Patterns on Triply Periodic Uniform Polyhedra

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Abstract

Artists have created patterns on closed polyhedra and on the hyperbolic plane, but no one to our knowledge has created patterns on triply periodic polyhedra. The goal of this paper is to exhibit a few such patterns and explain how they arose.

1. Introduction

In this paper we show patterns on triply periodic polyhedra, which are polyhedra that have translation symmetries in three independent directions in Euclidean 3-space. Figure 1 shows (a piece of) such a polyhedron decorated with angular fish and colored backbone lines. Each of the polyhedra we discuss is composed of

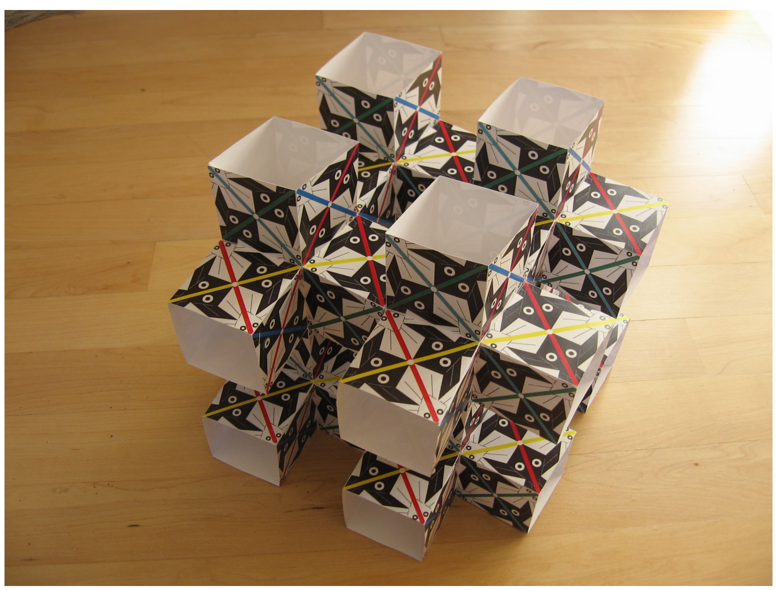

Figure 1: A piece of the $\{4, 6|4\}$ polyhedron decorated with angular fish.

copies of a regular polygon, either a square or an equilateral triangle. These polyhedra have negative curvature, and are related to regular tessellations of the hyperbolic plane. Similarly, the patterns we place on the polyhedra are related to patterns of the hyperbolic plane based on regular tessellations.

This work was inspired by Luecking's polyhedral approximation of one of Sherk's minimal surfaces shown at ISAMA 2011 [Luecking], and by Chuang, Jin, and Wei's beaded approximation to Schwarz's D and Schoen's G surfaces shown at the Art Exhibit of the 2012 Joint Mathematics Meeting [Chuang]. It turns out that the polyhedra we discuss are related to triply periodic minimal surfaces.

The Dutch artist M.C. Escher drew patterns on several closed polyhedra [Schattschneider04]. Later Doris Schattschneider and Wallace Walker designed non-convex rings of polyhedra, called Kaleidocycles, which could be rotated [Schattschneider05]. The purpose of this paper is to investigate patterns on polyhedra that could theoretically extend to infinity in all three directions. Thus we were naturally led to using triply periodic polyhedra.

We will begin with a discussion of regular tessellations and triply periodic polyhedra, explaining how they are related via minimal surfaces. This relationship also extends to patterns on the respective surfaces. Then we show two patterns on what is probably the simplest triply periodic polyhedron, followed by a pattern on a more complicated polyhedron. Finally, we indicate possibilities for other patterns on other surfaces.

2. Regular Tessellations and Triply Periodic Polyhedra

We use the Schläfli symbol $\{p, q\}$ to denote the regular tessellation formed by regular p-sided polygons or *p-gon* with q of them meeting at each vertex. If $(p-2)(q-2) > 4$, $\{p, q\}$ is a tessellation of the hyperbolic plane, otherwise it is Euclidean or spherical. Figure 2 shows the tessellation $\{4, 6\}$ in the Poincaré disk model of hyperbolic geometry. Figure 3 shows a pattern of angular fish based on that tessellation.

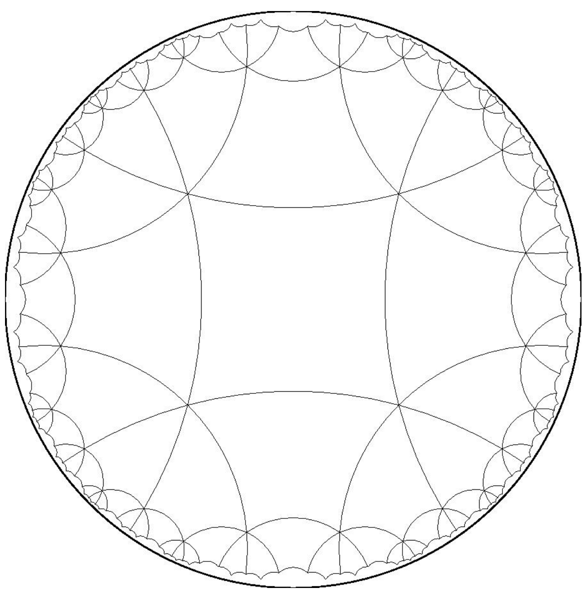

Figure 2: The $\{$4,6$\}$ tessellation

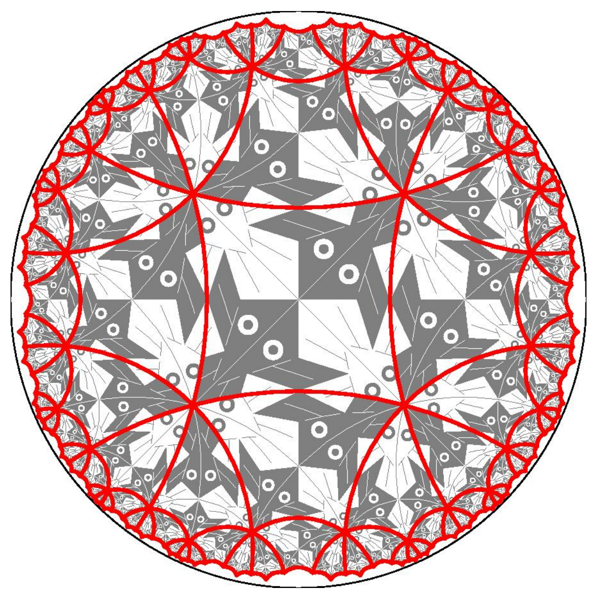

Figure 3: The $\{4, 6\}$ superimposed on a pattern of fish.

We will discuss *regular triply periodic polyhedra* which have a p-gon for each of its faces, has translation symmetries in three independent directions, and its symmetry group is transitive on vertices (i.e. it is uniform) at which q p-gons meet. The meaning of the Schläfli symbol $\{p, q\}$ can be extended to include these polyhedra. If the symmetry group is flag-transitive, then one obtains a more symmetric subfamily called *infinite skew polyhedron*. There are exactly three of them, which was discovered by John Petrie in 1926 [Wiki]. H.S.M. Coxeter designated them by the extended Schläfli $\{p, q|n\}$, indicating that there are q p-gons around each vertex and n-gonal holes [Coxeter73, Coxeter99]. Figure 1 shows $\{4, 6|4\}$ with a pattern of fish on it. The other possibilities are $\{6, 4|4\}$ and $\{6, 6|3\}$.

There is a 2-step connection between some regular triply periodic polyhedra $\{p, q\}$ and the corresponding regular tessellation $\{p, q\}$. First, some of the periodic polyhedra are approximations for (and closely related to) triply periodic minimal surfaces (TPMS). Alan Schoen has done extensive investigations into TPMS [Schoen]. Second, each smooth surface has a *universal covering surface*: a simply connected surface with a covering map onto the original surface which is a sphere, the Euclidean plane, or the hyperbolic plane. Since each TPMS has negative curvature, its universal covering surface is the hyperbolic plane. Similarly, we might call a hyperbolic pattern based on the tessellation $\{p, q\}$ the "universal covering pattern" for the related pattern on the polyhedron $\{p, q\}$.

3. A Pattern of Angels and Devils on the $\{4, 6|4\}$ Polyhedron

One can see by examining Figure 1 or Figure 5 below that the $\{4, 6|4\}$ polyhedron is based on the cubic lattice in 3-space. In fact it divides 3-space into two complementary congruent solids. Each of the solids is composed of "hub" cubes with "strut" cubes on each of its faces; each strut cube connects two hub cubes. M.C. Escher's hyperbolic print *Circle Limit IV* of angels and devils was based on the $\{6, 4\}$ tessellation. Figure 4 shows an angels and devils pattern based on the $\{4, 6\}$ tessellation. Figure 5 shows the corresponding pattern on the $\{4, 6|4\}$ polyhedron.

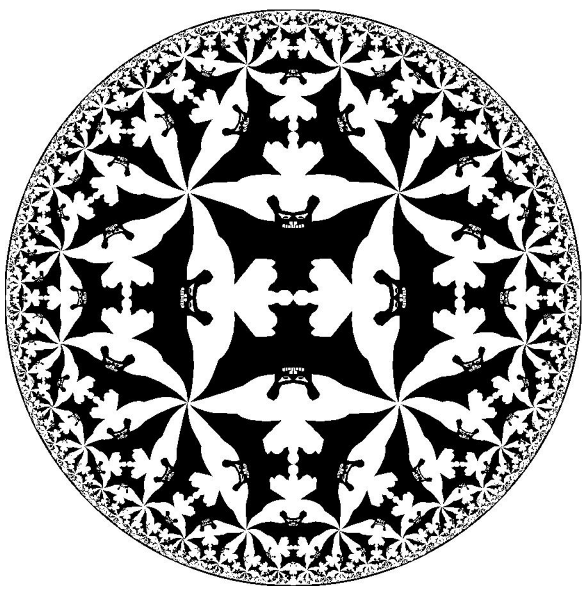

Figure 4: Angels and Devils on the $\{4, 6\}$ tessellation.

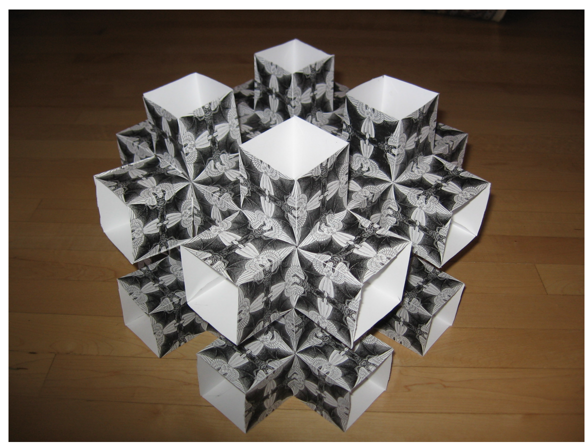

Figure 5: Angels and Devils on the $\{4, 6|4\}$ polyhedron.

The lines of bilateral symmetry of the angels and devils on the polyhedron form squares around the "waists" of the struts and "strut holes". If these squares are relaxed to circles (touching at the feet of the angels and devils), and the space between the circles is spanned by a minimal surface, one obtains Schwarz's P-surface, a TPMS. We will look at this more closely in the next section.

4. A Pattern of Fish on the $\{4, 6|4\}$ Polyhedron

Figure 2 above shows a pattern of angular fish in the hyperbolic plane — in fact it is the "universal covering pattern" of the fish on the $\{4, 6|4\}$ polyhedron shown in Figure 1. This pattern of fish is related to Escher's hyperbolic pattern *Circle Limit I* in the same way that Figure 5 is related to Escher's *Circle Limit IV* — Figures 2 and 5 are both based on the $\{4, 6\}$ tessellation, whereas Escher's hyperbolic patterns are both based on the $\{6, 4\}$ tessellation.

In the pattern of Figure 1, the backbones of the fish in a horizontal plane lie along parallel red lines or parallel yellow lines that are perpendicular to the red lines. Similarly for planes facing the lower left, fish backbones lie along green and cyan lines; and for planes facing the lower right, the backbones lie along blue and magenta lines. Also there are two kinds of vertices: those where red, blue, and green backbone lines intersect (at 60 degree angles), and those where cyan, magenta, and yellow lines intersect. Figure 6 shows a close-up view of the latter kind of vertex. Figure 7 shows the universal covering pattern of Figure 1, including the colored backbone lines. The backbone lines of the fish in Figures 1 and 6 are closely related to Schwarz's P-surface, since they all lie on that surface. Figure 8 shows the P-surface with the embedded lines [IndianaP]. One can see that they are the same as the backbone lines of Figure 1.

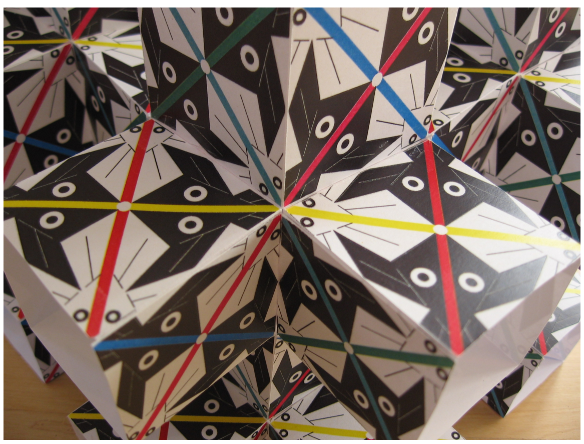

Figure 6: A vertex view of the fish pattern on the $\{4, 6|4\}$ polyhedron.

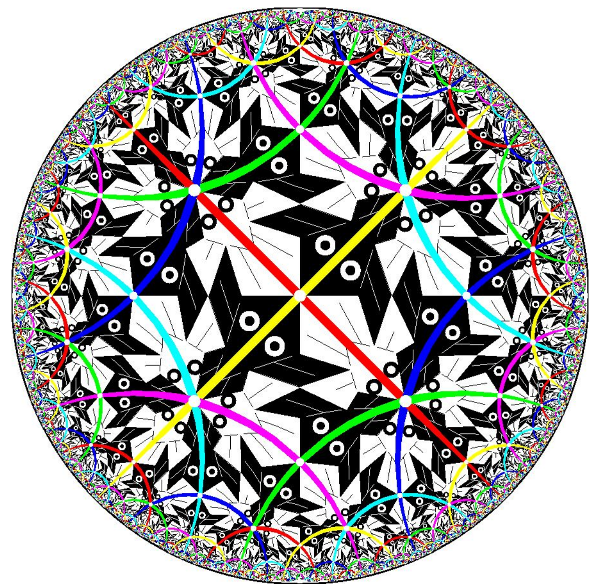

Figure 7: A hyperbolic pattern corresponding to Figure 1.

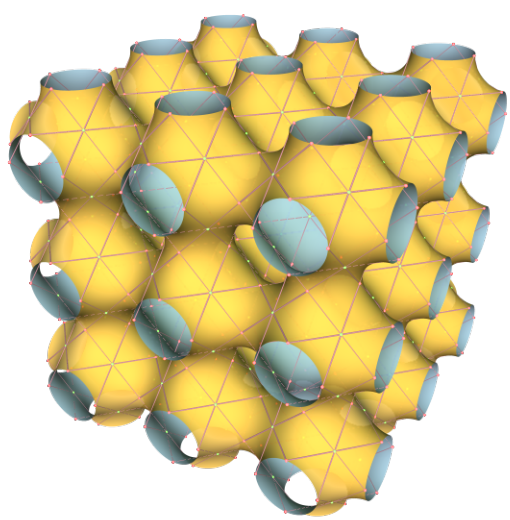

Figure 8: Schwarz's P-surface showing embedded lines.

5. A Pattern of Fish on a $\{3, 8\}$ Polyhedron

Figure 9 shows a triply periodic $\{3, 8\}$ polyhedron (Figure 5(2) of [Hyde]). The symmetry group is not flag-transitive, since there are two kinds of equilateral triangles. This surface can also be described in terms of hubs and struts, both of which are regular octahedra. A hub octahedron has strut octahedra on alternate faces, so that four hub triangles are covered by struts and four remain exposed. Each strut octahedron connects two hubs, and thus has two of its faces covered by hubs. The exposed hub triangles are different from the exposed strut triangles. Figure 10 shows a pattern of fish on the $\{3, 8\}$ polyhedron.

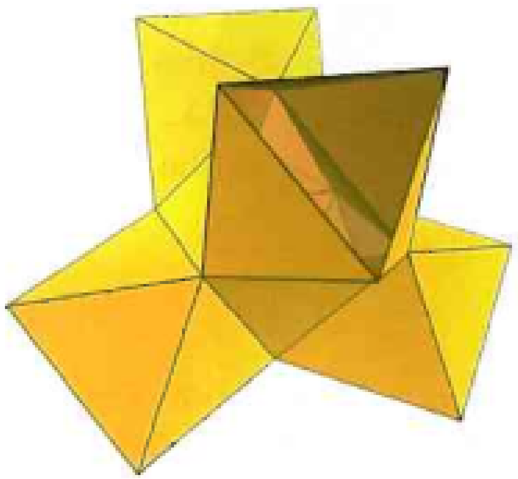

Figure 9: A triply periodic $\{3, 8\}$ polyhedron.

Figure 10: A pattern of fish on the $\{3, 8\}$ polyhedron.

The fish were inspired by those in Escher's *Circle Limit III* print. Figure 11 shows the corresponding $\{3, 8\}$ tessellation superimposed on a computer generated rendition of *Circle Limit III*.

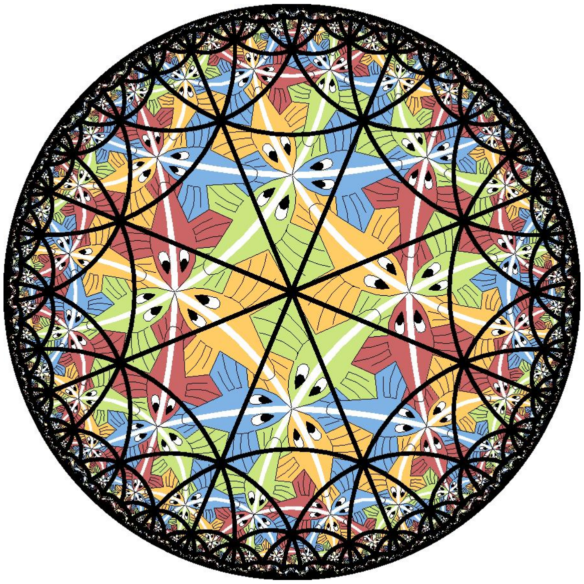

Figure 11: A rendition of *Circle Limit III* with the $\{3, 8\}$ tessellation superimposed.

The $\{3, 8\}$ polyhedron is an approximation to Schwarz's D-surface, a TPMS which also has embedded lines. The "D" is used to denote this surface since it has the shape of a thickened diamond lattice. The red, green, and yellow fish of Figure 9 swim along polygonal approximations to those embedded lines. The blue fish swim around hexagons which form the "waists" of the strut octahedra.

More than 30 years ago I tried piecing together small paper octagons with colored lines on them corresponding to the lines of fish in *Circle Limit III*. I was naturally led to a diamond lattice structure, and thought it would work, but was not sure until I undertook this project.

6. Observations and Future Work

We have shown some patterns on two triply periodic polyhedra. In fact we showed two different patterns on the $\{4, 6|4\}$ polyhedron. One of our goals was to find patterns that would emphasize interesting mathematical properties of the polyhedra — having linear elements that correspond closely to embedded lines in related minimal surfaces, for instance.

Although it has been known for 85 years that there are only three infinite skew polyhedron, the more general uniform triply periodic polyhedra have not been classified, but a number of examples are known. It would be challenging not only to place mathematically and artistically interesting patterns on the known polyhedra, but even more challenging to discover new polyhedra.

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