# Transforming "Circle Limit III" Patterns - First Steps

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**Abstract**

M.C. Escher's *Circle Limit III*, a repeating pattern of fish, is often considered to be the most appealing of his four "Circle Limit" prints. The fish meet four at each right fin tip, three at each left fin tip, and three nose-to-nose. The concept of this pattern has been generalized to allow any number of fish at those meeting points. But currently there is no computer program to draw such general patterns. In this paper we make progress toward that goal and show some new patterns in this *Circle Limit III* family.

## Introduction

Figure 1 below shows a computer rendition of the Dutch artist M.C. Escher's pattern *Circle Limit III* which he realized in the Poincaré disk model of hyperbolic geometry. Figure 2 shows another pattern from the *Circle Limit III* family. In the next section, we review a bit of hyperbolic geometry. Then we describe the

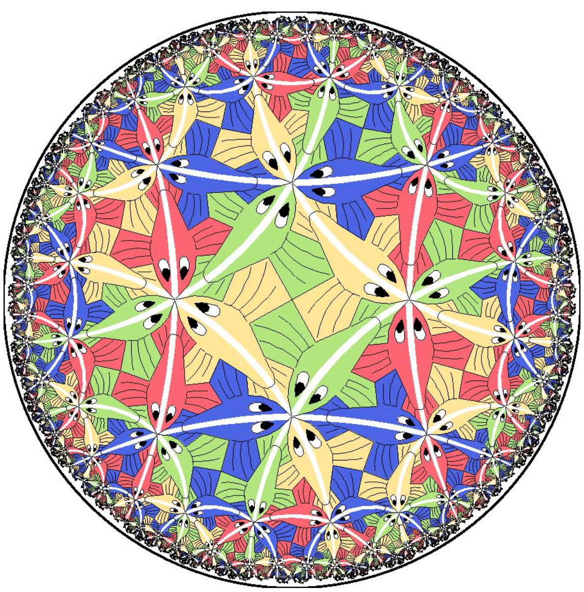

**Figure 1:** A rendition of Escher's *Circle Limit III*.

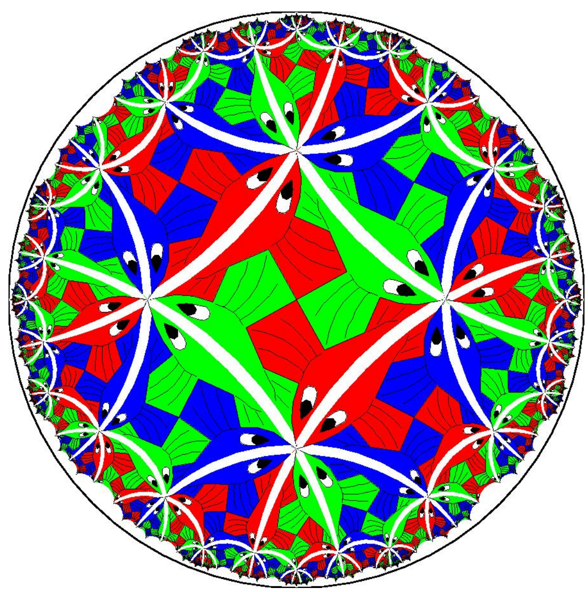

**Figure 2:** A $(4, 4, 3)$ pattern in the *Circle Limit III* family.

3-parameter family of *Circle Limit III* patterns. Next, we show how to create patterns from two special subfamilies. Finally, we discuss a possible attack on the problem of drawing general patterns.

## The Family of *Circle Limit III* Patterns

In a 2006 paper [Dun06], I introduced the concept of a 3-parameter family of *Circle Limit III* patterns indexed the numbers $p, q$, and $r$ of fish meeting at right fin tips, left fin tips, and noses respectively. Such a pattern was denoted by the triple $(p, q, r)$, where $p, q$, and $r$ should all be greater than or equal to 3. So the patterns of Figures 1 and 2 would be denoted $(4, 3, 3)$ and $(4, 4, 3)$ respectively. Following Escher, we place some restrictions on the patterns in this family. First, $r$ should be odd so that the fish swim head-to-tail. Second, right fin tips should be at the center of the bounding circle. Another condition is that colors of the fish should obey the map-coloring principle: fish that share an edge should be different colors. The fish should also be colored symmetrically and fish along the same "backbone line" should be the same color. Figures 3 and 4 show $(3, 4, 3)$ and $(5, 3, 3)$ patterns. Note the differences between Figure 3 and *Circle Limit III*. In particular, requiring that a right fin tip be at the center allows us to distinguish between $(p, q, r)$ and $(q, p, r)$ when $p \neq q$.

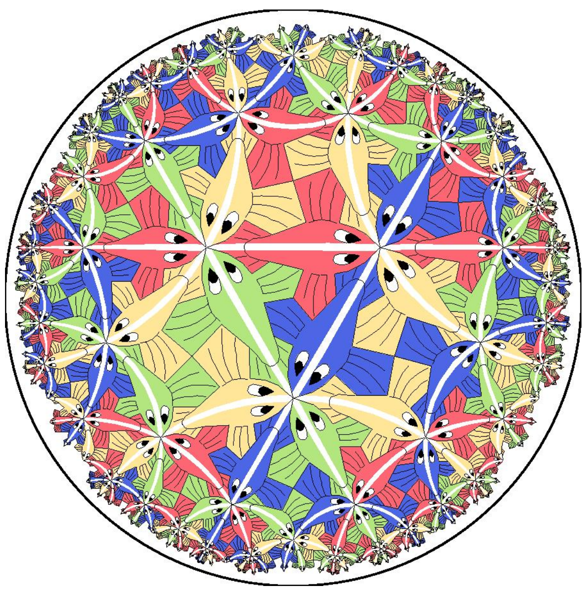

**Figure 3:** A $(3, 4, 3)$ fish pattern.

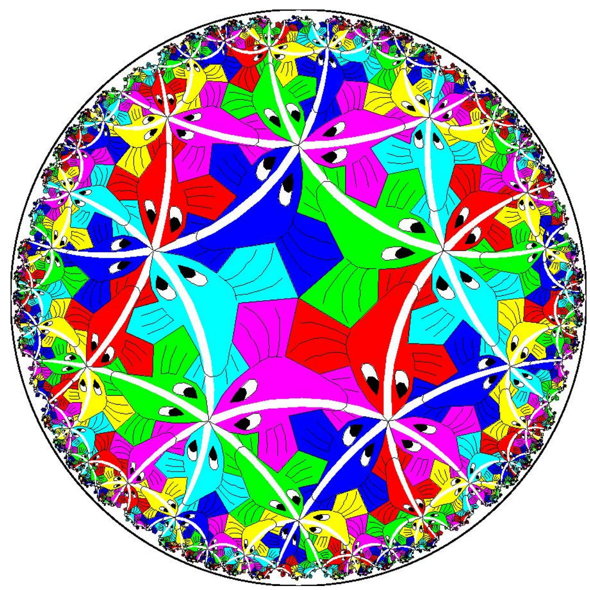

**Figure 4:** A $(5, 3, 3)$ fish pattern.

## Hyperbolic Geometry

Escher's "Circle Limit" patterns realized his goal of representing an infinite repeating pattern in a finite area. He used Euclidean constructions to make his patterns, but they could also be interpreted as repeating patterns in the Poincaré disk model of hyperbolic geometry. We must rely on models of hyperbolic geometry in which Euclidean constructs have interpretations as hyperbolic points and lines, since there is no smooth embedding of the hyperbolic plane into Euclidean 3-space [Hil01]. The hyperbolic points in this model are just the (Euclidean) points within a Euclidean bounding circle. Hyperbolic lines are represented by circular arcs orthogonal to the bounding circle (including diameters). For example, the backbone lines of the fish lie along hyperbolic lines in Figure 2. Also, equal hyperbolic distances correspond to ever smaller Euclidean distances toward the edge of the disk. For example, all the fish in Figure 1 are hyperbolically the same size, as are all the fish in Figure 2.

One might guess that the backbone arcs of the fish in Figure 1 (*Circle Limit III*) are also hyperbolic lines, but this is not the case. They are *equidistant curves* in hyperbolic geometry: curves at a constant hyperbolic distance from the hyperbolic line with the same endpoints on the bounding circle. For each hyperbolic line and a given distance, there are two equidistant curves, called *branches*, one each side of the line at that distance from it. In the Poincaré disk model, those two branches are represented by circular arcs making the same (non-right) angle with the bounding circle and having the same endpoints as the corresponding hyperbolic line. Equidistant curves are the hyperbolic analog of small circles in spherical geometry: a small circle of latitude in the northern hemisphere is equidistant from the equator (a great circle or "line" in spherical geometry), and has a corresponding small circle of latitude in the southern hemisphere the same distance from the equator. For more on hyperbolic geometry see [Gre08].

## Creating Patterns from the $p$ = $q$ Subfamily

When $p = q$, the backbones lie along hyperbolic lines, not equidistant curves, so that the fish are symmetric by reflection across their backbones. Thus, in this case we only need to usd half a fish as a motif since we can get the other half by reflection. Figure 2 shows the $(4, 4, 3)$ pattern, an example from this subfamily.

To transform one half-fish motif to another in this subfamily, we can proceed as follows. A half-fish motif (or more accurately, pieces of it) with right fin at the center of the bounding circle and backbone line perpendicular to the positive $x$-axis fits into a hyperbolic isosceles triangle symmetric across the $x$-axis, with apex at the origin. When such an isosceles triangle is transformed into the Klein disk model of hyperbolic geometry, it become a Euclidean isosceles triangle, since the hyperbolic lines in the Klein model are represented by Euclidean chords of the bounding circle (a chord in the Klein model corresponds to the orthogonal circular arc in the Poincaré model with the same endpoints as the chord) [Gre08]. Any one of these Euclidean isosceles triangles can be mapped onto another by a differential scaling — having different $x$- and $y$-scale factors. So the transformation process is: map the original half-fish from the Poincaré to the Klein model, apply the differential scaling to get the half-fish into the new Euclidean isosceles triangle, and finally map the new isosceles triangle and its half-fish back to the Poincaré model. Figures 5 and 6 show $(5, 5, 3)$ and $(3, 3, 5)$ patterns that were created this way.

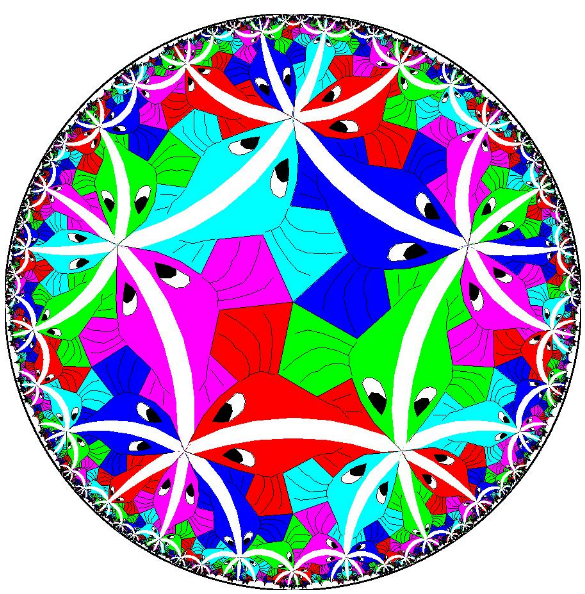

**Figure 5:** A $(5, 5, 3)$ pattern.

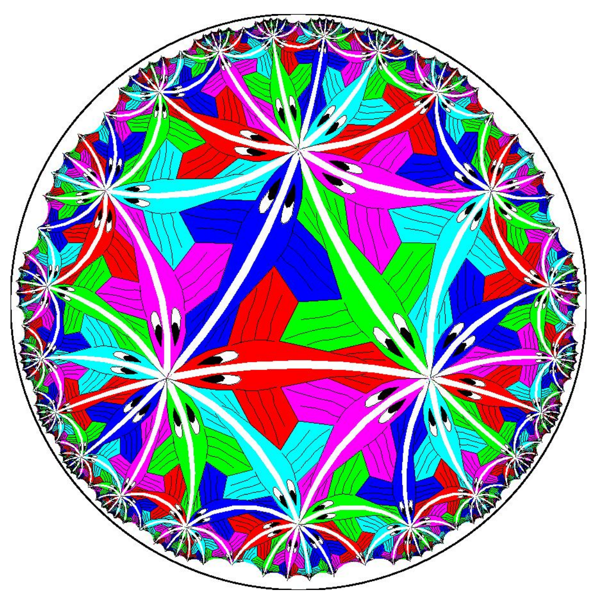

**Figure 6:** A $(3, 3, 5)$ pattern.

## Creating Patterns from the $p = r = 3$ Subfamily

When $p = r = 3$, the backbone lines nearest the origin in the Poincaré model form a Euclidean equilateral triangle, as can be seen in Figure 3 above in the case of $(3, 4, 3)$ and Figure 7 below which shows $(3, 5, 3)$. This fact can be exploited to generate new patterns using the Euclidean scaling mentioned above — but only for right halves of the fish. Figure 8 shows the right halves of the fish of Figure 7.

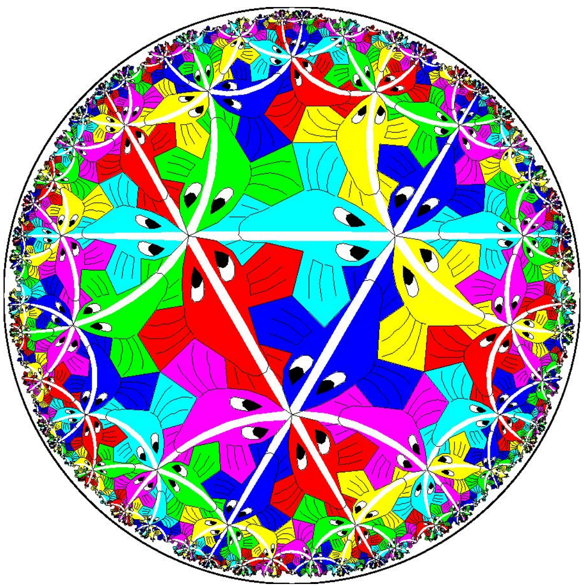

**Figure 7:** A $(3, 5, 3)$ pattern.

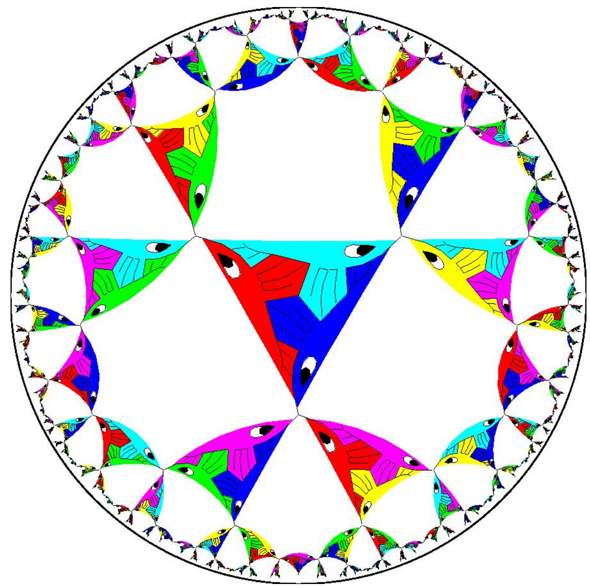

**Figure 8:** The right sides of the fish in the $(3, 5, 3)$ pattern.

Figure 9 below shows the right sides of the fish in the $(3, 4, 3)$ pattern of Figure 3. Figure 10 shows the right sides of fish in the $(3, 6, 3)$ pattern. Actually, the "seed" right fish-half was taken from the $(3, 4, 3)$ pattern of Figure 3 and it was scaled to form the right fish-halves of Figures 8 and 10. Close examination reveals that the fish halves in Figure 8 are slightly different than the corresponding ones in Figure 7.

## A Possible Solution for the General Case

We have seen in cases of the two subfamilies of the previous sections that transforming a hyperbolic motif so that it fits inside a Euclidean isosceles triangle allows us to transform that motif so that it can form any pattern in that subfamily. But in both cases, only half a fish was transformed. Thus it would seem that to transform a fish from a $(p, q, r)$ pattern to a $(p', q', r')$ would require separate processes to transform the left half and the right half. One possible idea would be to find a model of hyperbolic geometry the right "distance" in between the Poincaré model and the Klein model so that the backbone line (equidistant curve) would "flatten out" to a Euclidean line. This would handle the right half-fish. To take care of the left half-fish, we could hyperbolically translate its fin tip to the origin, find the correct "in between" hyperbolic model (probably different than for the right half), apply the transformation, then hyperbolically translate back.

## Conclusions and Future Work

For two subfamilies of $(p, q, r)$ *Circle Limit III* patterns, we have shown how one "seed" half-fish motif can be transformed to create any pattern in that subfamily. In the case $p = q$, the backbone lines are hyperbolic

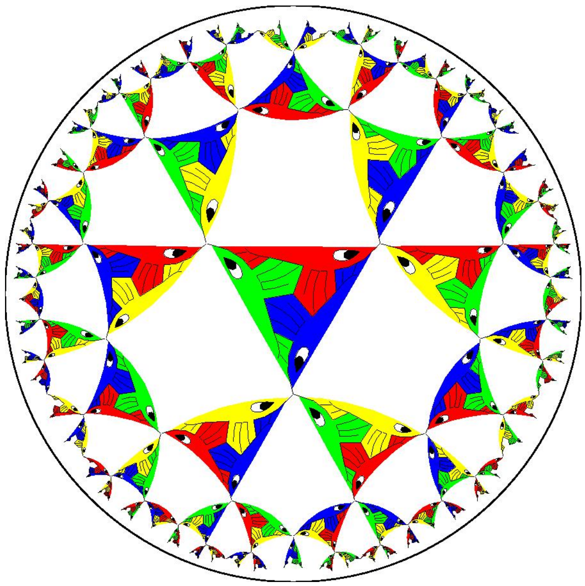

**Figure 9:** A $(3,4,3)$ pattern of right fish-halves.

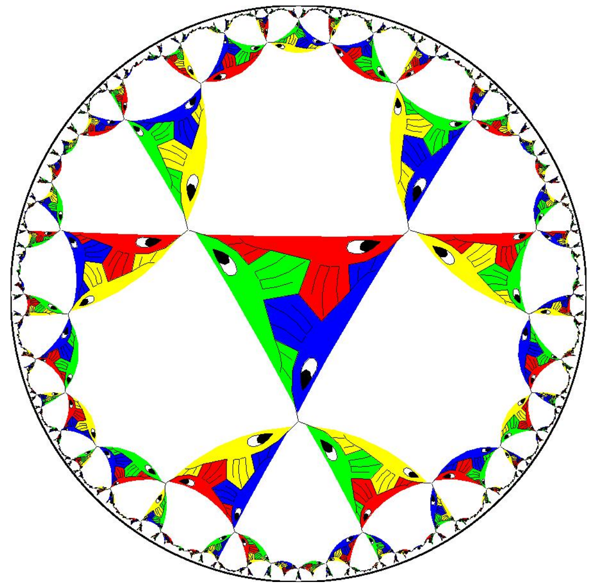

**Figure 10:** A $(3,6,3)$ pattern of right fish-halves.

lines, so that we can get the other half-fish by reflecting across those lines, and thus obtain the entire pattern. We have also indicated a possible direction of attack for the general case of transforming from and $(p, q, r)$ pattern to any other such pattern.

Another seemingly difficult problem is to automate the process of coloring a $(p, q, r)$ pattern so that it has the same color along any line of fish and adheres to the map-coloring principle that adjacent fish have different colors. I determined the colorings of all the patterns above "by hand", except for Escher's *Circle Limit III* pattern (and the related patterns of Figures 3 and 9). Although it may be possible to program symmetric colorings of any repeating pattern, the requirement that fish along a backbone line be the same color adds an extra degree of difficulty to coloring $(p, q, r)$ patterns.

## Acknowledgments

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## References

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