Transforming “Circle Limit III” Patterns - First Steps

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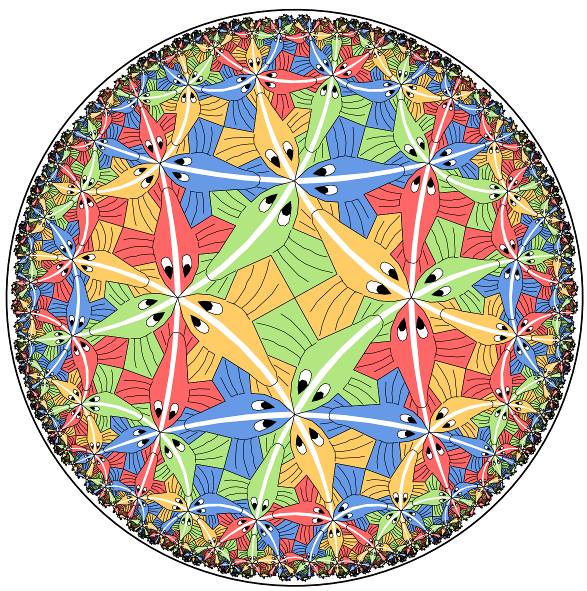


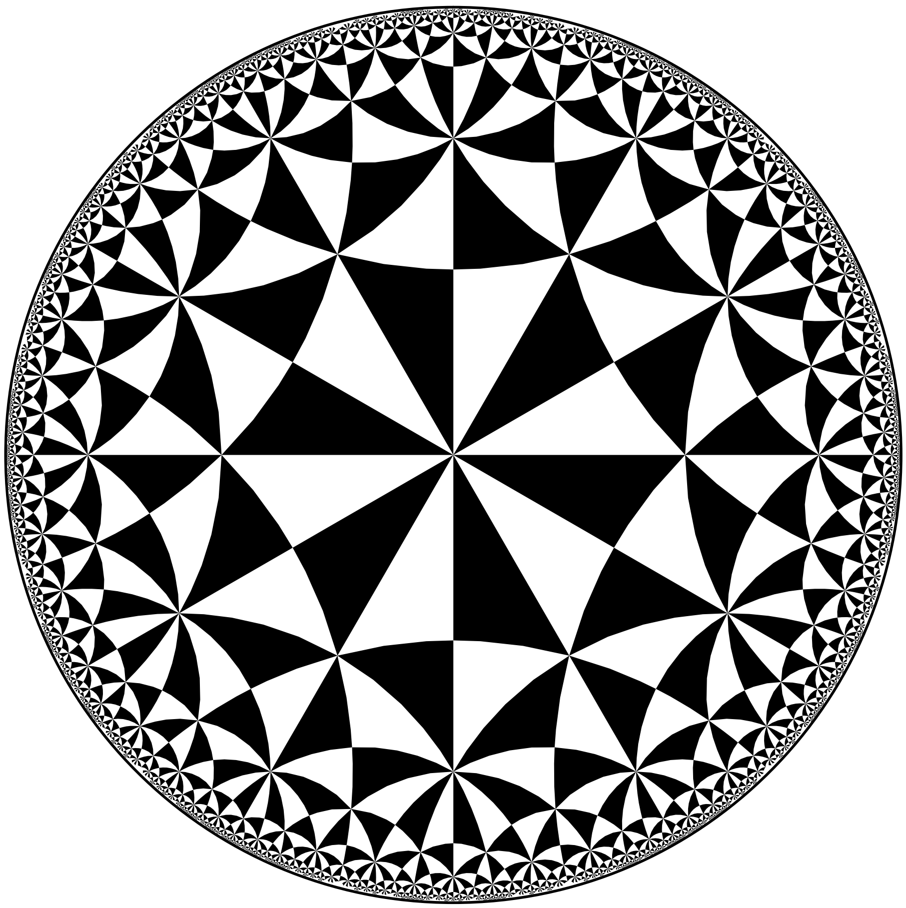
Outline

- History
- Theory of general (p, q, r) "Circle Limit III" patterns and hyperbolic geometry.
- The $p = q$ subfamily.
- The $p = r = 3$ subfamily.
- A possible solution for the general case.
- Future work.

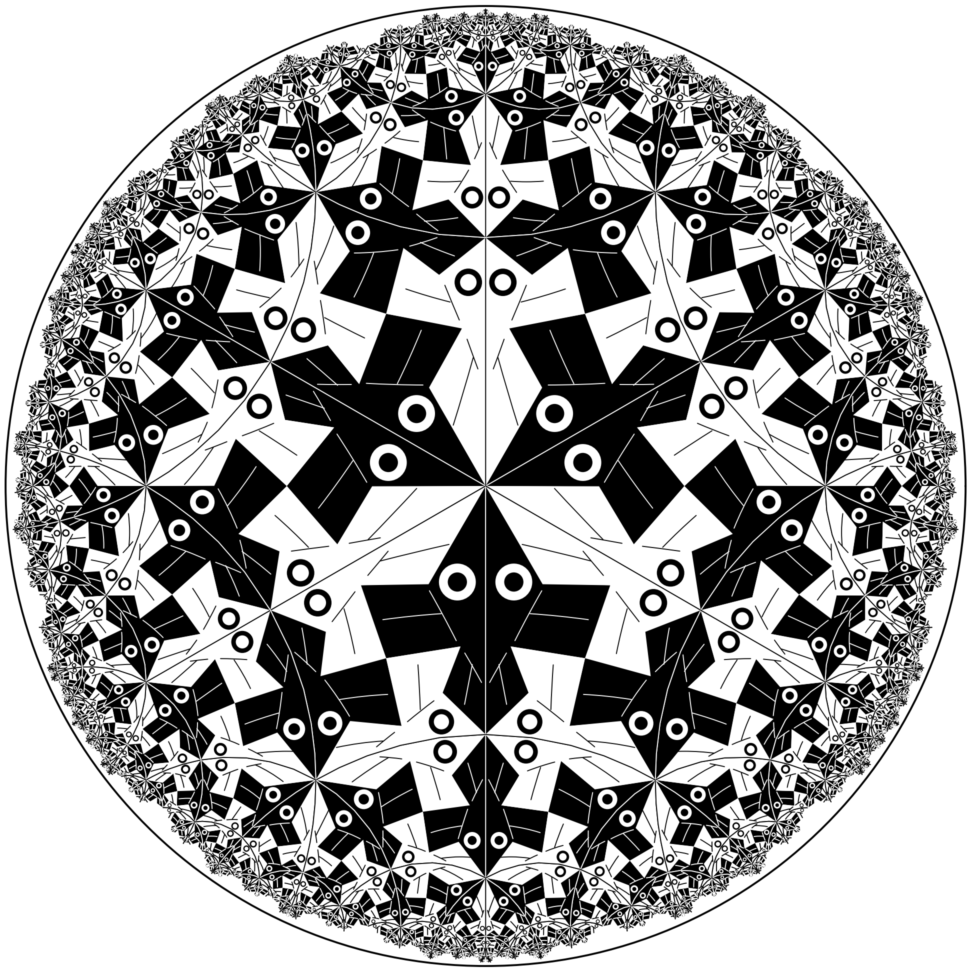
History

- Early 1958: H.S.M. Coxeter sends M.C. Escher a reprint containing a hyperbolic triangle tessellation.
- Later in 1958: Inspired by that tessellation, Escher creates *Circle Limit I*.
- Late 1959: Solving the "problems" of *Circle Limit I*, Escher creates *Circle Limit III*.
- 1979: In a *Leonardo* article, Coxeter uses hyperbolic trigonometry to calculate the "backbone arc" angle.
- 1996: In a *Mathematical Intelligencer* article, Coxeter uses Euclidean geometry to calculate the "backbone arc" angle.
- 2006: In a *Bridges* paper, D. Dunham introduces general (p, q, r) "Circle Limit III" patterns and gives an "arc angle" formula for $(p, 3, 3)$.
- 2007: L. Tee derives an "arc angle" formula for general (p, q, r) patterns, reported in *Bridges 2008.*

The hyperbolic triangle pattern in Coxeter's paper



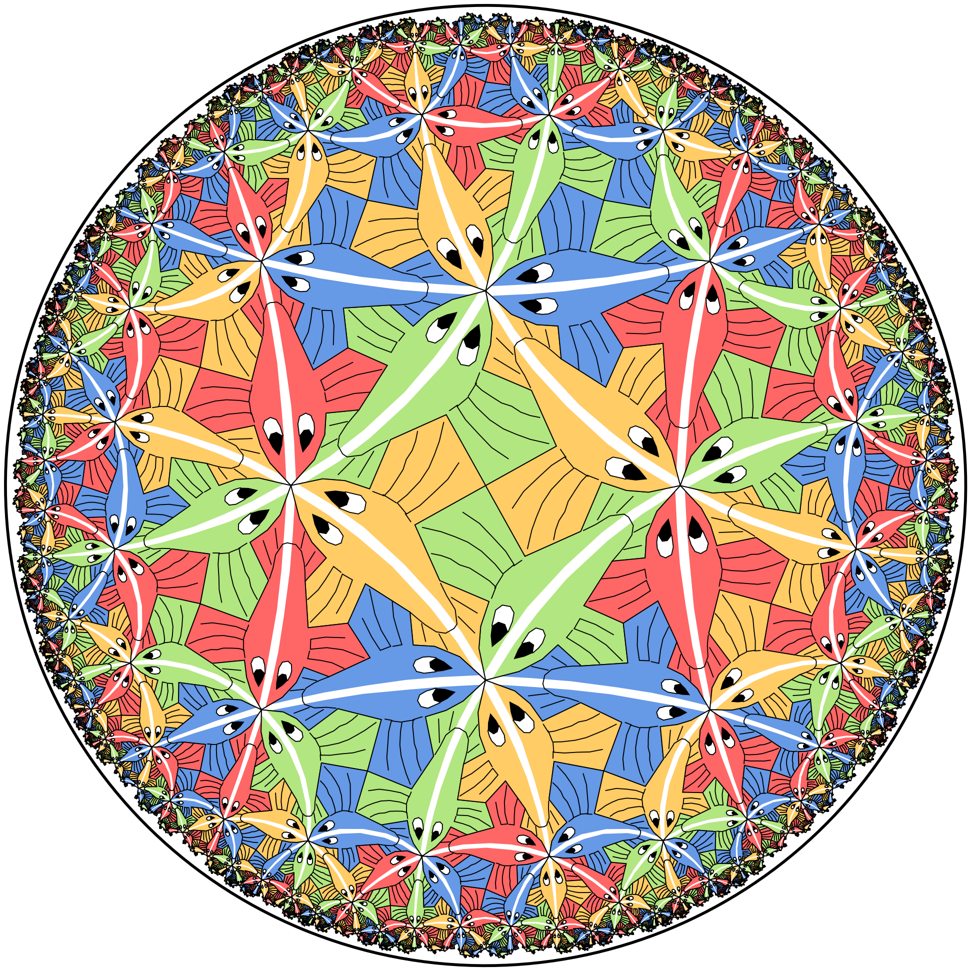
A Computer Rendition of *Circle Limit I*



Escher: Shortcomings of *Circle Limit I*

"There is no continuity, no 'traffic flow', no unity of colour in each row ..."

A Computer Rendition of *Circle Limit III*



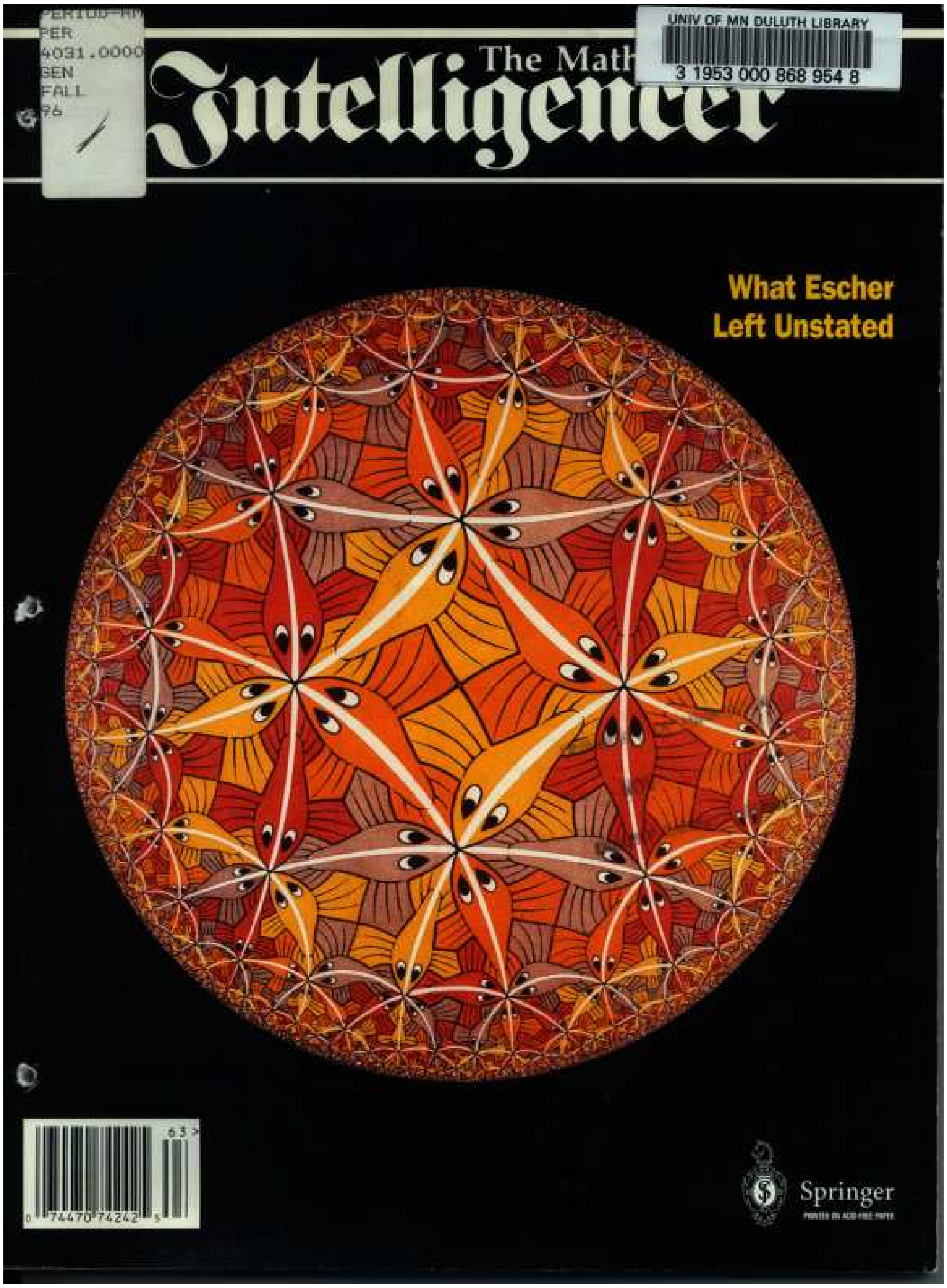
Coxeter's *Leonardo* and *Intelligencer* Articles

In *Leonardo* 12, (1979), pages 19–25, Coxeter used hyperbolic trigonometry to find the following expression for the angle ω that the backbone arcs make with the bounding circle in *Circle Limit III*.

$$\cos(\omega) = (2^{1/4} - 2^{-1/4})/2 \quad \text{or} \quad \omega \approx 79.97^\circ$$

Later Coxeter derived the same result using elementary Euclidean geometry in *The Mathematical Intelligencer* 18, No. 4 (1996), pages 42–46.

Mathematical Intelligencer Cover



Mathematical Intelligencer Contents Page

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Indexed in *Wilson General Science Abstracts* and *General Science Index* (since 1984)

On the Cover:
Coxeter's enthusiasm for the gift M.C. Escher gave him, a print of Circle Limit III, is understandable. So is his continuing curiosity. See the articles on pp. 35-46. He has not, however, said of what general theory this pattern is a special case. Not as yet. (© 1996 M.C. Escher/Cordon Art–Baarn–Holland. All rights reserved.) Photograph by David Vatcher.

“On The Cover:”

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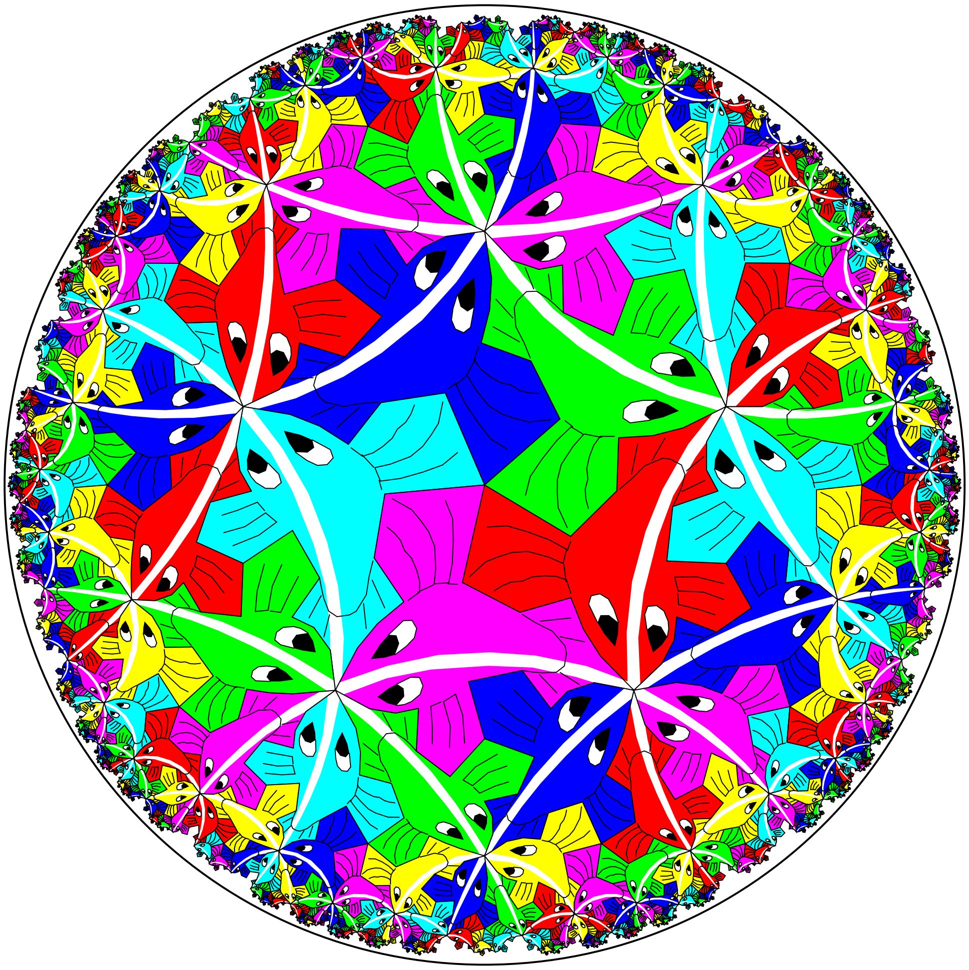
Annonymous Editor

A General Theory

We use the symbolism **(*p,q,r*)** to denote a pattern of fish in which ***p*** meet at right fin tips, ***q*** meet at left fin tips, and ***r*** fish meet at their noses. Of course ***p*** and ***q*** must be at least three, and ***r*** must be odd so that the fish swim head-to-tail (as they do in *Circle Limit III*).

The *Circle Limit III* pattern would be labeled (4,3,3) in this notation.

A (5,3,3) Pattern



A General Formula for the Intersection Angle

The general formula for the angle of intersection between the backbone arcs and the bounding circle for a (p, q, r) pattern (which agrees with Coxeter's result for *Circle Limit III*).

$$\cos(\omega) = \frac{\sin(\frac{\pi}{2r})\ (\cos(\frac{\pi}{p}) - \cos(\frac{\pi}{q}))}{\sqrt{\cos(\frac{\pi}{p})^2 + \cos(\frac{\pi}{q})^2 + \cos(\frac{\pi}{r})^2 + 2\cos(\frac{\pi}{p})\cos(\frac{\pi}{q})\cos(\frac{\pi}{r}) - 1)}}$$

An alternative formula:

$$\cot(\omega) = \frac{\tan(\frac{\pi}{2r})(\cos(\frac{\pi}{q}) - \cos(\frac{\pi}{p}))}{\sqrt{(\cos(\frac{\pi}{p}) + \cos(\frac{\pi}{q}))^2 + 2\cos(\frac{\pi}{r}) - 2}}$$

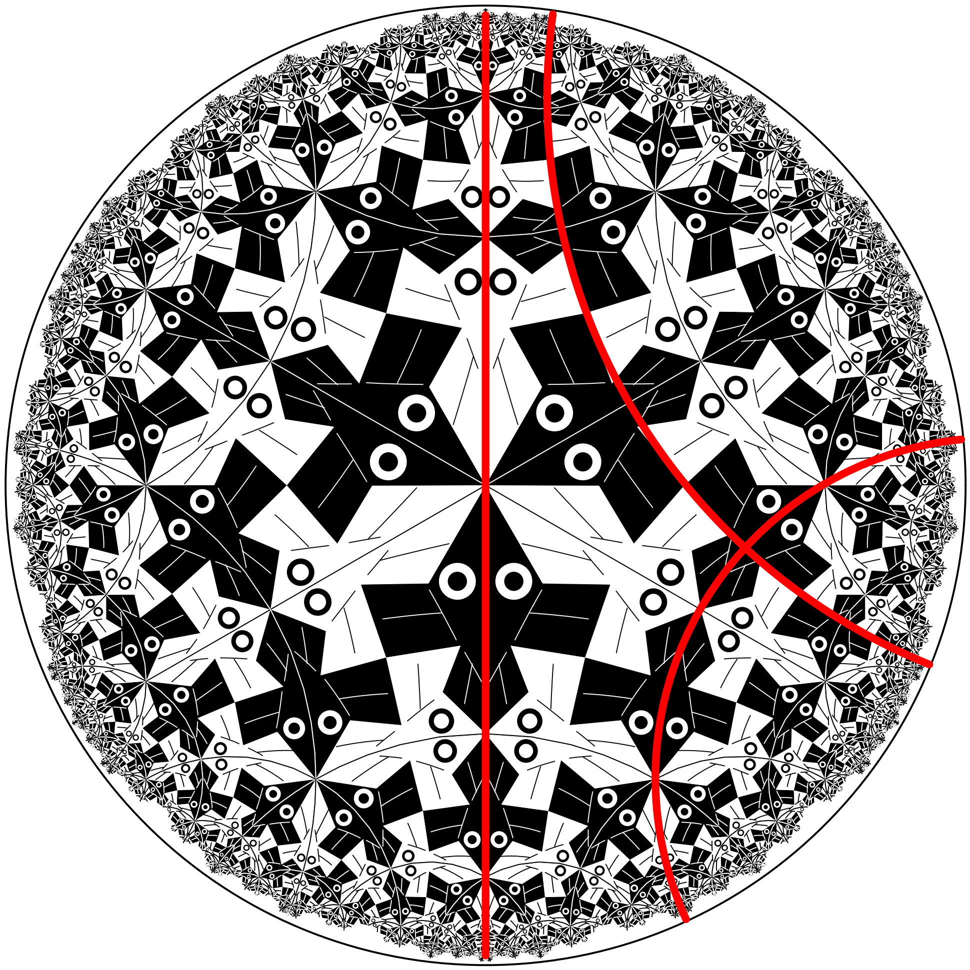
The Need for Models of Hyperbolic Geometry

In 1901 David Hilbert proved that (unlike the sphere) there was no smooth embedding of the hyperbolic plane in Euclidean 3-space.

Thus we must use Euclidean *models* of hyperbolic geometry.

Three useful models are the Poincaré circle model (used by Escher), the Klein model, and the Weierstrass model.

The Poincaré Circle Model of Hyperbolic Geometry



- **Points:** points within the **bounding circle**
- **Lines:** circular arcs perpendicular to the bounding circle (including diameters as a special case)

The Klein Model of Hyperbolic Geometry

- **Points:** points within the **bounding circle**
- **Lines:** chords of the bounding circle (including diameters as a special case)
- The chord corresponds to the Poincaré circular arc with the same endpoints on the bounding circle.

Weierstrass Model of Hyperbolic Geometry

- **Points:** points on the upper sheet of a hyperboloid of two sheets: $x^2 + y^2 - z^2 = -1$, $z \geq 1$.
- **Lines:** the intersection of a Euclidean plane through the origin with this upper sheet (and so is one branch of a hyperbola).

  A line can be represented by its **pole**, a 3-vector $\begin{bmatrix} \ell_x \\ \ell_y \\ \ell_z \end{bmatrix}$ on the dual hyperboloid $\ell_x^2 + \ell_y^2 - \ell_z^2 = +1$, so that the line is the set of points satisfying $x\ell_x + y\ell_y - z\ell_z = 0$.

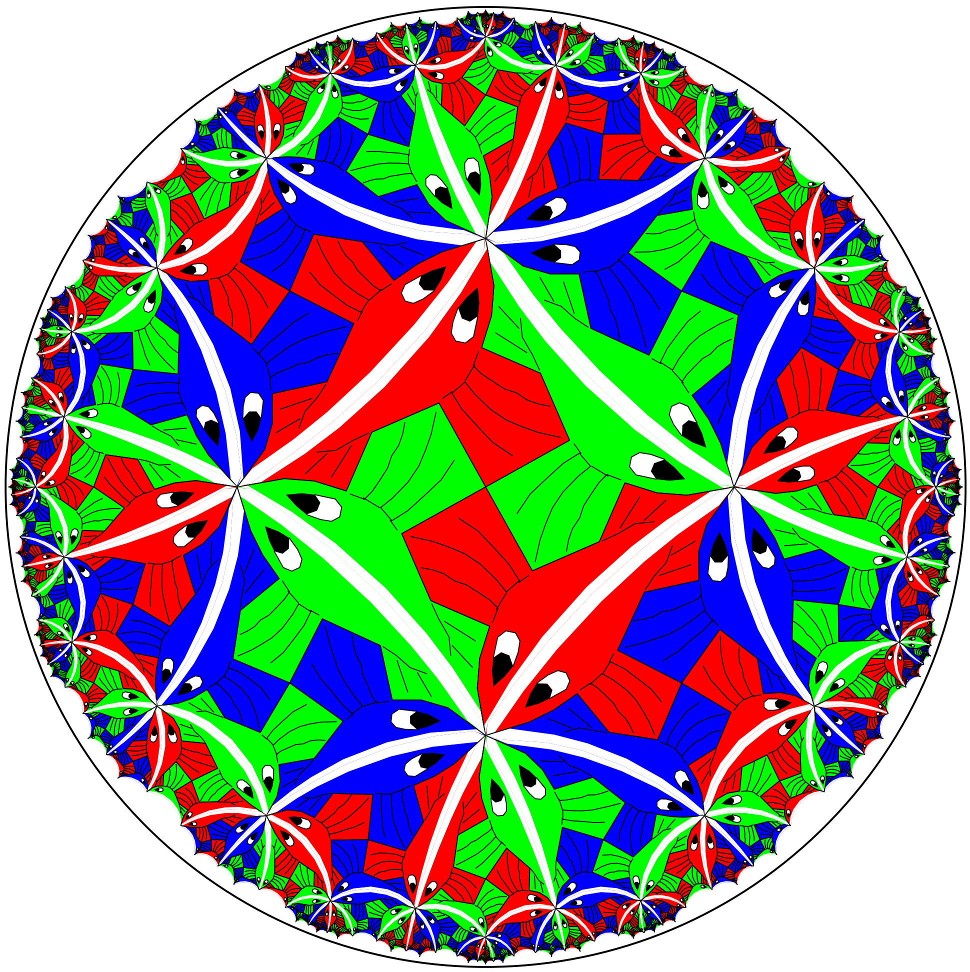
The Relation Between the Poincaré and Weierstrass Models

The models are related via stereographic projection from the Weierstrass model onto the (unit) Poincaré disk in the xy-plane toward the point $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$,

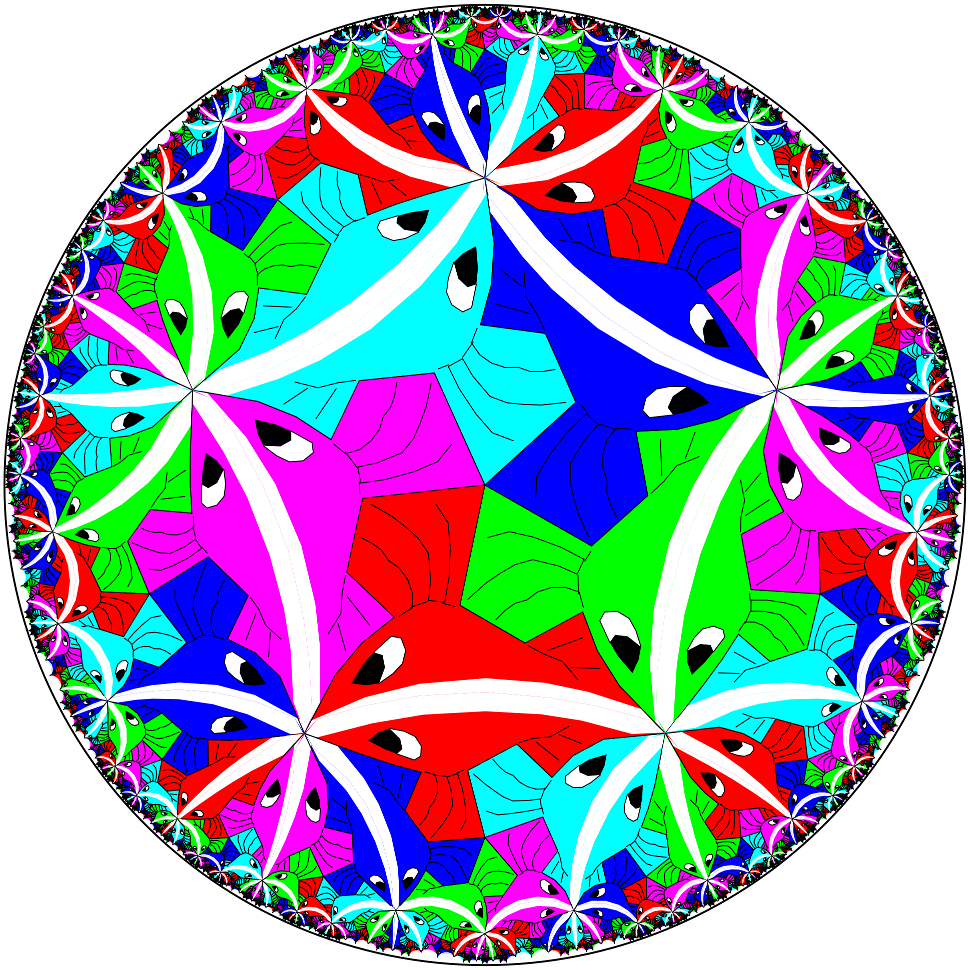
Given by the formula: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x/(1+z) \\ y/(1+z) \\ 0 \end{bmatrix}$.

Patterns with p = q

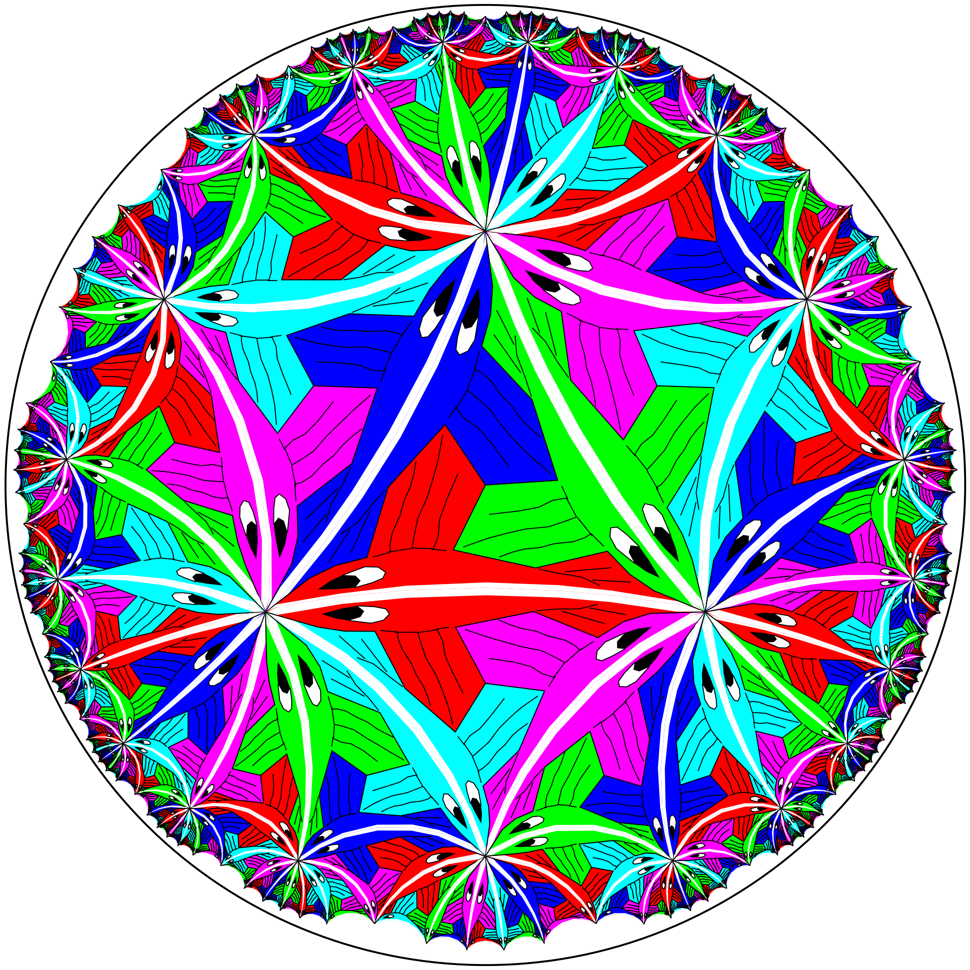
When $p = q$, the fish are symmetric, so half a fish can serve as the fundamental region for the pattern since the other half of the fish may be obtained by reflection. The figure shows a $(4, 4, 3)$ pattern.



A (5,5,3) Pattern

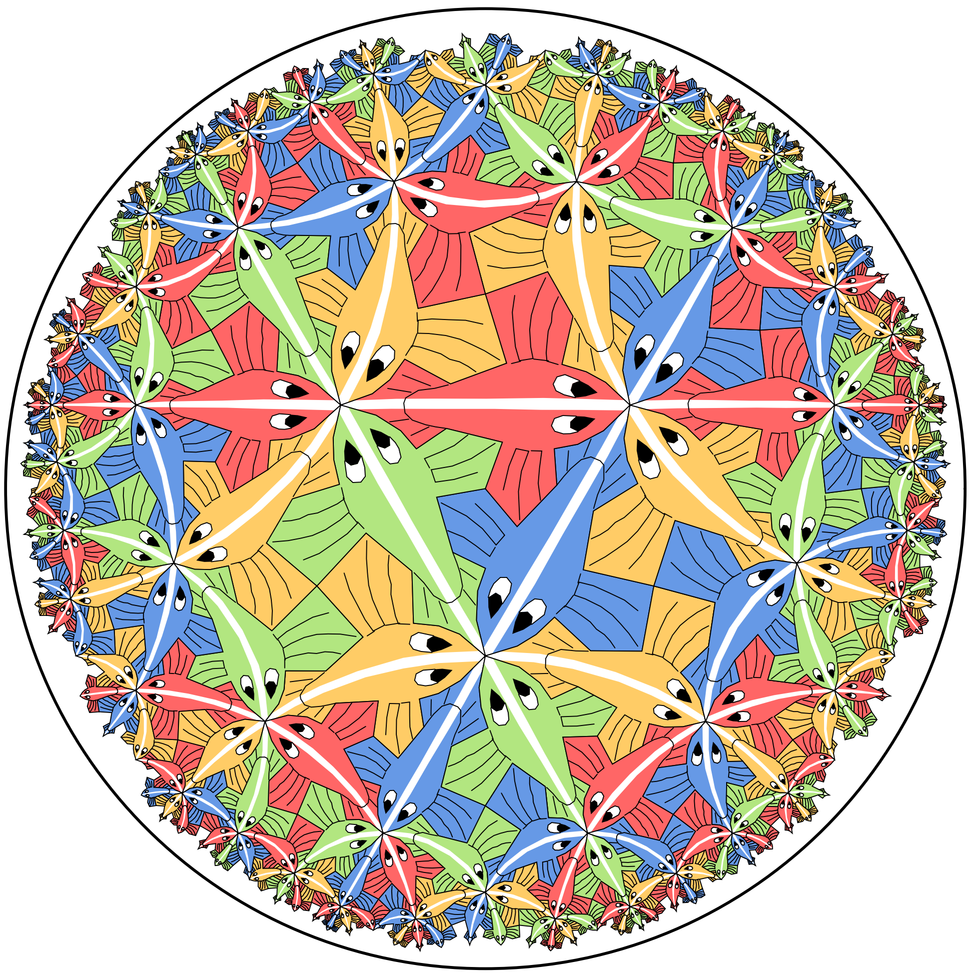


A (3,3,5) Pattern

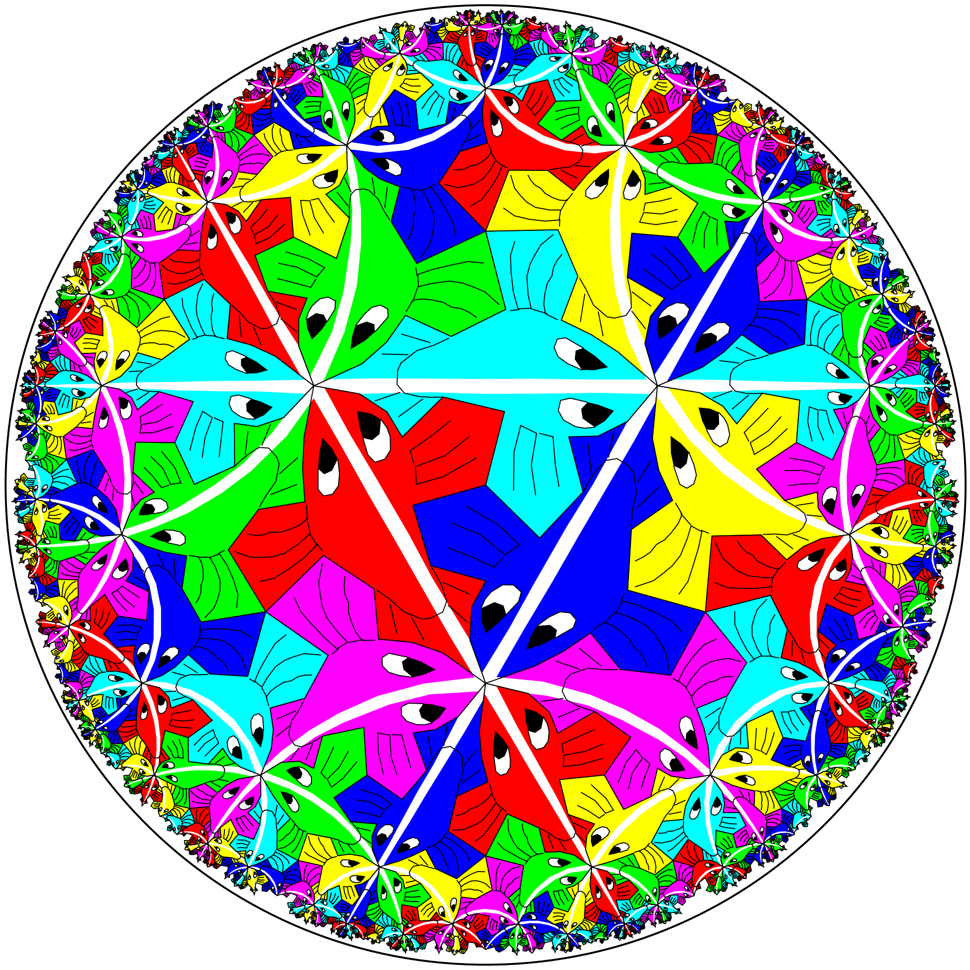


Patterns with p = r = 3

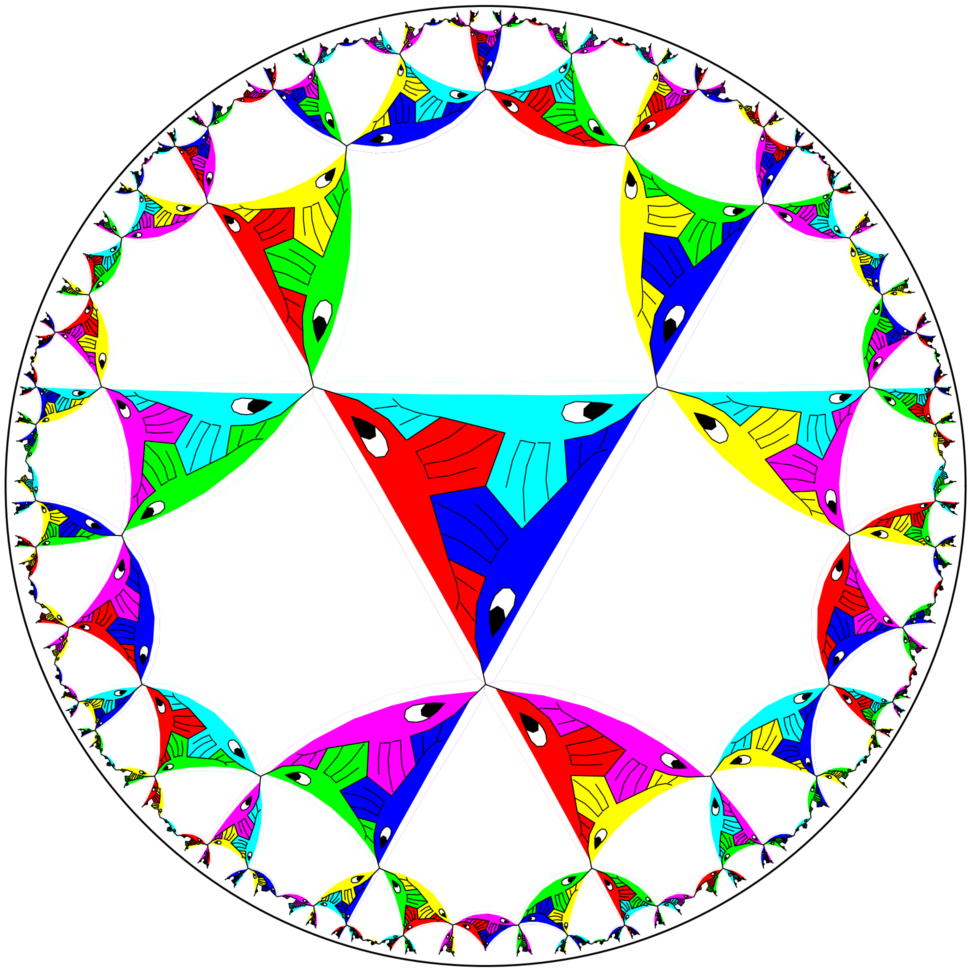
When $p = r = 3$, the backbone lines of the three center fish form a Euclidean equilateral triangle. This equilateral triangle can be scaled to correspond to different values of q. Unfortunately, this only transforms the right sides of the fish correctly. The figure shows a $(3, 4, 3)$ pattern



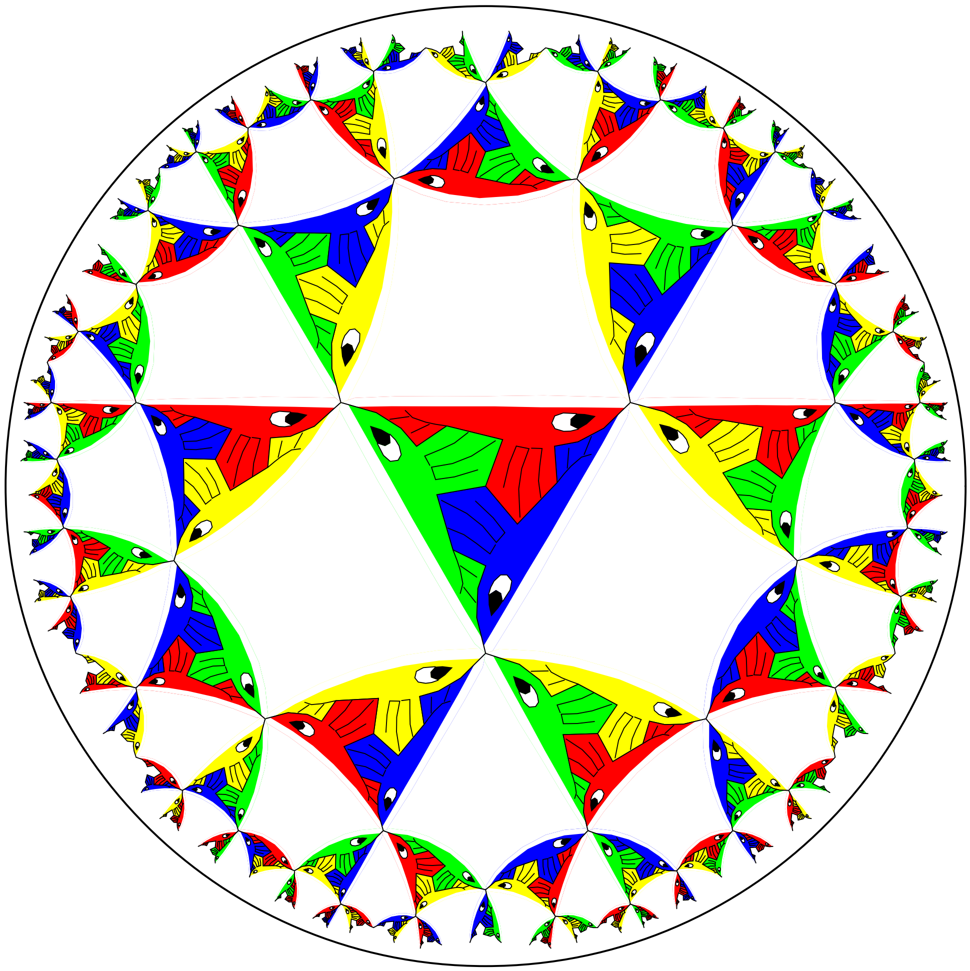
A (3,5,3) Pattern



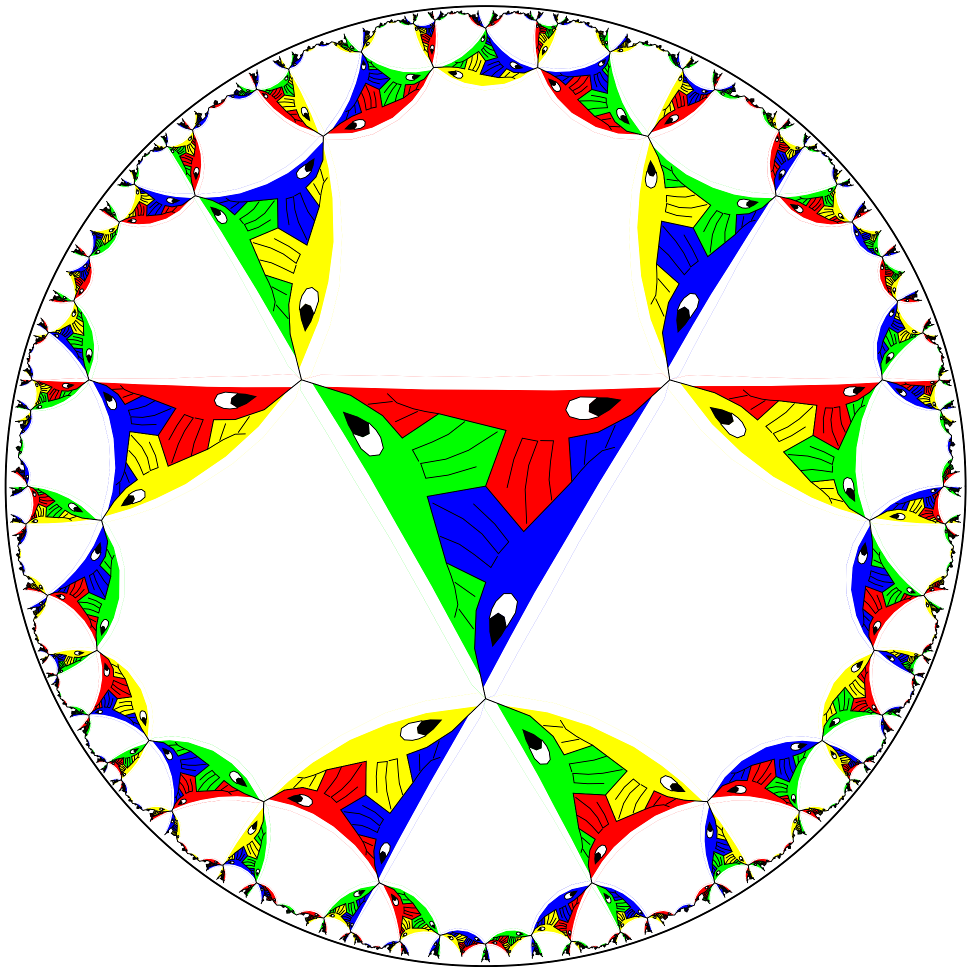
The Right Halves of the Fish of the (3,5,3) Pattern



A (3,4,3) Pattern of Right Fish Halves



A (3,6,3) Pattern of Right Fish Halves



A Possible Solution for the General Case

- In the two subcases above, the transformations worked because half a fish motif could be made to fit inside a Euclidean isosceles triangle, and one isosceles triangle can be transformed into another by (differential) scaling.
- Thus, in the general case, to transform the fish motif of a (p, q, r) pattern to a (p', q', r') fish motif might involve separate processes to transform the left and right halves of the fish.
- To transform right fish halves, one possible idea would be to find a model of hyperbolic geometry the right "distance" in between the Poincaré model and the Klein model so that the backbone line (equidistant curve) would "flatten out" to a Euclidean line. Then the transformation would just be a Euclidean scaling.
- To transform a left fish half, we could hyperbolically translate its fin tip to the origin (making it like a right fish half), find the correct "in between" hyperbolic model (probably different than for the right half), apply the transformation, then hyperbolically translate back.

Future Work

- Find a general method, possibly the one outlined above, to transform the fish motif of a (p, q, r) pattern to a (p', q', r') fish motif.
- Find an algorithm to automatically color (p, q, r) patterns with the minimum number of colors as Escher's did in *Circle Limit III*: all fish along a backbone line are the same color, and adjacent fish are different colors (the "map-coloring principle").

The End

Many thanks to Nat, Ergun, and the other organizers of ISAMA ’09!