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Enumerations of Hyperbolic Truchet Tiles

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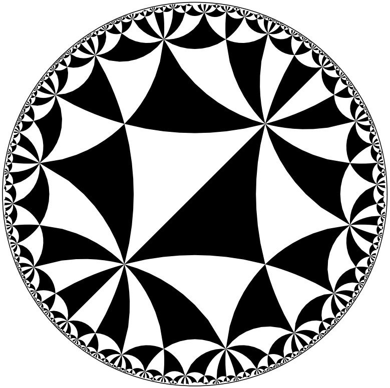


Outline

- A brief history of Truchet tilings
- Examples of Truchet tilings
- Hyperbolic geometry and regular tessellations
- Hyperbolic Truchet tilings
- Random hyperbolic Truchet tilings
- Truchet tiles with multiple triangles per p-gon
- Truchet tilings with other motifs.
- Future research

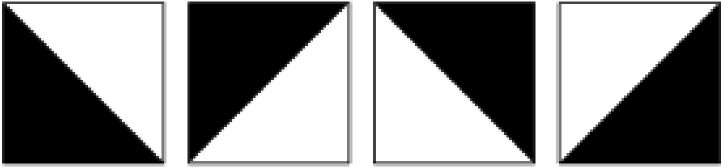
Sébastian Truchet

Brief History of Truchet Tilings

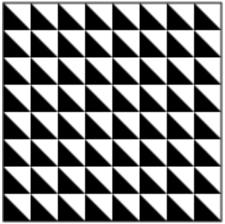
- Sébastian Truchet was born in Lyon, France 1657, died 1729.
- Interests: mathematics, hydraulics, graphics, and typography.
- Also invented sundials, weapons, and methods for transporting large trees within the Versailles gardens.
- In 1704 he published "Memoir sur les Combinaisons" in *Memoires de l'Académie Royale des Sciences* enumerating possible pairs of juxtaposed squares divided by a diagonal into a black and a white triangle. The "Memoir" contained 7 plates, the first four showed 24 simple pattern, labeled A to Z and & (no J, K, W); the last three showed six more complicated patterns.
- In 1942 M.C. Escher enumerated 2×2 tiles of squares containing simple motifs, thus extending Truchet's idea for 2×1 tiles.
- In 1987 Truchet's "Memoir" was translated in English by Pauline Bouchard with comments and "circular arc" tiles by Cyril Smith in *Leonardo*, igniting renewed interest in these tilings.

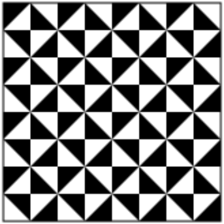
Examples of Truchet Tilings

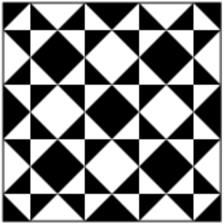
- Truchet triangle tilings
- Based on a square divided in two into a black and white triangle — 4 orientations.
- Either repeating patterns or random patterns.



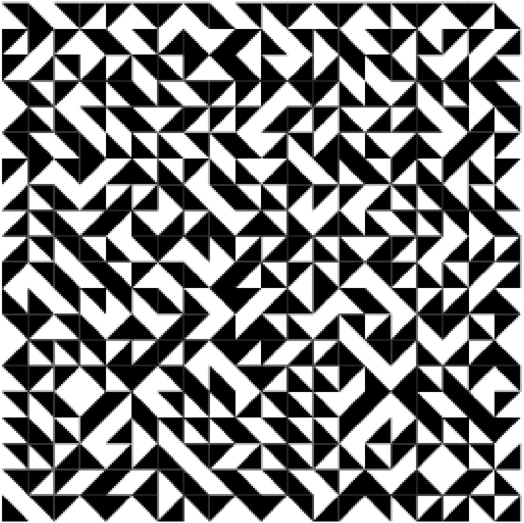
Regular Truchet Tilings







A Random Truchet Tiling



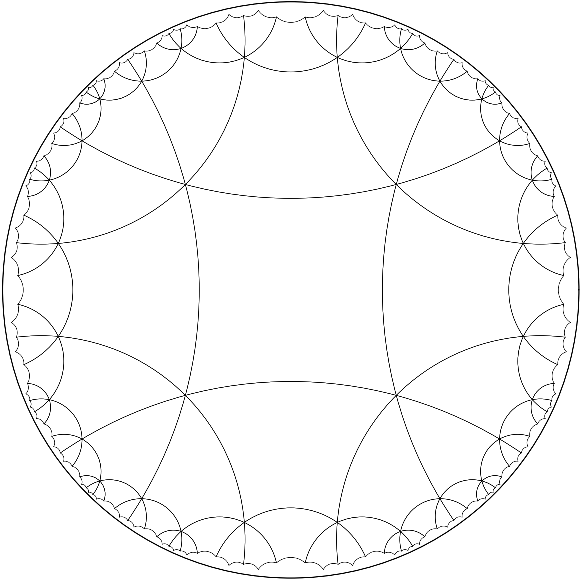
Hyperbolic Geometry and Regular Tessellations

- In 1901, David Hilbert proved that, unlike the sphere, there was no isometric (distance-preserving) embedding of the hyperbolic plane into ordinary Euclidean 3-space.
- Thus we must use *models* of hyperbolic geometry in which Euclidean objects have hyperbolic meaning, and which must distort distance.
- One such model is the *Poincaré disk model*. The hyperbolic points in this model are represented by interior point of a Euclidean circle — the *bounding circle*. The hyperbolic lines are represented by (internal) circular arcs that are perpendicular to the bounding circle (with diameters as special cases).
- This model is appealing to artests since (1) angles have their Euclidean measure (i.e. it is conformal), so that motifs of a repeating pattern retain their approximate shape as they get smaller toward the edge of the bounding circle, and (2) it can display an entire pattern in a finite area.

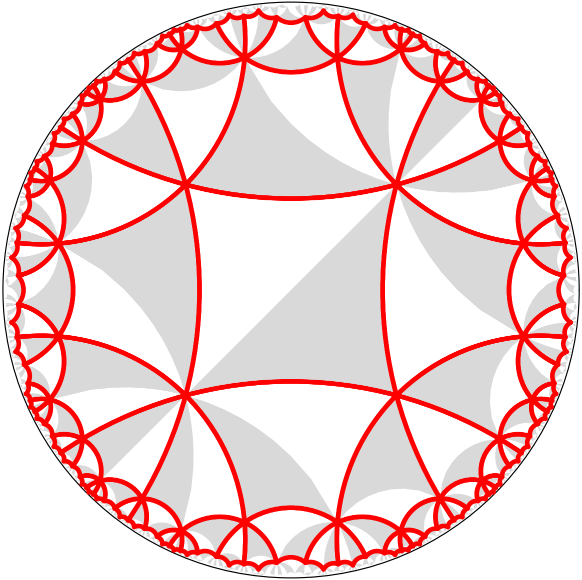
Repeating Patterns and Regular Tessellations

- A *repeating pattern* in any of the 3 "classical geometries" (Euclidean, spherical, and hyperbolic geometry) is composed of congruent copies of a basic subpattern or *motif*.
- The *regular tessellation*, $\{p, q\}$, is an important kind of repeating pattern composed of regular p-sided polygons meeting q at a vertex.
- If $(p-2)(q-2) < 4$, $\{p, q\}$ is a spherical tessellation (assuming $p > 2$ and $q > 2$ to avoid special cases).
- If $(p-2)(q-2) = 4$, $\{p, q\}$ is a Euclidean tessellation.
- If $(p-2)(q-2) > 4$, $\{p, q\}$ is a hyperbolic tessellation. The next slide shows the $\{6, 4\}$ tessellation.
- Escher based his 4 "Circle Limit" patterns, and many of his spherical and Euclidean patterns on regular tessellations.

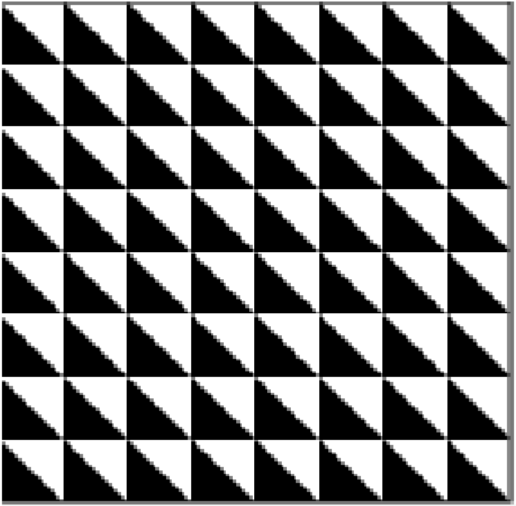
The Regular Tessellation $\{4, 6\}$ Underlying the Title Slide Image



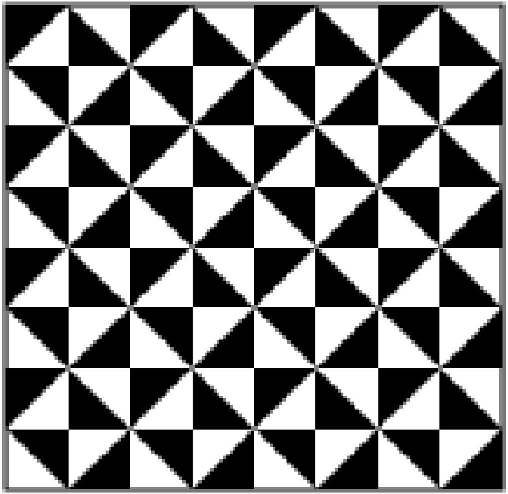
The tessellation $\{4, 6\}$ superimposed on the title slide pattern



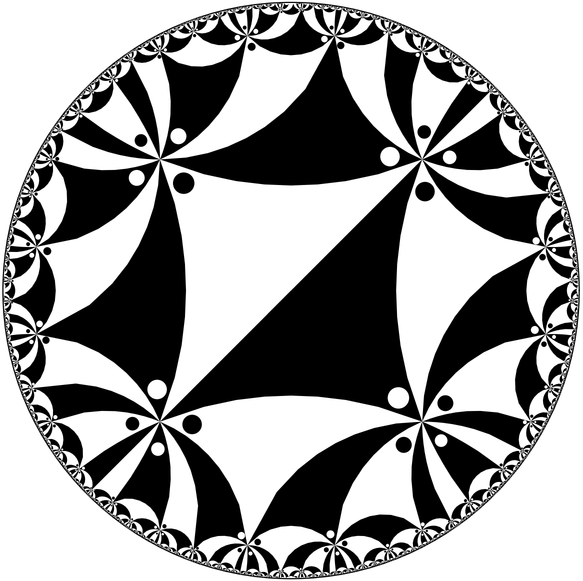
Truchet's "translation" tiling A.



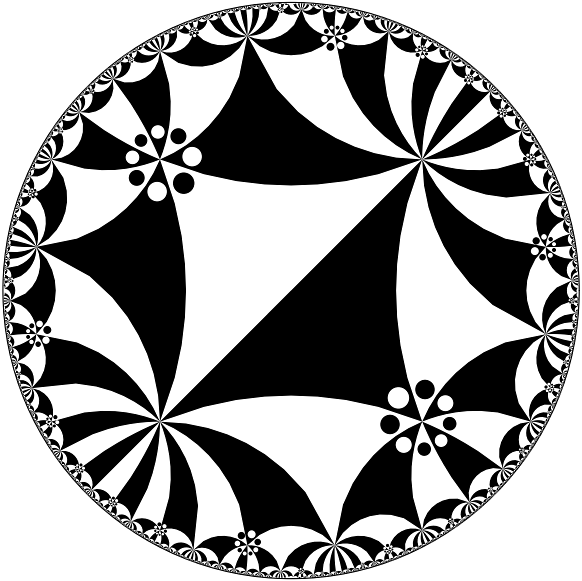
Truchet's "rotation" tiling D.



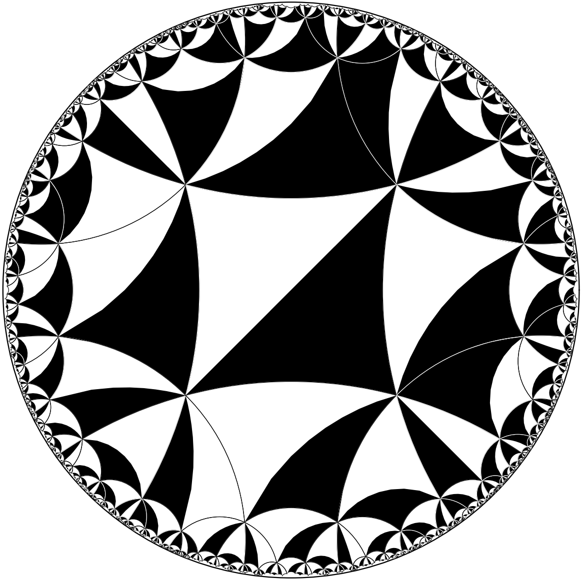
A hyperbolic "translation" Truchet tiling based on the $\{4, 8\}$ tessellation.



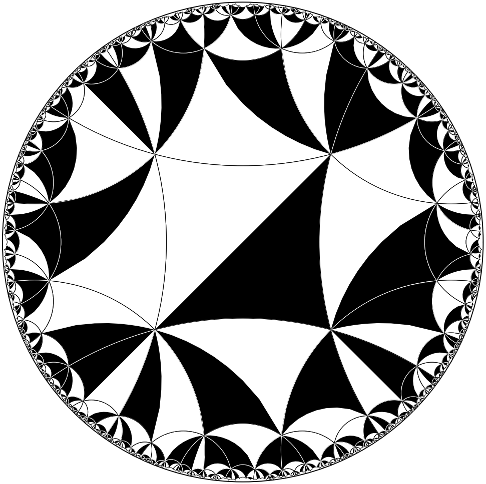
A hyperbolic "rotation" Truchet tiling based on the $\{4, 8\}$ tessellation.



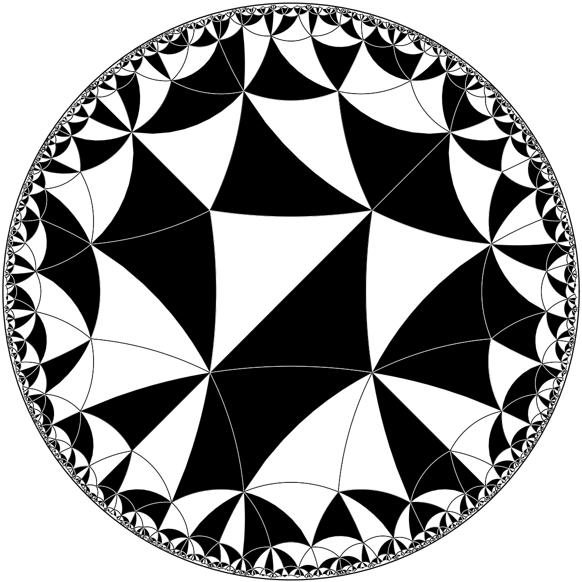
A Non-Regular Hyperbolic Truchet Tiling
(based on the $\{4,5\}$ tessellation)



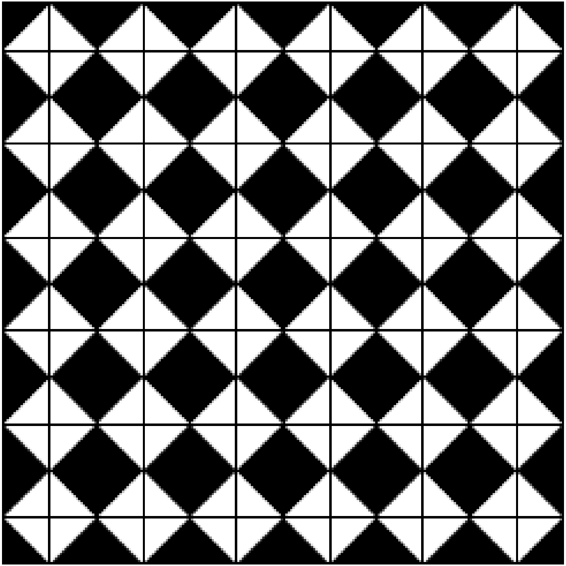
Random Hyperbolic Truchet Tilings
(One based on the $\{4, 6\}$ tessellation)



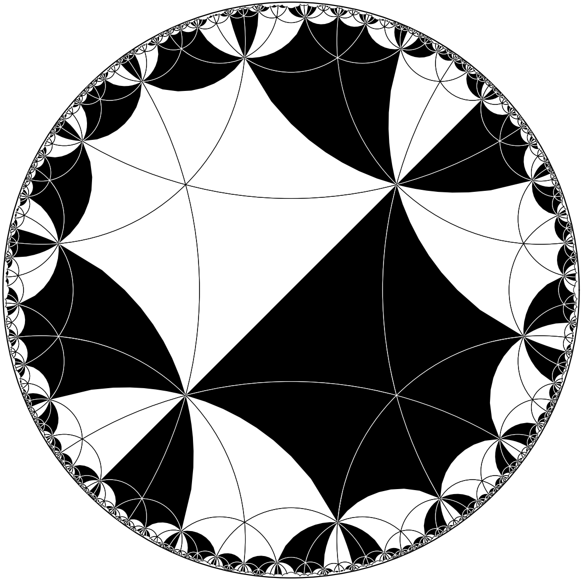
Another Random Hyperbolic Truchet Tiling
(based on the $\{4, 5\}$ tessellation)



Truchet's pattern F, which does not adhere to the map-coloring principle



A hyperbolic Truchet pattern corresponding to Truchet's pattern F (based on the $\{4, 6\}$ tessellation)



Truchet Tiles with Multiple Triangles per p-gon

- Truchet considered 2×1 rectangles composed of two squares, which easily tile the Euclidean plane.
- Problem: it is more difficult to tile the hyperbolic plane by "rectangles" — quadrilaterals with congruent opposite sides.
- Solution: the p-gons of $\{p, q\}$ tile the hyperbolic plane.
- We divide the p-gons of a $\{p, q\}$ divided into black and white $\frac{\pi}{p}$ - $\frac{\pi}{q}$ - $\frac{\pi}{2}$ *basic triangles* by radii and apothems.
- To satisfy the map-coloring principle, the basic triangles should alternate black and white, giving only two possible tilings.
- If we don't require map-coloring, there are $N_2(2p)$ possible ways to fill a p-gon with black and white basic triangles, where $N_k(n)$ is the number of n-bead necklaces using beads of k colors:

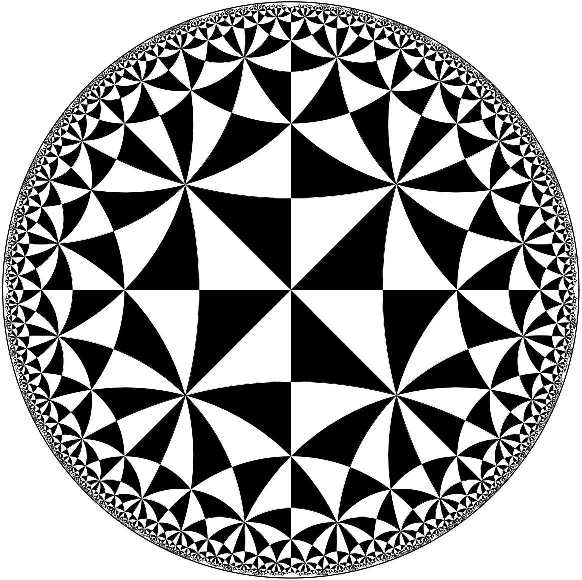
$$N_k(n) = \frac{1}{n} \sum_{d|n} \varphi(d) k^{n/k}$$

where $\varphi(d)$ is Euler's totient function.

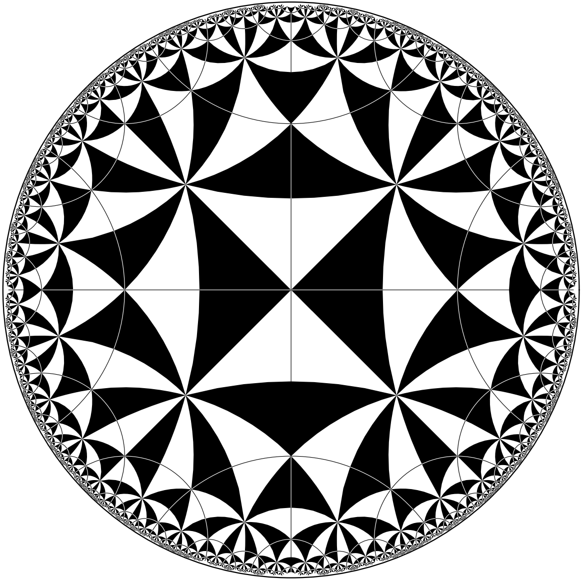
Truchet Tiles with Multiple Triangles per p-gon (continued)

- If we consider our "necklaces" to be equivalent by reflection across a diameter or apothem of the p-gon, there are fewer possibilities, given by $B_k(n)$ the number of n-bead "bracelets" made with k colors of beads. The value of $B_k(n)$ is $1/2$ that of $N_k(n)$ with added adjustment terms that depend on the parity of n.
- It seems to be a difficult problem to enumerate all the ways such a p-gon pattern of triangles could be extended across each of its edges, though an upper bound would be $(2p)^p N_2(2p)$

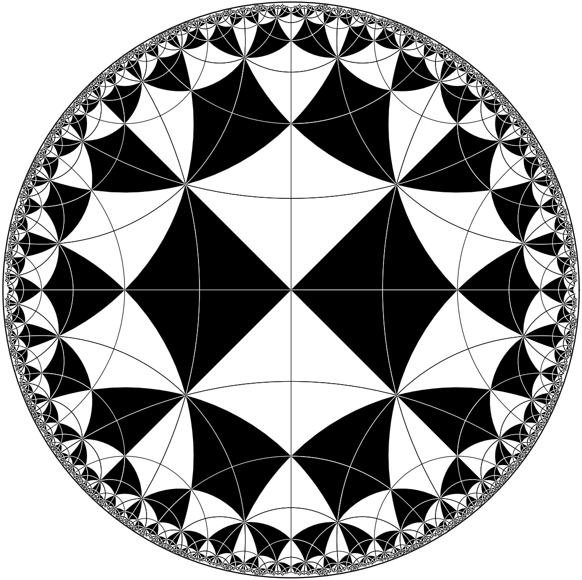
A pattern generated by alternate black and white triangles in a 4-gon, a p-gon analog of Truchet's pattern A.



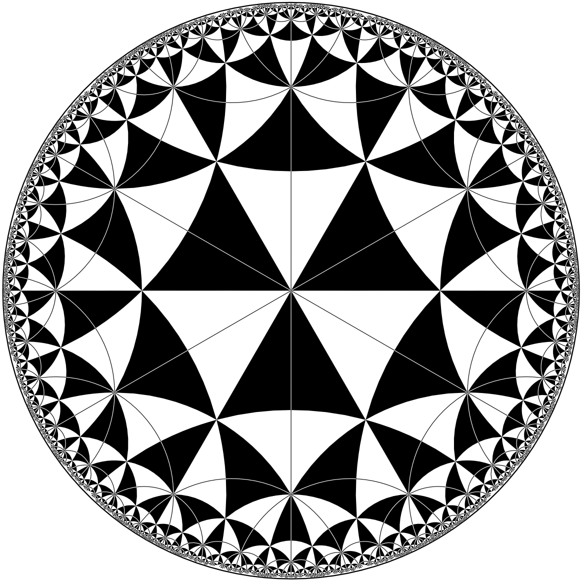
A pattern generated by paired black and white triangles in a 4-gon, analogous to Truchet's pattern E.



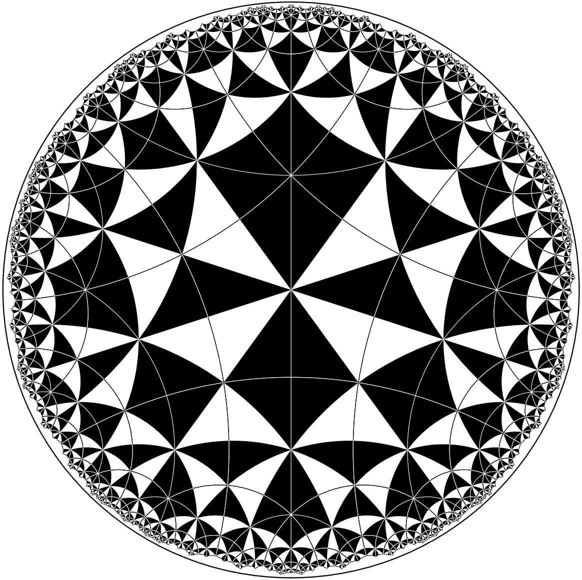
Another pattern generated by paired black and white triangles in a 4-gon, analogous to Truchet's pattern F.



A pattern based on the $\{6,4\}$ tessellation, similar to Truchet's pattern E.

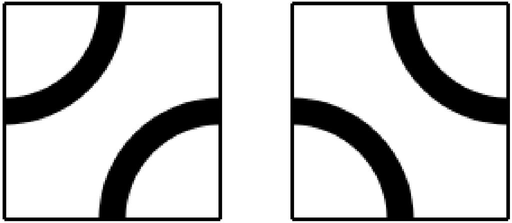


A Truchet-like pattern based on the $\{5,4\}$ tessellation.

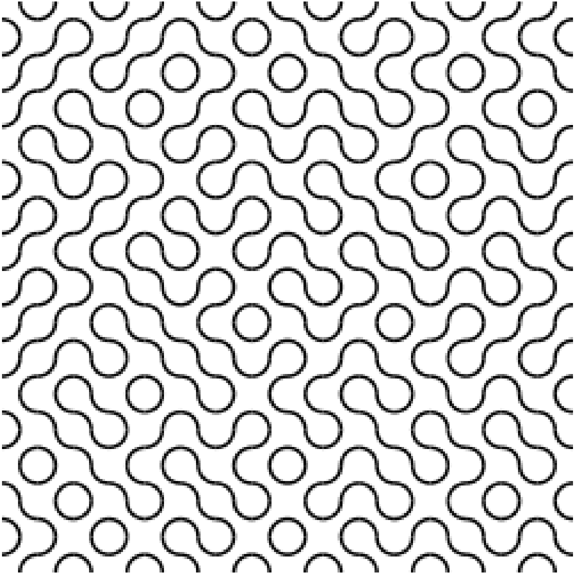


Truchet Tilings with other Motifs — Circular Arcs

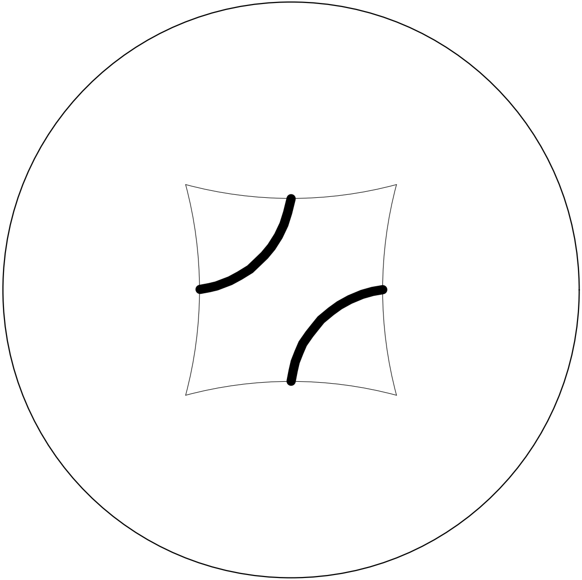
- Based on a square with circular arcs connecting adjacent sides — 2 orientations.
- Either repeating patterns or random patterns.



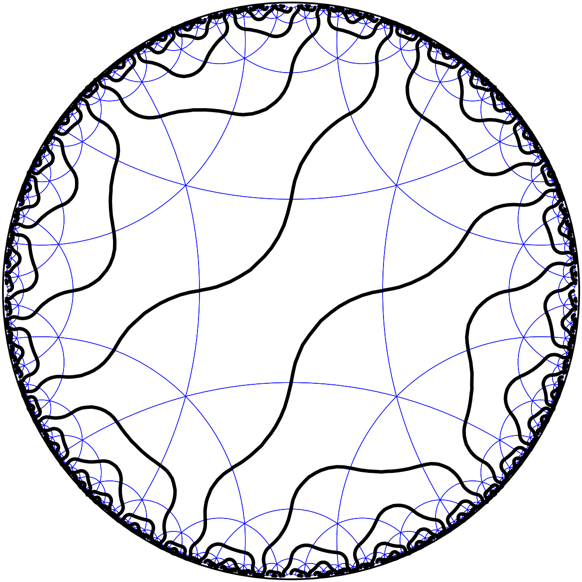
A Random Truchet Arc Tiling
(based on the Euclidean $\{4,4\}$ tessellation by squares)



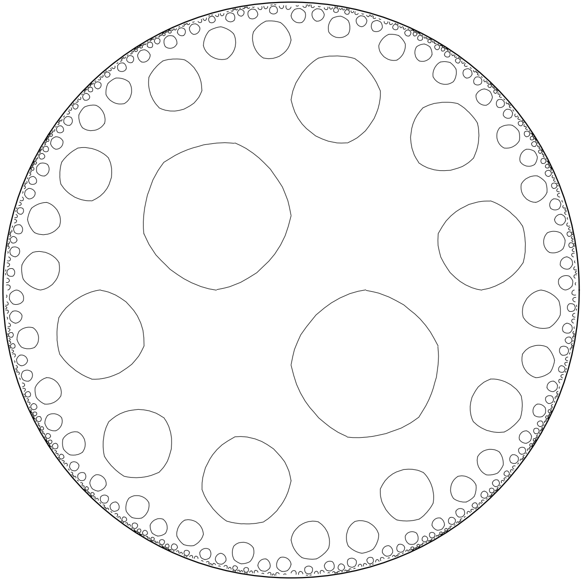
A Hyperbolic Arc Tile (based on the $\{4, 6\}$ tessellation)



A Hyperbolic Arc Pattern (based on the $\{4, 6\}$ tessellation)



A Hyperbolic Arc Pattern of Circles (based on the $\{4, 5\}$ tessellation)



Counting Circular Arc Patterns Based on p-gons

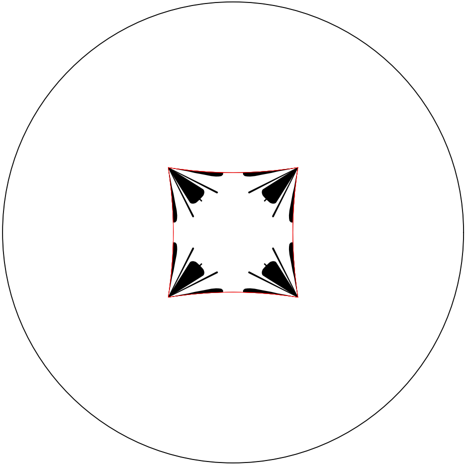
- We generalize Truchet arc patterns from Euclidean squares to p-gons by connecting the midpoints of the edges of a $2n$-gon ($p = 2n$ must be even).
- The number of possible $2n$-gon tiles is the same as the number of ways to connect $2n$ points on a circle with non-intersecting chords. It is the Catalan number:

$$C(n) = 2n!/[n!(n+1)!]$$

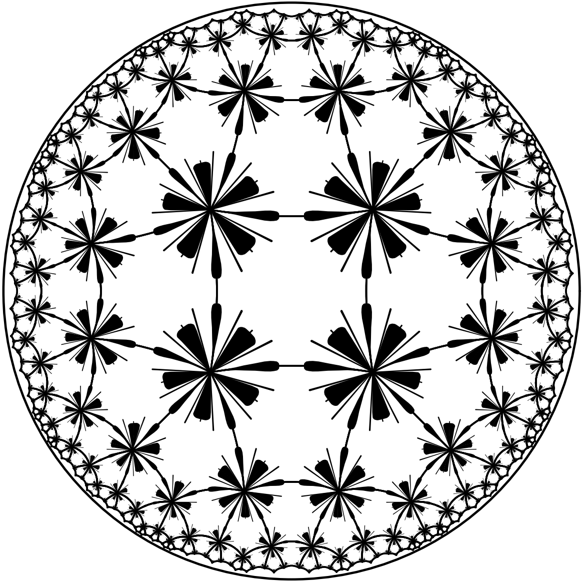
- As is the case with the triangle-decorated p-gons, the number of possible patterns is bounded above by $(2n)^{2n}C(n)$, but again, it seems difficult to get an exact count.

Truchet Tilings with other Motifs — “Wasps”

Four wasps at the corners of a square
— wasp motif designed by Pierre Simon Fournier (mid 1700’s)



A Truchet Pattern of Wasps
(based on the $\{4, 5\}$ tessellation)



Future Work

- Investigate colored hyperbolic Truchet triangle patterns.
- Implement a hyperbolic circular arc tool in the program.
- Investigate more hyperbolic Truchet arc patterns with more arcs per p-gon.

Thank You!

Nat, ISAMA, the organizers at Columbia College