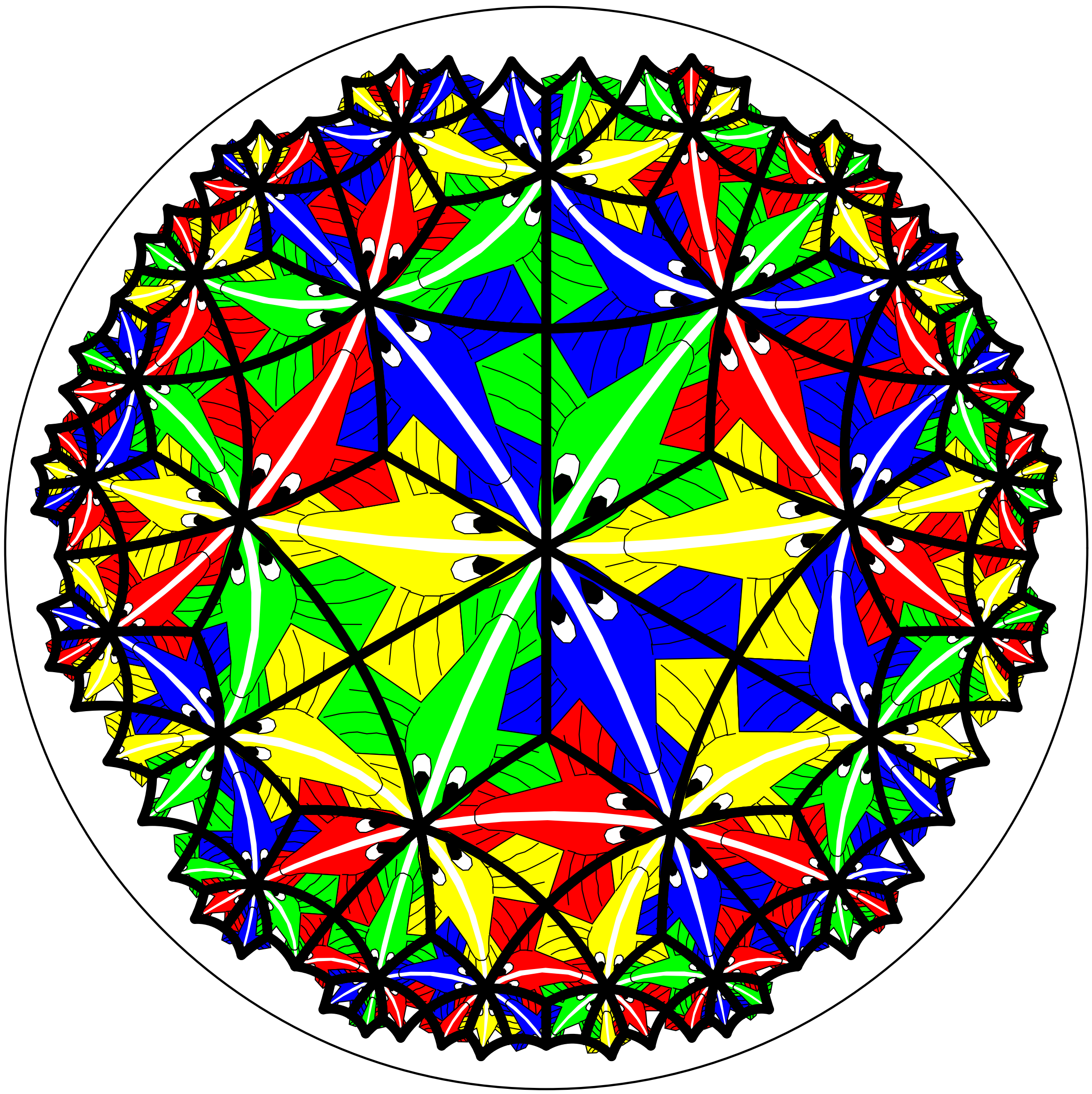
In Escher patterns the motif can usually be used as a fundamental region.

For a pattern with a finite motif, the fundamental region can be taken to be a convex polygon. This polygon will contain exactly the right pieces of the motif to reconstruct it.

Replication using copies of such a fundamental polygon will also create the entire pattern of motifs.

A Fundamental Polygon Tessellation

A quadrilateral can used as the fundamental region for the *Circle Limit III* pattern, as shown below



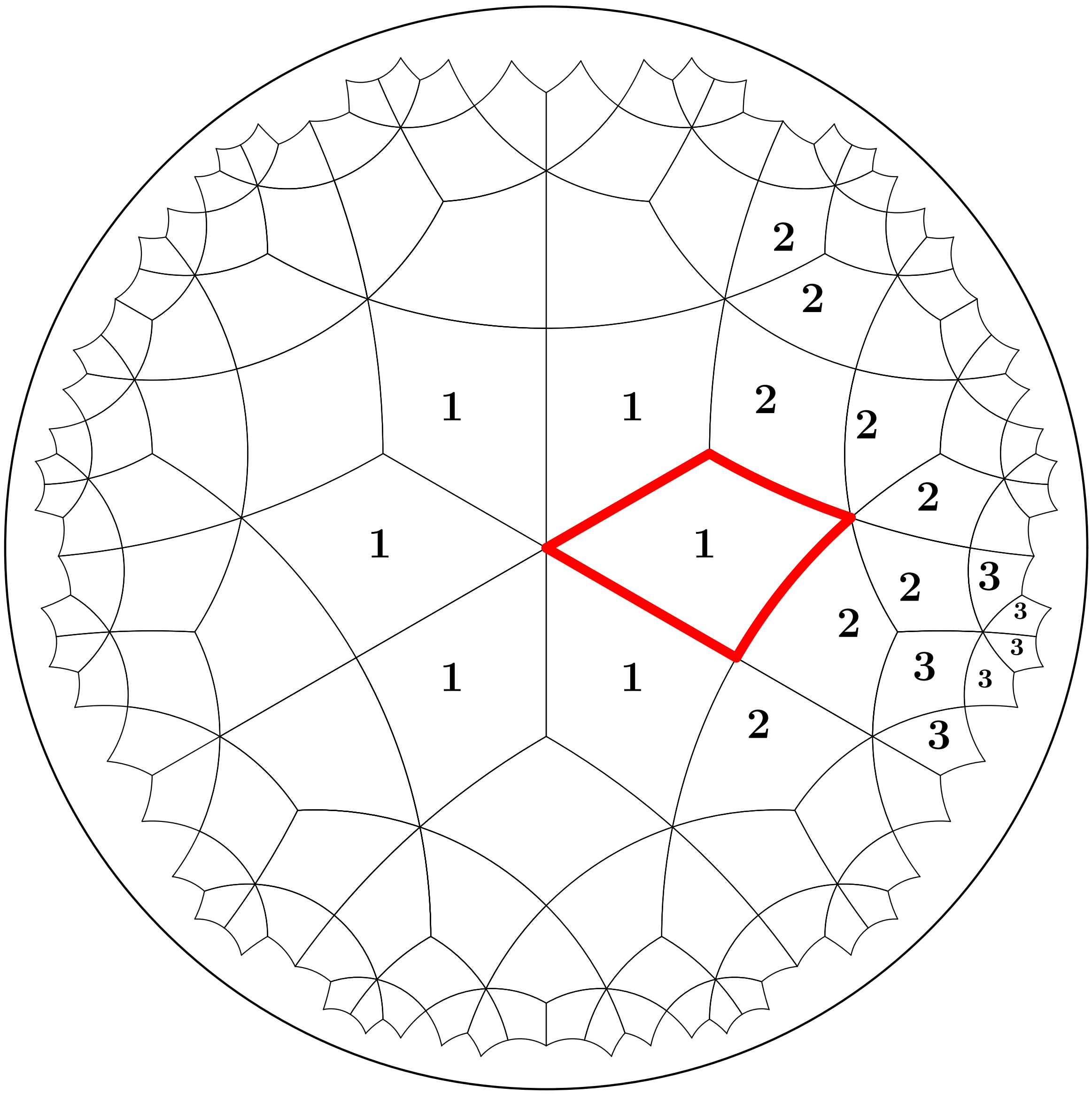
Layers of Fundamental Polygons

The fundamental polygons are arranged in *layers* (also called *coronas* in tiling literature), which are defined inductively.

The first layer **consists of all polygons with a vertex at the center of the bounding circle.**

The *boldmath* $k+1^{st}$ *layer* consists of all polygons sharing an edge or vertex with the k^{th} layer (and no previous layers).

A Polygon Tessellation Showing Layers



The polygon tessellation, with a fundamental polygon emphasized and parts of layers 1, 2, and 3 labeled.

Specification of the Fundamental Polygon

We use $\{p; q_1, q_2, \ldots, q_p\}$ to denote the fundamental polygon with p sides and q_i polygons meeting at vertex i (so the interior angle at the i^{th} vertex is $2\pi/q_i$).

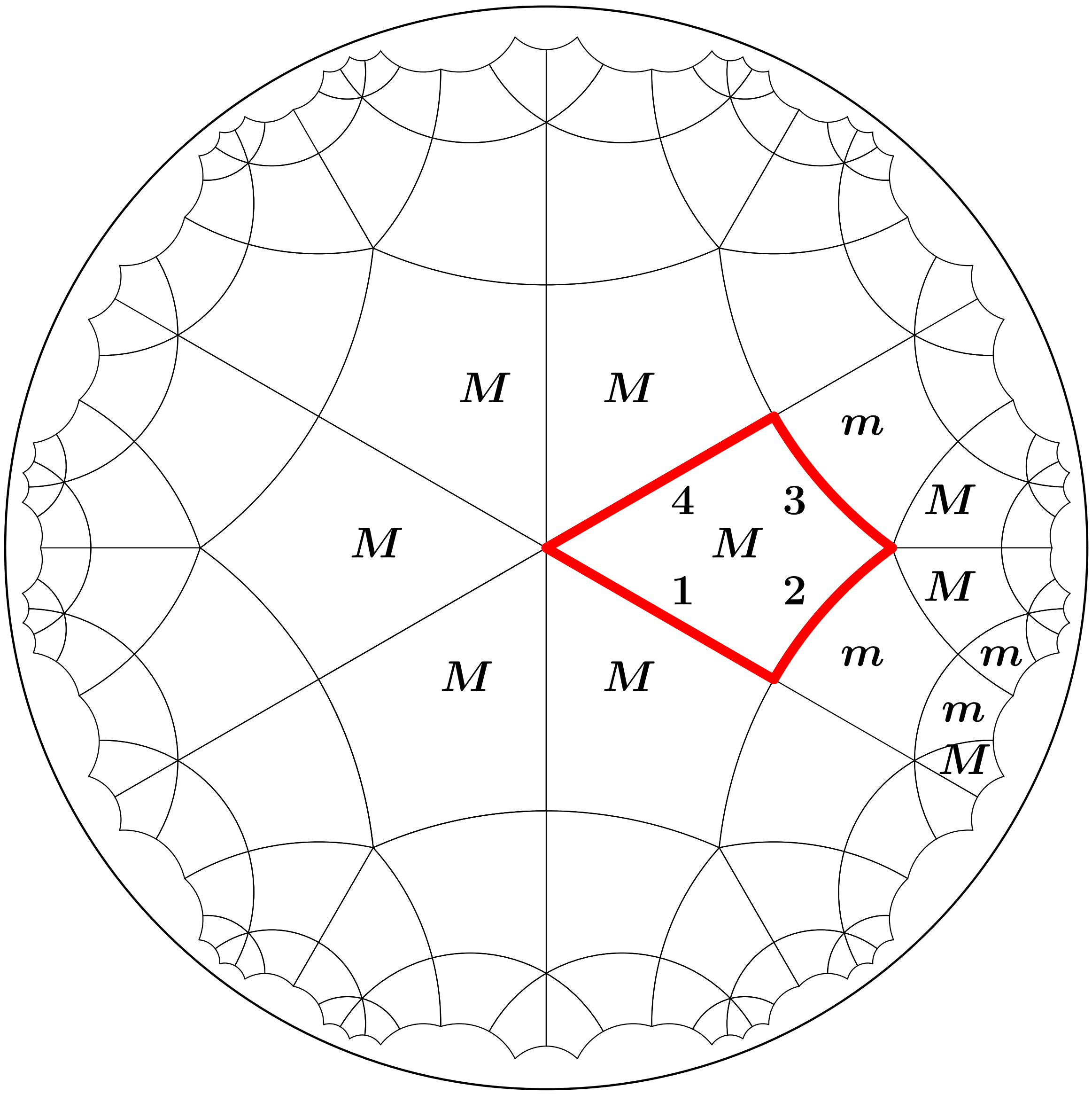
The condition that a polygon is a fundamental polygon for a hyperbolic tessellation is that:

$$\sum_{i=1}^{p} \frac{1}{q_i} < \frac{p}{2} - 1$$

(which generalizes the condition $(p-2)(q-2) > 4$ for regular tessellations). If the “$<$” is replaced with “=” or “$>$”, one obtains a Euclidean or spherical tessellation respectively.

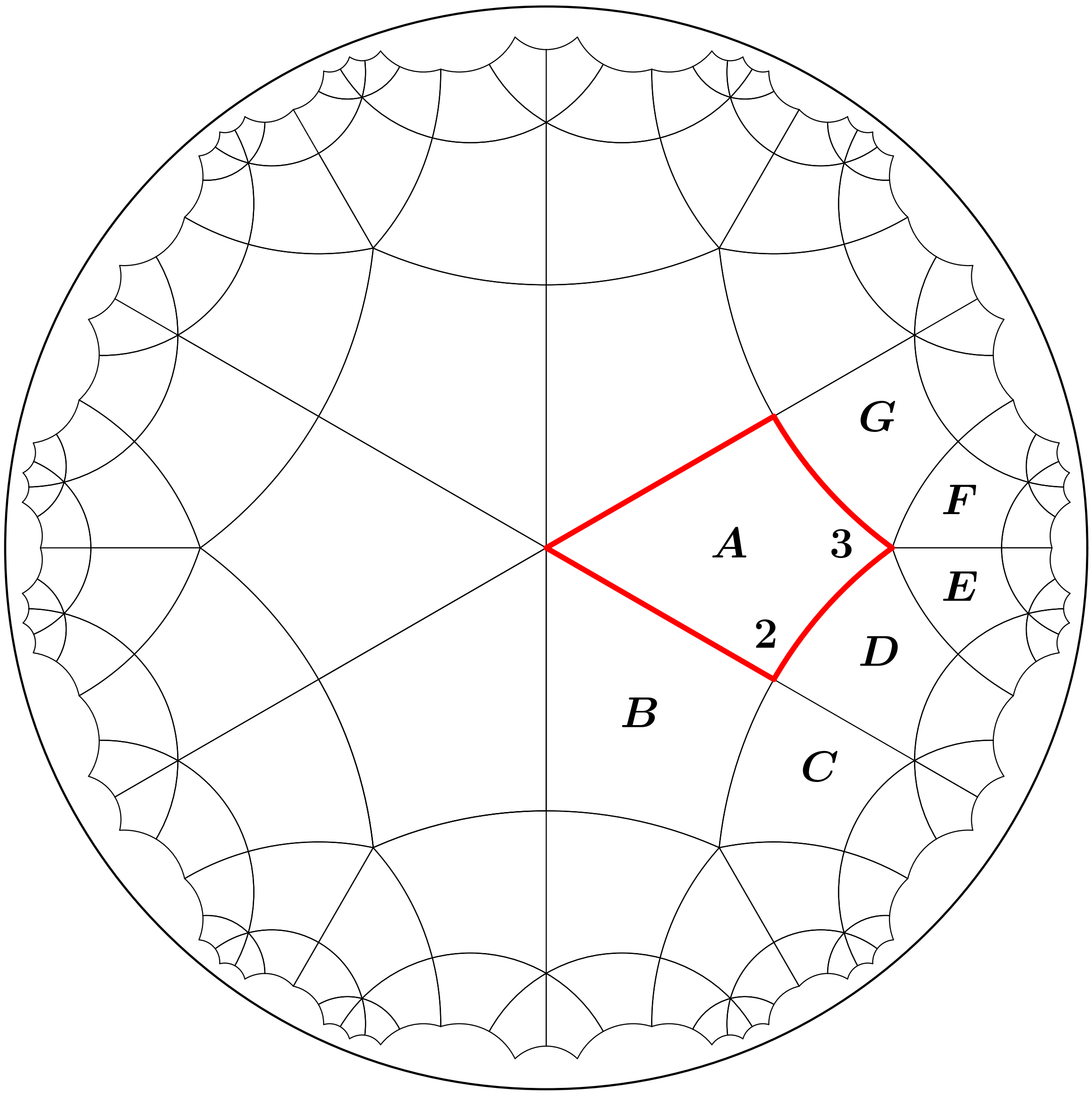
We say a polygon of a tessellation has ***minimal exposure*** if it shares an edge with a previous layer; we say it has ***minimal exposure*** if it shares a vertex with a previous layer.

Minimal and Maximal Exposure



The polygons with minimal exposure are marked with *m*'s, and those with maximal exposure are marked with *M*'s.

Some Polygons and Replication



This figure shows how recursive calls in the replication work starting at polygon A. Polygon vertices are numbered in counter-clockwise order with vertex i at the right end of edge i looking outward.

The Top-level “Driver” for Replication

The replication process starts with the following top-level “driver”, which calls the recursive routine `replicateMotif()` **to create the rest of the pattern.**

```
replicate ( motif )
{
  for ( j = 1 to q[1] )
  {
    qTran = edgeTran[1] ;

    replicateMotif(motif,qTran,2,MAX_EXP);

    qTran = addToTran ( qTran, -1 ) ;
  }
}
```

Utilities to Support Replication

Functions to compute transformations, based on `tranMult()` which multiplies two transformations and returns the product.

```
addToTran ( tran, shift )
{
   if ( shift % p == 0 ) return tran ;
   else return computeTran (tran, shift);
}

computeTran ( tran, shift )
{
   newEdge = (tran.pPosition +
            tran.orientation*shift) %p ;
   return tranMult(tran,
                   edgeTran[newEdge] ) ;
}
```

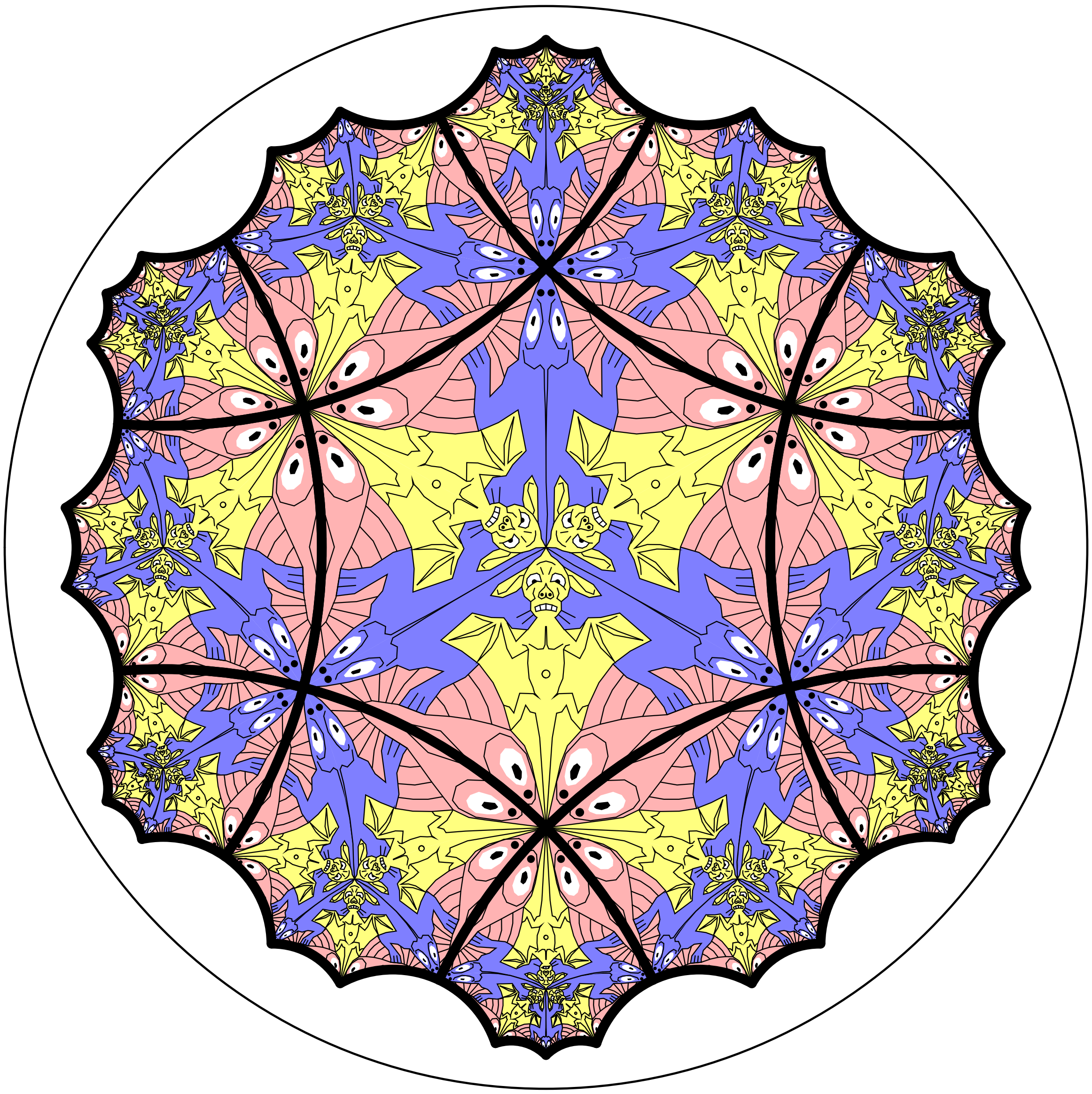
Arrays that control replication.

```
pShiftArray[]          = { 1,  0 } ;
verticesToSkipArray[] = { 3,  2 } ;
qShiftArray[]          = { 0, -1 } ;
polygonsToSkipArray[] = { 2,  3 } ;
exposureArray[] = { MAX_EXP, MIN_EXP } ;
```

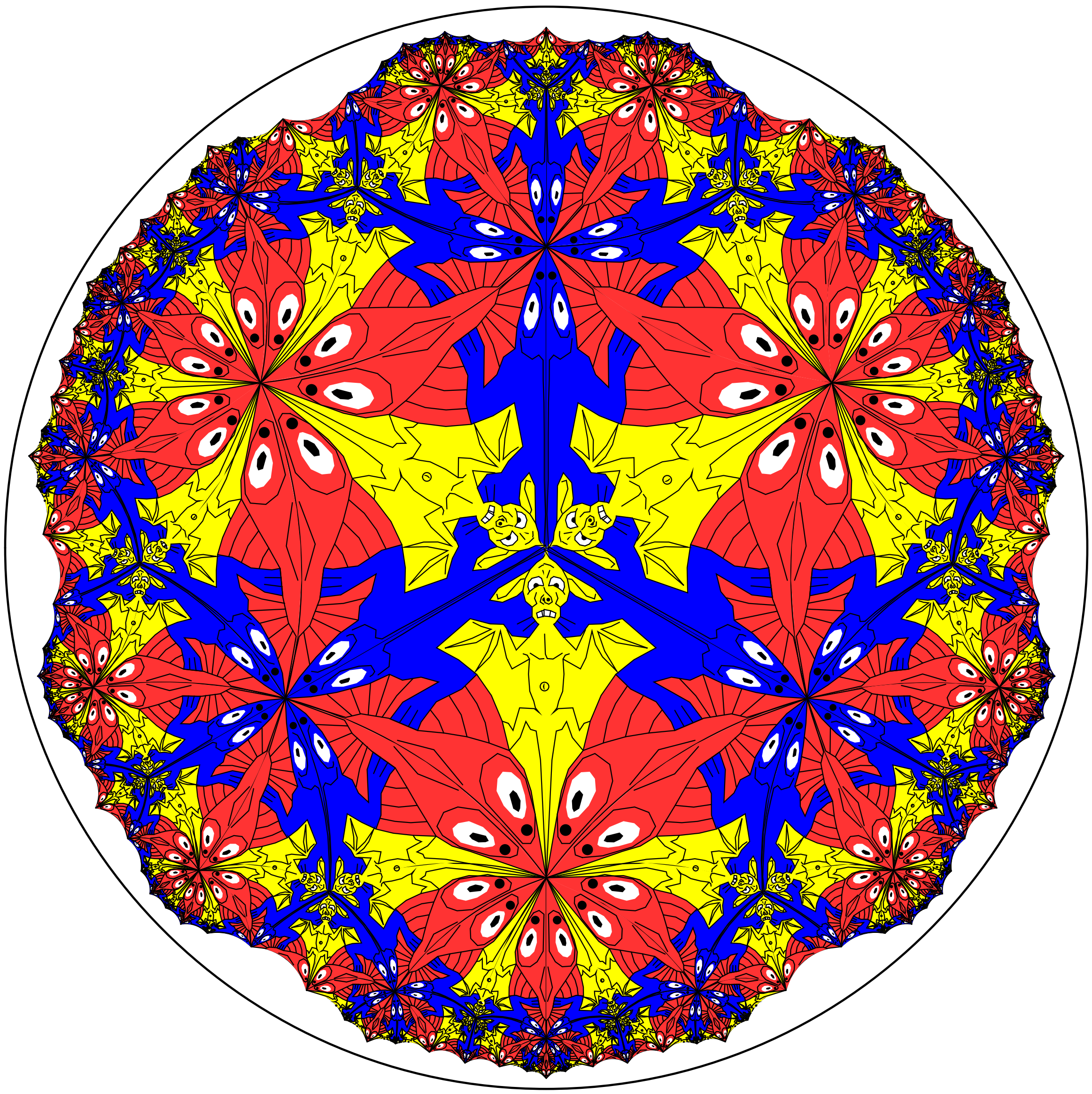
The Recursive `replicateMotif()`

```
replicateMotif(motif, inTran, layer, exposure)
{
  drawMotif ( motif, inTran ) ;
  if ( layer < maxLayers )
  {
    pShift = pShiftArray[exposure] ;
    verticesToDo = p -
                  verticesToSkipArray[exposure] ;

    for ( i = 1 to verticesToDo )
    {
      pTran = computeTran(initialTran, pShift) ;
      first_i = ( i == 1 ) ;
      qTran = addToTran(pTran, qShiftArray[first_i]) ;
      if ( pTran.orientation > 0 )
         vertex = (pTran.pposition-1) % p ;
      else
         vertex =  pTran.pposition ;
      polygonsToDo = q[vertex] -
                  polygonsToSkipArray[first_i] ;

      for ( j = 1 to polygonsToDo )
      {
        first_j = ( j == 1 ) ;
        newExpose = exposureArray[first_j] ;

        replicateMotif(motif, qTran, layer+1, newExpose) ;
        qTran = addToTran ( qTran, -1 ) ;
      }
      pShift = (pShift + 1) % p ;
    }
  }
}
```

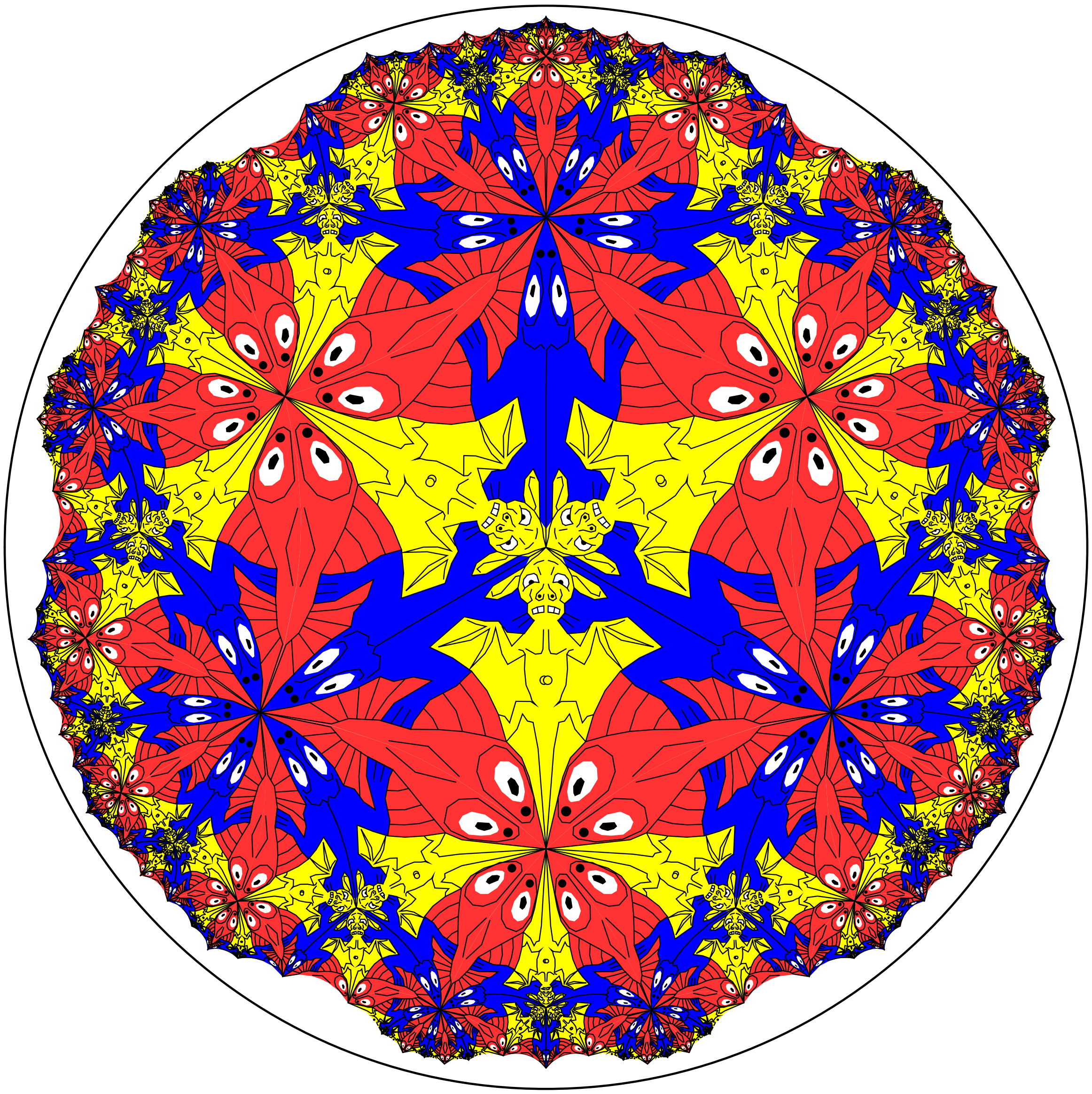
A “Three Element” Pattern Using {6,4}



A “Three Element” Pattern with Different Numbers of Animals Meeting at their Heads



A “Three Element” Pattern with 3 Bats, 5 Lizards, and 4 Fish Meeting at their Heads



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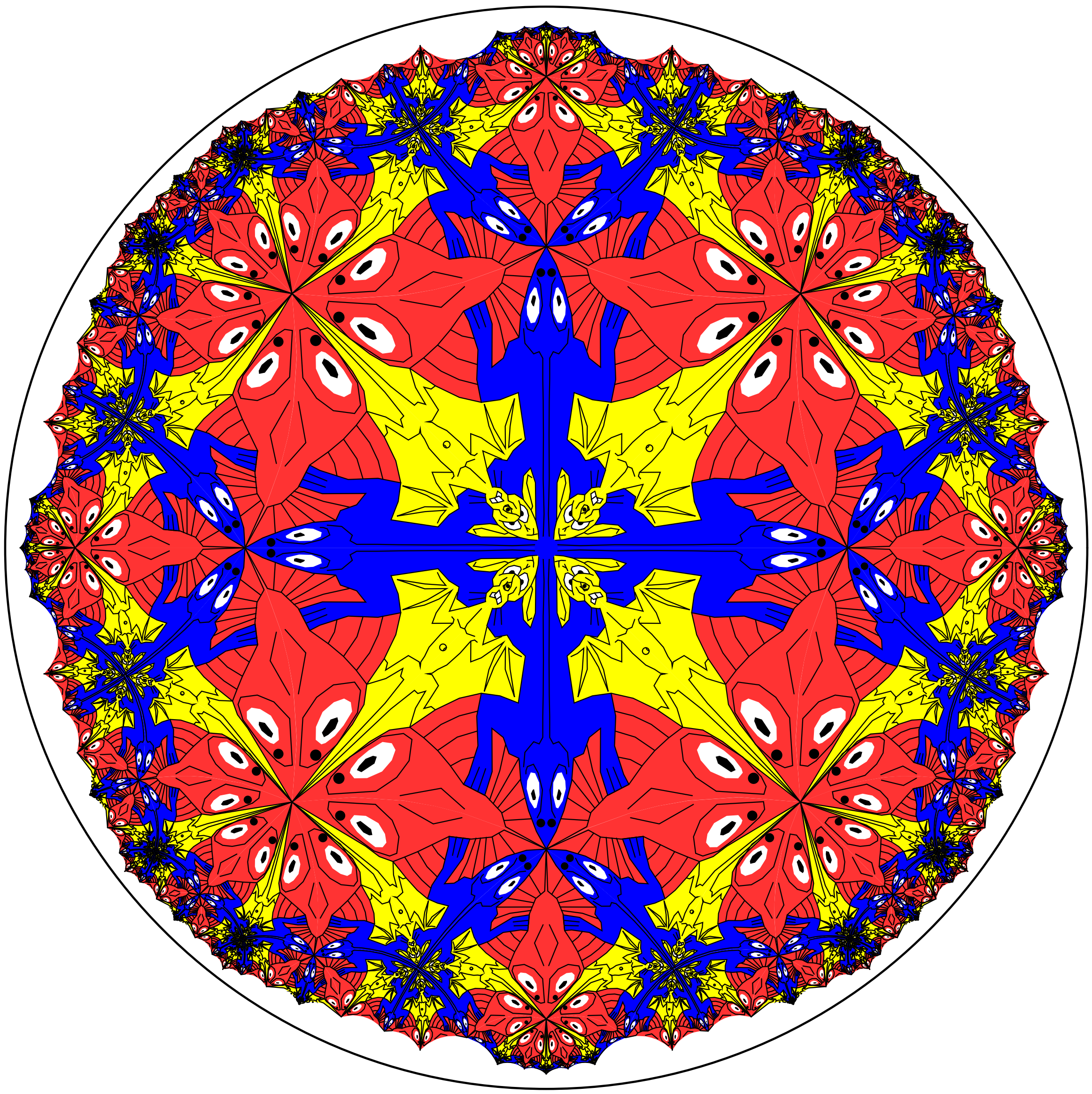

Future Work

- **Allow vertices at infinity.**
- **Create a program to transform between different fundamental polygons.**
- **Automatically generate patterns with color symmetry.**