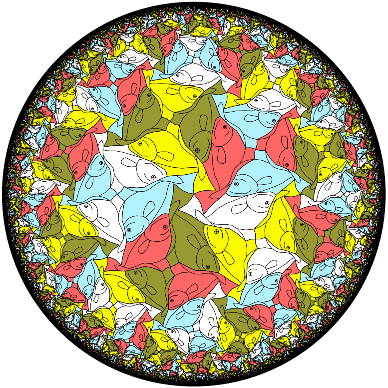
ISAMA 2010

Creating Repeating Patterns with Color Symmetry

Douglas Dunham
University of Minnesota Duluth
Duluth, Minnesota
USA



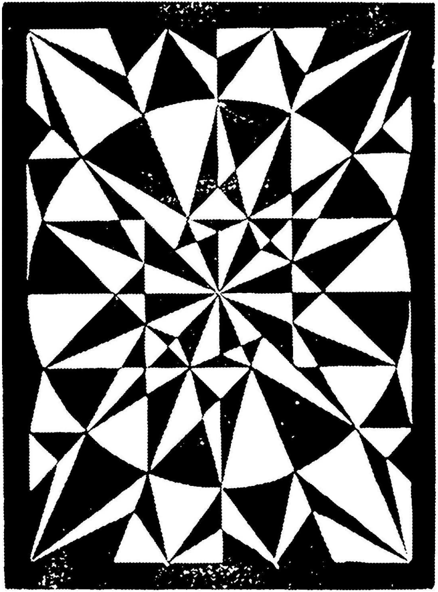
Outline

- Brief history of color symmetry
- Review of hyperbolic geometry
- Repeating patterns and regular tessellations
- Symmetries and color symmetry
- Color symmetry of a family of fish patterns
- Color symmetry of Escher's "Circle Limit" patterns
- Color symmetry of patterns related to *Circle Limit III*
- Color symmetry of butterfly patterns
- Future research

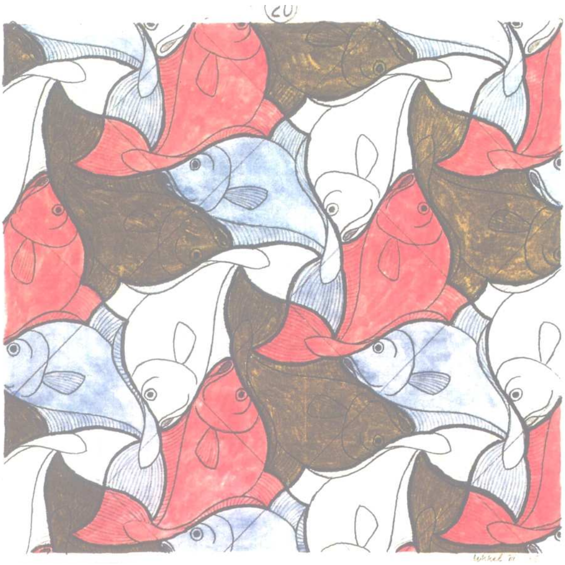
History

- People have created symmetrically colored patterns for hundreds and perhaps thousands of years.
- The Dutch artist M.C. Escher created a pattern with 2-color black-white) symmetry as early as 1921.
- H.J. Woods analyzes 2-color symmetry in "Counterchange Symmetry in Plane Patterns" in *Journal of the Textile Institute* (Manchester) in 1936.
- Escher created patterns with 3-color in the mid 1920's, and in 1938 he created Regular Division Drawing 20 with 4-color symmetry.
- From 1958 to 1960, Escher created his hyperbolic four "Circle Limit" patterns, two of which have color symmetry.
- In 1961, B.L. Van der Waerden and J.J. Burckhardt defined what we now call (perfect) color symmetry in "Farbgrupen" in *Zeitschrift für Kristallographie*.
- In the late 1970's and early 1980's computer programs were written to draw repeating hyperbolic patterns with color symmetry.

An Escher Pattern with 2-color Symmetry (1921)

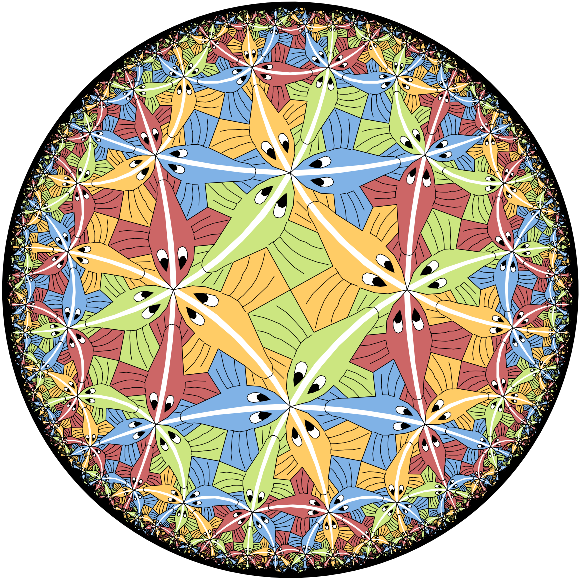


Escher's Notebook Drawing Number 20 with 4-color symmetry (1938)

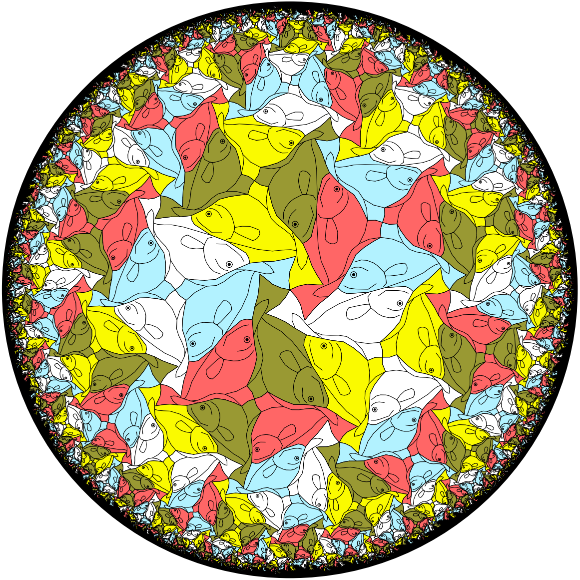


Escher's Circle Limit II pattern with 3-color symmetry (1959)

Escher's Circle Limit III pattern with 4-color symmetry (1959)



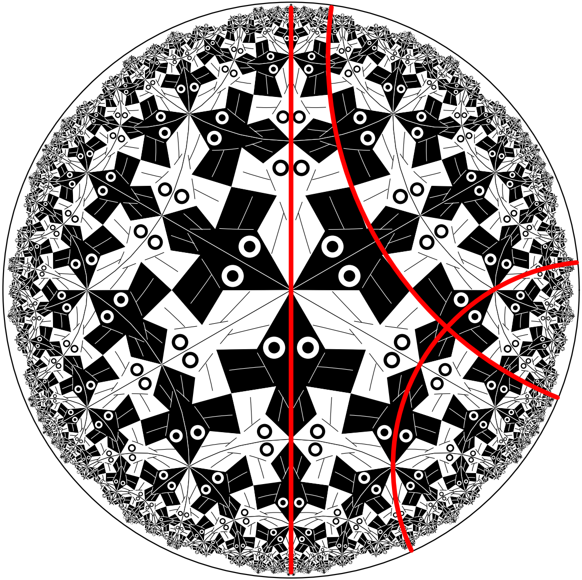
A computer generated fish pattern with 5-color symmetry (1980's))



Hyperbolic Geometry

- In 1901, David Hilbert proved that, unlike the sphere, there was no isometric (distance-preserving) embedding of the hyperbolic plane into ordinary Euclidean 3-space.
- Thus we must use *models* of hyperbolic geometry in which Euclidean objects have hyperbolic meaning, and which must distort distance.
- One such model, used by Escher, is the *Poincaré disk model*.
- The hyperbolic points in this model are represented by interior point of a Euclidean circle — the *bounding circle*.
- The hyperbolic lines are represented by (internal) circular arcs that are perpendicular to the bounding circle (with diameters as special cases).
- This model was preferred by Escher since (1) angles have their Euclidean measure (i.e. it is conformal), so that motifs of a repeating pattern retain their approximate shape as they get smaller toward the edge of the bounding circle, and (2) it could display an entire pattern in a finite area.

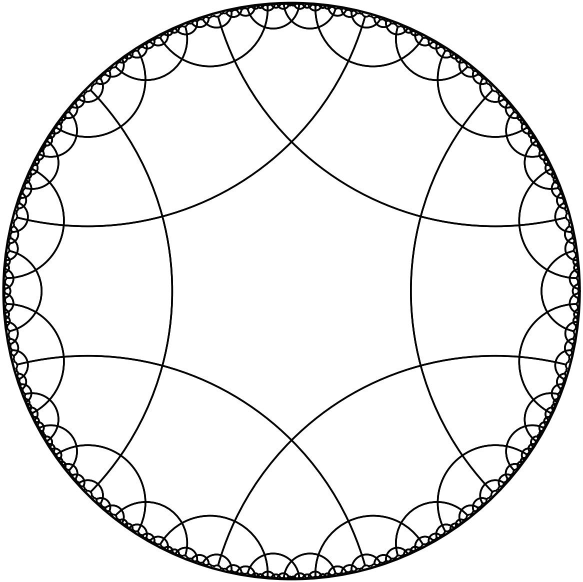
Escher's Circle Limit I showing hyperbolic lines.



Repeating Patterns and Regular Tessellations

- A *repeating pattern* in any of the 3 "classical geometries" (Euclidean, spherical, and hyperbolic geometry) is composed of congruent copies of a basic subpattern or *motif*.
- For example if we ignore color, one fish is a motif for the fish pattern on the title page.
- The *regular tessellation*, $\{p, q\}$, is an important kind of repeating pattern composed of regular p-sided polygons meeting q at a vertex.
- If $(p-2)(q-2) < 4$, $\{p, q\}$ is a spherical tessellation (assuming $p > 2$ and $q > 2$ to avoid special cases).
- If $(p-2)(q-2) = 4$, $\{p, q\}$ is a Euclidean tessellation.
- If $(p-2)(q-2) > 4$, $\{p, q\}$ is a hyperbolic tessellation. The next slide shows the $\{6, 4\}$ tessellation.
- Escher based his 4 "Circle Limit" patterns, and many of his spherical and Euclidean patterns on regular tessellations.

The $\{6, 4\}$ tessellation.



A Table of the Regular Tessellations

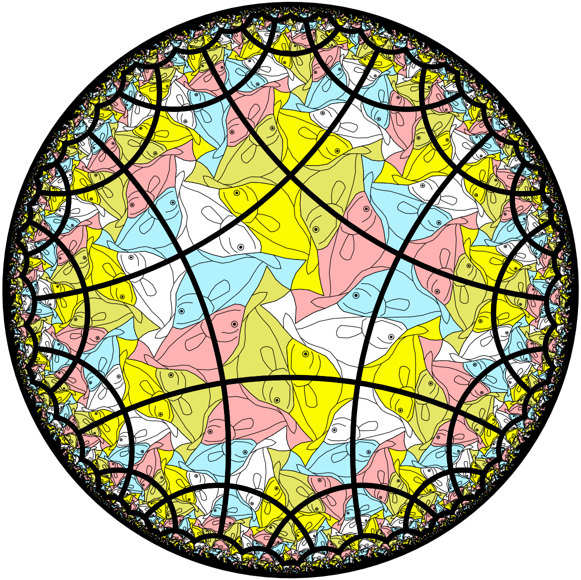
q							
⋮	⋮	⋮	⋮	⋮	⋮	⋮	
8	*	*	*	*	*	*	⋯
7	*	*	*	*	*	*	⋯
6	□	*	*	*	*	*	⋯
5	○	*	*	*	*	*	⋯
4	○	□	*	*	*	*	⋯
3	○	○	○	□	*	*	⋯
	3	4	5	6	7	8	⋯
				p			

□ - Euclidean tessellations

○ - spherical tessellations

* - hyperbolic tessellations

The $\{5, 4\}$ tessellation underlying the fish pattern



Families of Patterns

- If a pattern is based on an underlying $\{p, q\}$ tessellation, we can conceive of other patterns with the same motif (actually slightly distorted) based on a different tessellation $\{p', q'\}$.
- This observation leads us to consider an whole *family* of such patterns indexed by p and q.
- We use (p, q) to denote the pattern of the family that is based on $\{p, q\}$.
- For example, the previous fish pattern would be denoted $(5, 4)$.

Symmetries and Color Symmetry

- A *symmetry* of a repeating pattern is an isometry (distance-preserving transformation) that maps the pattern onto itselr. Thus each motif goes onto another copy of the motif.
- There are 5-fold (72°) rotation symmetries about fish tails in the preceding fish pattern, and also 4-fold rotations about dorsal fins.
- A reflection across a hyperbolic line in the Poincaré disk model is represented by an inversion in the circular arc representing that line. There are reflection symmetries across the backbone lines in the *Circle Limit I* pattern.
- As in Euclidean geometry, a hyperbolic rotation can be produced by successive reflections across intersecting lines. The rotation angle is twice the angle of intersection.

Symmetries and Color Symmetry (Continued)

- A *color symmetry* of a pattern of colored motifs is a symmetry of the uncolored pattern that takes all motifs of one color to motifs of a single color — that is, it permutes the colors of the motifs.
- Thus rotation about the center of the preceding fish pattern permutes the colors: red $\rightarrow$ yellow $\rightarrow$ blue $\rightarrow$ brown $\rightarrow$ white $\rightarrow$ red, and black remains fixed since it is used as an outline/detail color.

Implementation of Color Symmetry

- Symmetries of of uncolored patterns in the 3 classical geometries can be implemented as matrices in many programming languages.
- We use integers to represent colors. In the fish pattern, 0 $\leftrightarrow$ black, 1 $\leftrightarrow$ white, 2 $\leftrightarrow$ red, 3 $\leftrightarrow$ yellow, 4 $\leftrightarrow$ blue, and 5 $\leftrightarrow$ brown.
- We use arrays to represent permutations (more convenient than cycle notation). If α is the color permutation induced by the 72° central rotation of the fish pattern,

$$\alpha = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

  in two-line notation, then

$$\alpha[0] = 0, \quad \alpha[1] = 2, \quad \alpha[2] = 3, \quad \alpha[3] = 4, \quad \alpha[4] = 5, \quad \alpha[5] = 1.$$

Implementation of Color Symmetry (Continued)

- To multiply permutations α and β to obtain their product γ:

  for i $\leftarrow$ 0 to nColors - 1

  $\gamma[\texttt{i}] = \beta[\alpha[\texttt{i}]]$

- To obtain the inverse of a permutations α:

  for i $\leftarrow$ 0 to nColors - 1

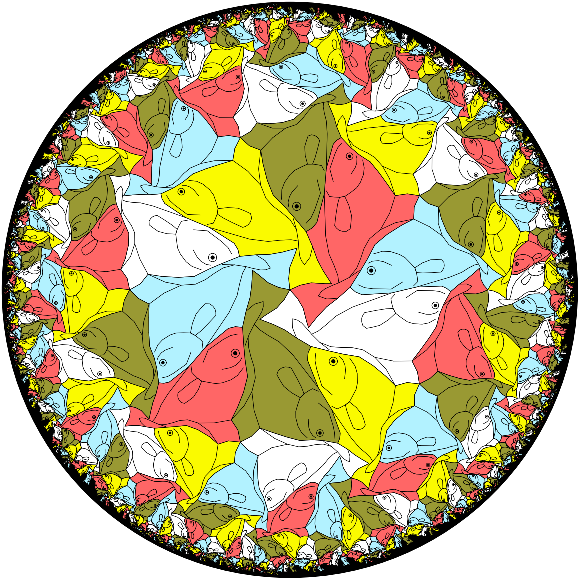
  $\alpha^{-1}[\alpha[\texttt{i}]] = \texttt{i}$

- It is useful to "bundle" the matrix representing a symmetry with its color permutation (as an array) into a single "transformation" structure (or class in an object oriented language).

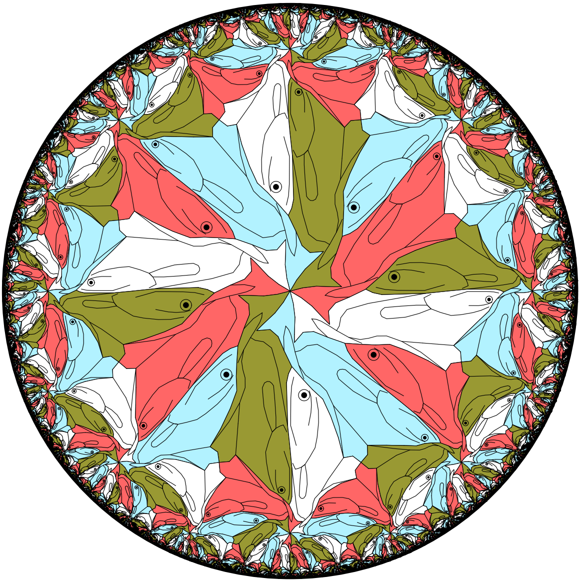
The Color Symmetry of Fish Patterns

- Theoretically, we can create a fish pattern based on $\{p, q\}$ like the one above for any values of p and q provided $p \geq 3$ and $q \geq 3$.
- For these patterns, p is the number of fish that meet their tails and q is the number of fish that meet at their dorsal fins.
- This family of fish patterns is based on Escher's 4-colored Notebook Drawing Number 20 above, which is based on the Euclidean "square" tessellation $\{4, 4\}$.
- For Notebook Drawing Number 20, at least three colors are needed to satisfy the map-coloring principle, and I think four colors are needed for color symmetry.
- The hyperbolic fish pattern based on the $\{5, 4\}$ tessellation requires at least five colors for color symmetry since five is prime.
- Large values of p or q or both usually do not produce aesthetically appealing patterns, since such values lead to distortion of the motif and/or push most of the pattern outward near the bounding circle.

A 5-colored fish pattern based on $\{5, 5\}$



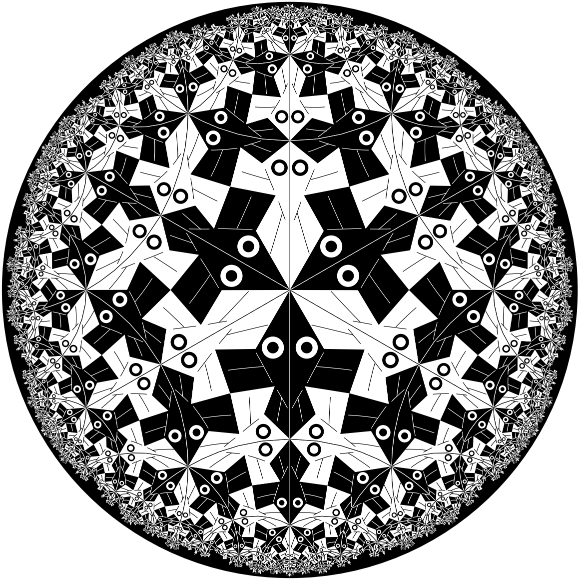
A 4-colored pattern of distorted fish based on $\{8, 4\}$



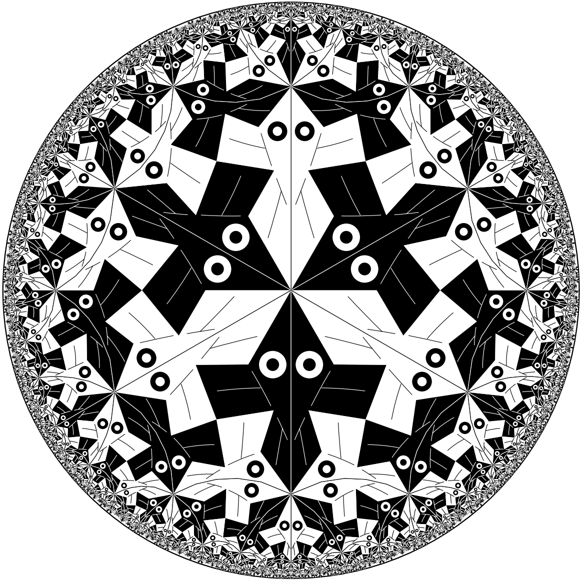
Color Symmetry of Escher's "Circle Limits"

- *Circle Limit I* does not have color symmetry, but related patterns do. For a pattern in the *Circle Limit I* family, p and q must be even due to reflection lines across the backbones of the fish. To obtain 2-color symmetry, p must equal q.
- *Circle Limit II* has 3-color symmetry, as seen above.
- *Circle Limit III* has 4-color symmetry, and cannot be symmetrically colored with fewer colors.
- Patterns in the *Circle Limit IV* family cannot have color symmetry.

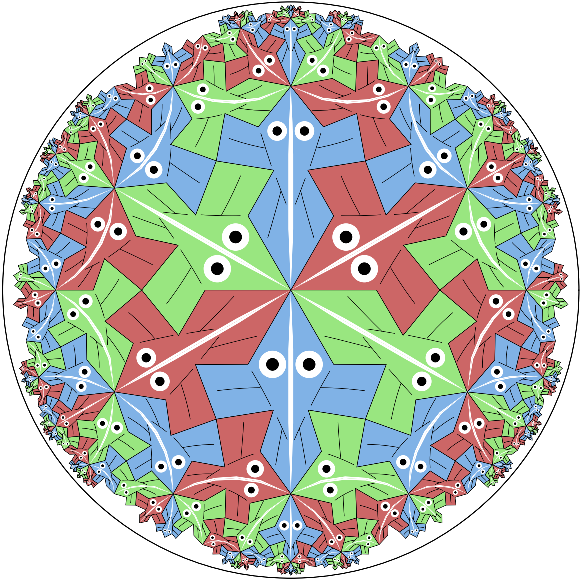
Escher's Circle Limit I $\{6, 4\}$ pattern
No color symmetry



A 2-colored Circle Limit I pattern
Based on the $\{6, 6\}$ tessellation



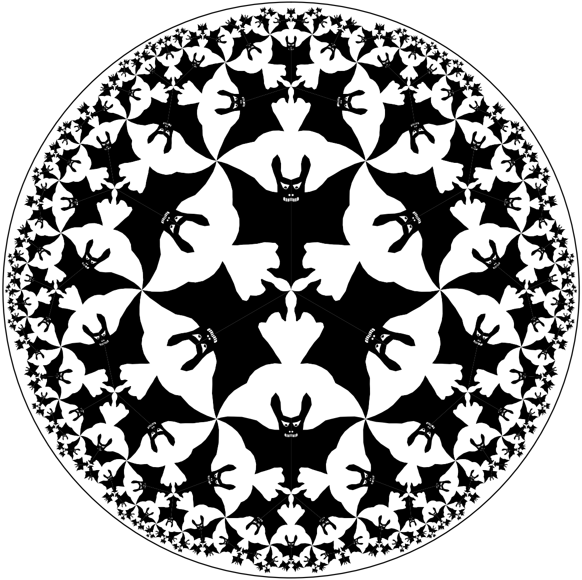
A 3-colored Circle Limit I pattern
Based on the $\{6, 6\}$ tessellation



Escher's Circle Limit II $\{8, 3\}$ pattern
3-colored (p must be even for these patterns)

A 2-colored Circle Limit II pattern
Based on the $\{8, 4\}$ tessellation

Escher's Circle Limit IV pattern
No color symmetry

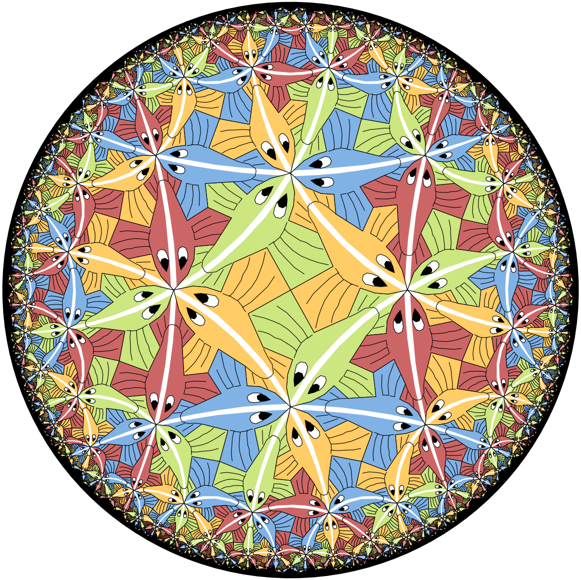


Color Symmetry of *Circle Limit III* Patterns

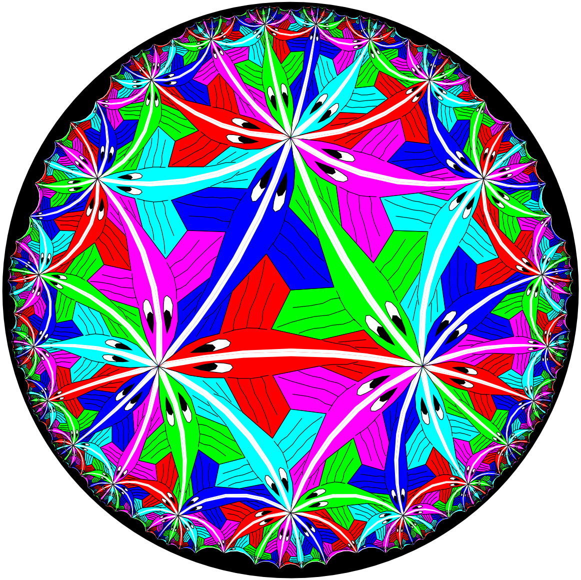
- As mentioned above, *Circle Limit III* has 4-color symmetry, and cannot be symmetrically colored with fewer colors.
- *Circle Limit III* solved the problems Escher saw in *Circle Limit I*:
  - There was no “traffic flow” — the fish alternated directions along a backbone line.
  - The fish alternated colors along a backbone line.
  - The fish were angular — not “fish shaped”.
- For other patterns in the *Circle Limit III* family, the restriction that fish along a backbone line be the same color adds another restriction to symmetric coloring.
- The *Circle Limit III* family of patterns depends on 3 numbers, $p, q,$, the numbers of fish meeting at right and left fin tips, and r the number of fish meeting at noses. So r must be odd so that the fish swim head-to-tail.
- We use (p, q, r) to denote such a pattern.

Escher's *Circle Limit III*

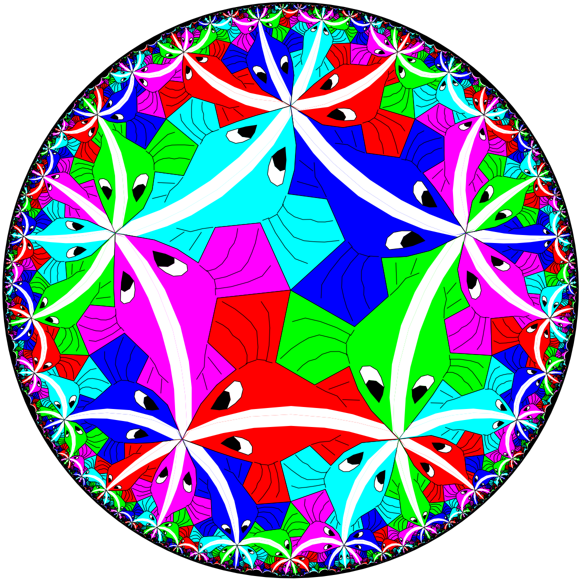
Needs 4 colors — a $(4, 3, 3)$ pattern



A 5-colored $(3, 3, 5)$ *Circle Limit III* pattern.

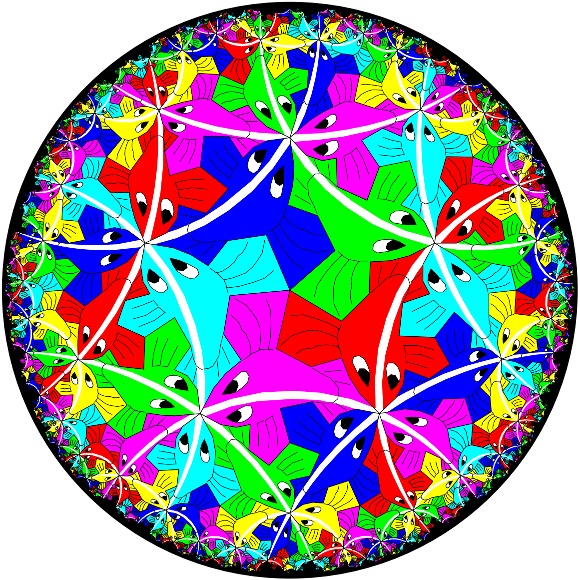


A 5-colored $(5, 5, 3)$ *Circle Limit III* pattern.

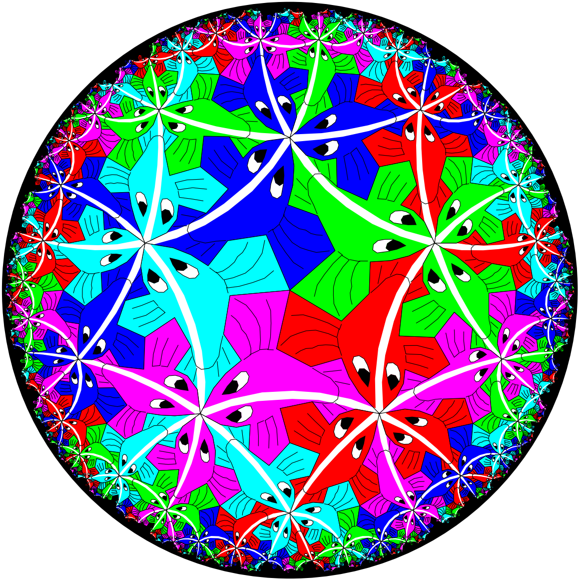


A $(5,3,3)$ *Circle Limit III* pattern

Needs 6 colors to maintain colors on backbones.



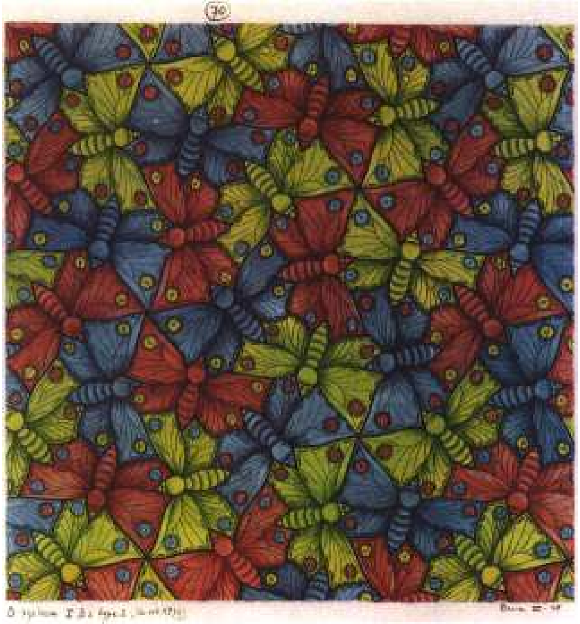
A 5-coloring of the $(5, 3, 3)$ pattern
Colors on backbone lines alternate



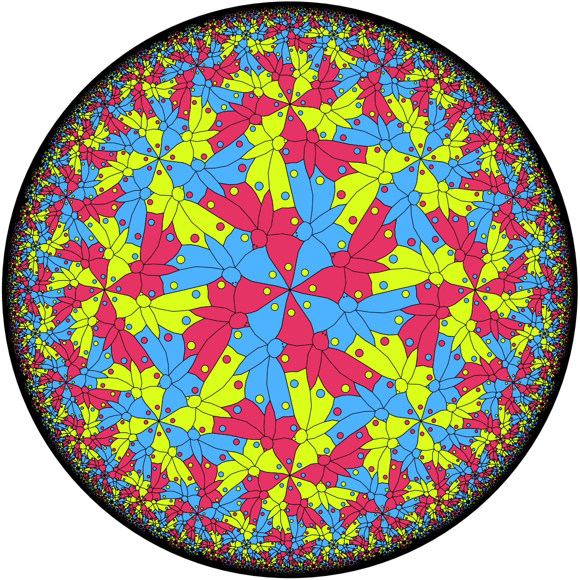
Color Symmetries of Butterfly Patterns

- The patterns in the butterfly family are based on the $\{p, q\}$ tessellations and there is no restriction except that they must be greater than or equal to 3.
- For these patterns, p is the number of butterflies meeting at left front wingtips, and q is the number of butterflies meeting at their left rear wings.
- Escher only created one pattern in this family, his Euclidean Notebook Drawing 70, which based on the $\{6, 3\}$ tessellation.
- Following Escher, we imposed an additional restriction that all circles on the butterfly wings around a p-fold meeting point of left wingtips be a color that is different from the butterflies meeting there.

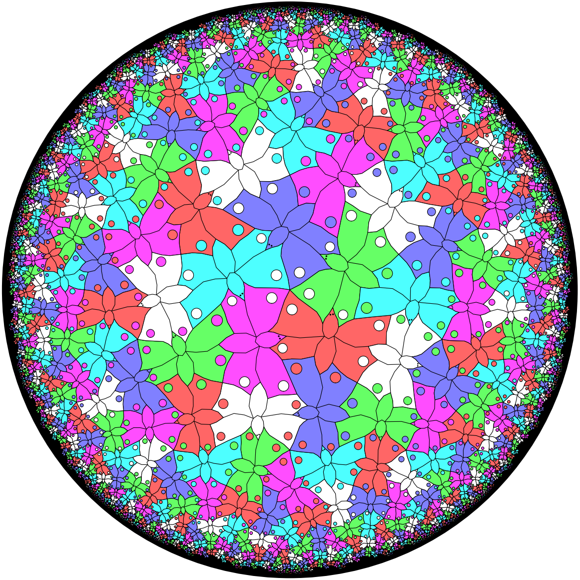
Escher's 3-colored butterfly pattern
Notebook Drawing Number 70



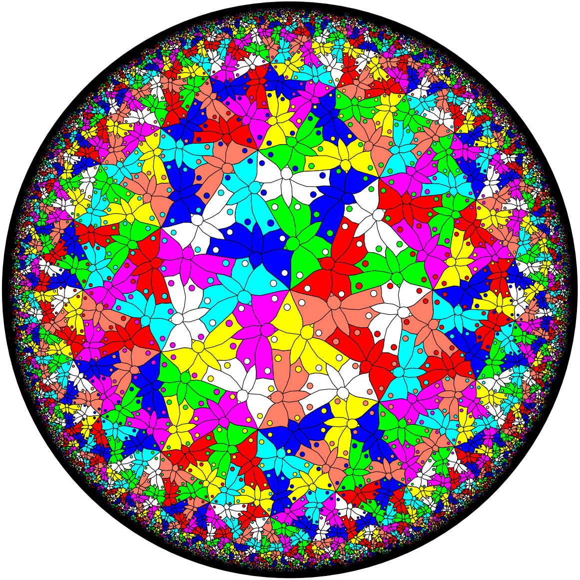
A 3-colored butterfly pattern
Based on the $\{8, 3\}$ tessellation Butterfly Pattern



A 6-colored butterfly pattern
Based on the $\{5, 4\}$ tessellation



An 8-colored butterfly pattern
Based on the $\{7, 3\}$ tessellation



Future Work

- Automatically generate the colors so that the pattern is symmetrically colored. Currently this must be done manually for each pattern in a family.
- Extend such a generation algorithm so that it can handle additional restrictions, such as using the same color for fish along each backbone line of a *Circle Limit III* pattern, or using a different color for the wing circles on a butterfly pattern.
- Make more patterns!

Thank You

Nat, Ergun

And the other organizers at DePaul University