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An Escher-like Fish Pattern on a Triply Periodic Polyhedron

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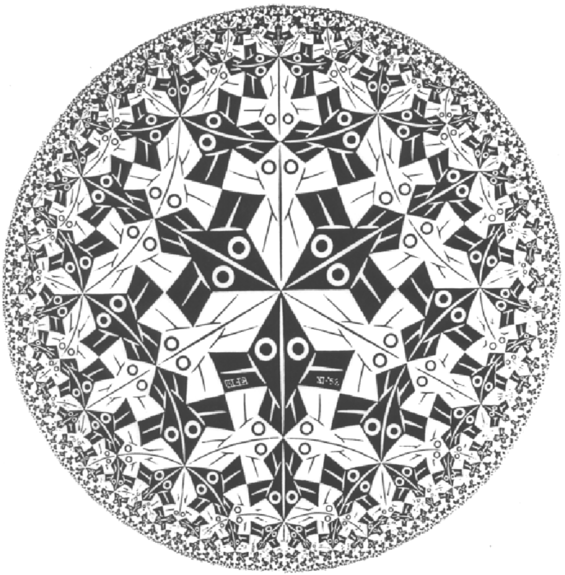
Outline

- Background and motivation
  - M.C. Escher's *Circle Limit I* and *Circle Limit III*
  - Regular $\{p, q \mid r\}$ triply periodic polyhedra
  - Previous polyhedra and their problems
- The papercrafted part of a $\{4, 6 \mid 4\}$ polyhedron
- A part of the $\{6, 6 \mid 3\}$ polyhedron that solves all the problems
- Future work
- Contact information

Escher's Woodcut Circle Limit I



Problems Circle Limit I per Escher

1. The fish were not consistently colored along backbone lines — they alternated from black to white and back every two fish lengths.
2. The fish also changed direction every two fish lengths — thus there was no "traffic flow" (Escher's words) in a single direction along the backbone lines.
3. The fish are very angular and not "fish-like"

Escher's Woodcut Circle Limit III

— solved the problems

Regular Triply Repeating Polyhedra

In 1926 H.S.M. Coxeter defined *regular skew polyhedra* (apeirohedra) to be infinite polyhedra repeating in three independent directions in Euclidean 3-space.

Coxeter denoted them by the extended Schläfli symbol $\{p, q \mid r\}$ which denotes the polyhedron composed of p-gons meeting q at each vertex, with regular r-sided polygonal holes.

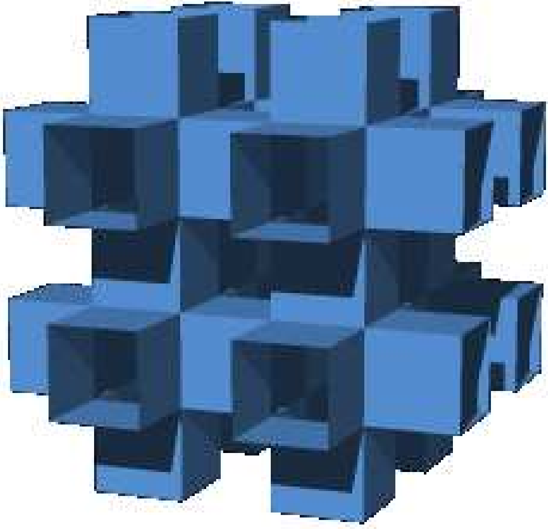
Coxeter and John Flinders Petrie proved that there are exactly three of them: $\{4, 6 \mid 4\}$, $\{6, 4 \mid 4\}$, and $\{6, 6 \mid 3\}$.

Since the sum of the vertex angles is greater than 2π, they are considered to be the hyperbolic analogs of the Platonic solids and the regular Euclidean tessellations $\{3, 6\}$, $\{4, 4\}$, and $\{6, 3\}$

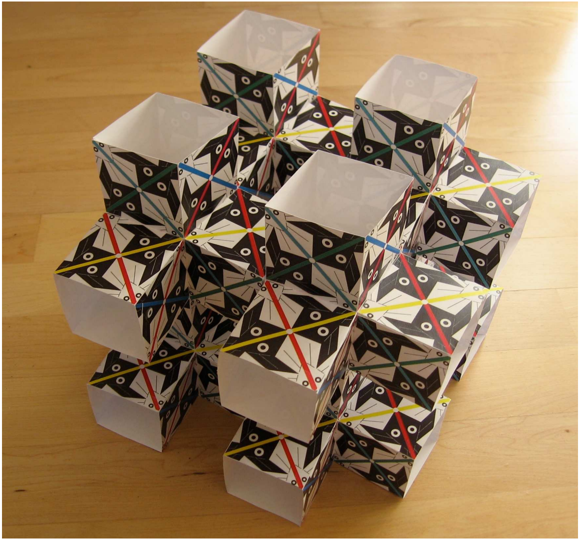
In 2012 Dunham was the first person to decorate those solids with Escher-inspired patterns.

The simplest regular skew polyhedron: $\{4, 6 \,|\, 4\}$

Also called the *Mucube* (for Multi-cube). It consists of invisible “hub” cubes connected by “strut” cubes, hollow cubical cylinders with their open ends connecting neighboring hubs.



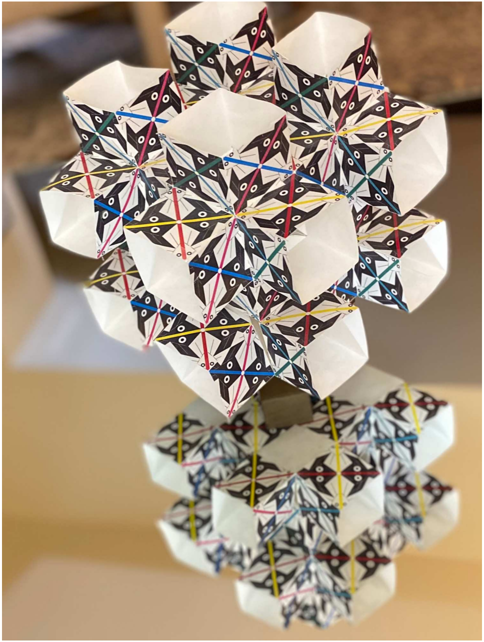
An old patterned $\{4, 6 \,|\, 4\}$ with fish



Problems with the old fish polyhedron

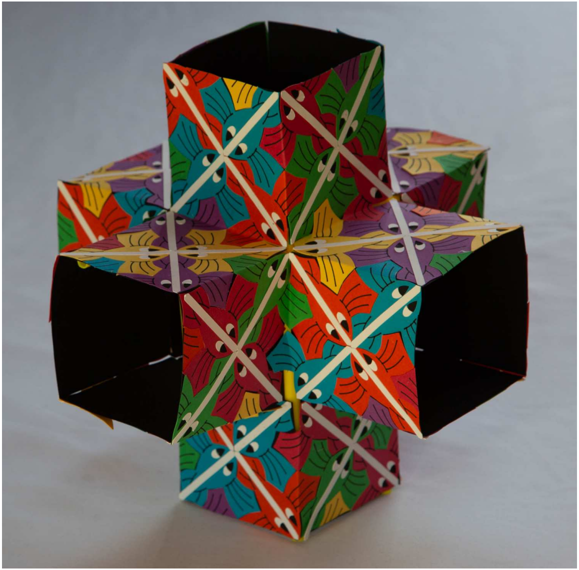
1. The same three problems Escher saw in *Circle Limit I*.
2. A fourth problem: the backbone lines of a particular color are not parallel — which can be seen in a mirror.

The old fish polyhedron on a mirror



A new papercrafted fish pattern on the $\{4, 6 \,|\, 4\}$ polyhedron

Fixes the first and third problems.



The papercrafted $\{4, 6 \,|\, 4\}$ polyhedron on a mirror

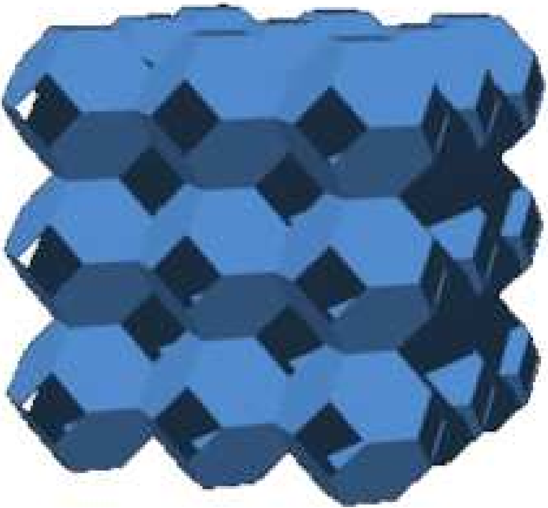
Fixes the fourth problem too, but not the second one.

Colors of fish on the $\{4, 6 \,|\, 4\}$ polyhedron

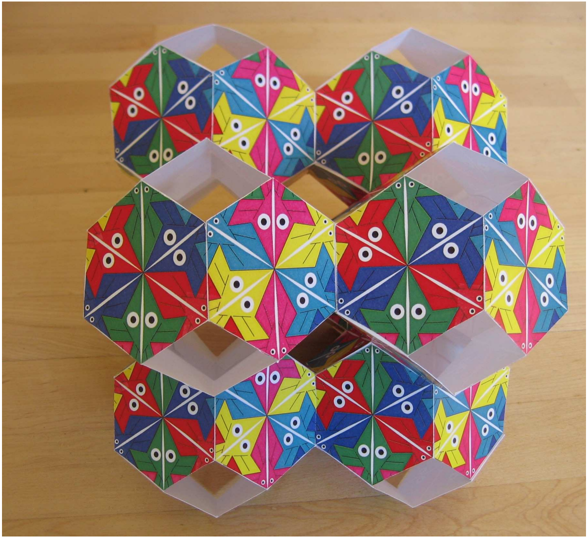
1. There are six families of fish backbone lines that are parallel to the face diagonals of a cube.
2. All the fish in one family are the same color.

The dual of the Mucube is the $\{6, 4 \mid 4\}$ polyhedron

Also called the *Muoctahedron* (for Multi-octahedron). It consists of truncated octahedra in a cubic lattice arrangement, connected on their invisible square faces (which are also the square holes between the truncated octahedra).

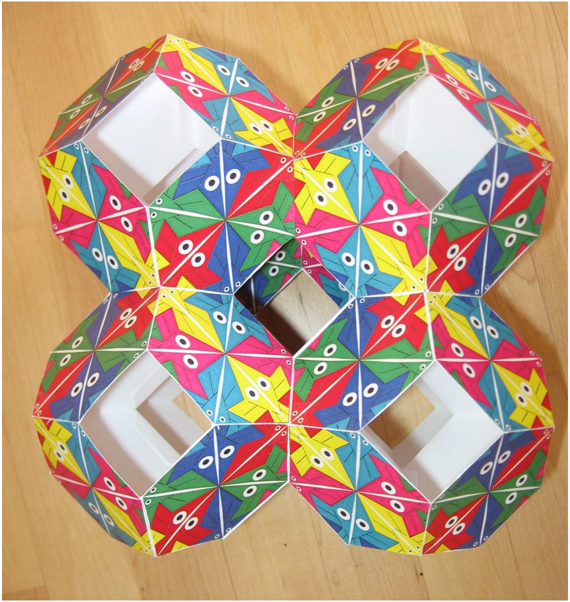


An angular fish pattern on the $\{6, 4 \,|\, 4\}$ polyhedron



A top view of the fish pattern on the $\{6, 4 \,|\, 4\}$ polyhedron

It solves Escher's first problem, but still has problems two and three.



The $\{6, 6 \,|\, 3\}$ polyhedron is self-dual

Also called the *Mutetrahedron* (for Multi-tetrahedron). It consists of truncated tetrahedra in a diamond lattice arrangement, connected by their missing triangular faces to faces of invisible regular tetrahedra between them.

The new $\{6, 6\,|\,3\}$ patterned polyhedron

Also fixes the second, "traffic flow", problem.

Colors of fish on the $\{6, 6 \mid 3\}$ polyhedron

1. Again, there are six families of fish backbone lines that go through the centers of the hexagon faces of the $\{6, 6 \mid 3\}$ polyhedron.
2. And again, the fish in one family are the same color.
3. Each of the families is parallel to one of the sides of a tetrahedron — which can be one of the truncated tetrahedra, since all the (patterned) truncated tetrahedra in the $\{6, 6 \mid 3\}$ polyhedron are translates of one another.
4. In each family half the lines of fish go one direction, and the other half go the opposite direction — so that fish of one color on one truncated tetrahedron go in opposite directions on adjacent faces.

Future Work

- We would like to make a papercrafted version of the new $\{6, 6 \mid 3\}$ patterned polyhedron.
- We would like to explore putting other patterns on the $\{p, q \mid r\}$ polyhedra, and on less regular triply periodic $\{p, q\}$ polyhedra.

Acknowledgements and Contact

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