MAA North Central Section, Fall, 2012

Patterns on Triply Periodic Polyhedra

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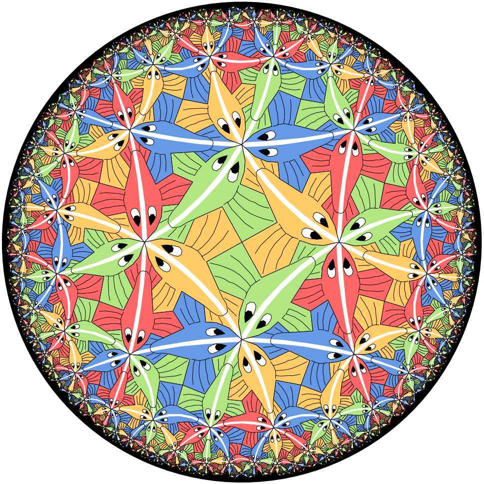


Outline

- Some history of hyperbolic patterns
- M.C. Escher's hyperbolic patterns
- Hyperbolic geometry and regular tessellations
- More hyperbolic patterns inspired by Escher
- Triply periodic polyhedra
- Relation between periodic polyhedra and regular tessellations
- Patterns triply periodic polyhedra
- Future research

H.S.M. Coxeter's 1957 Figure

In Figure 7 we see another such group, with the important difference that now the angle-sum of each triangle is less than two right angles and the number of triangles is infinite. The group is again generated by inversions in three circles, but the figure is no longer a picture of something in space. We do not find it as easy as before to imagine that the smaller peripheral triangles are the same size as those in the middle. But in so far as we succeed in stretching our imagination to this extent, we are visualizing the non-Euclidean plane of Gauss, Bolyai and Lobatschewsky.

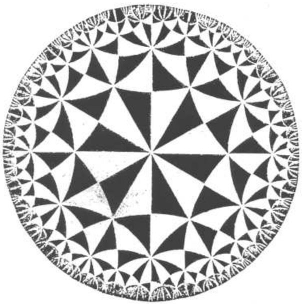
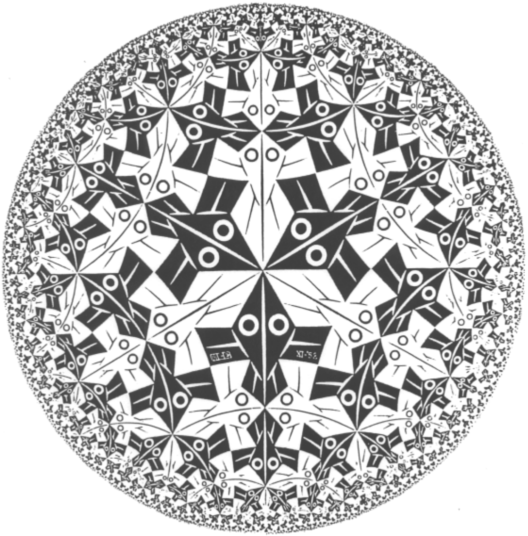


FIGURE 7

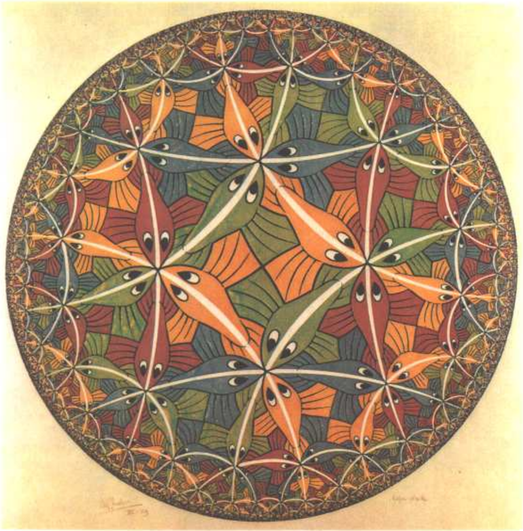
This is one way to generalize the idea of symmetry. Another is to increase he number of dimensions. Plate IV shows a wire model made by Mr. P. S. Donchian of Hartford, Conn.[3] This represents an orthogonal projection of a four-dimensional hyper-solid bounded by 120 regular dodecahedra. The

Escher's *Circle Limit I*

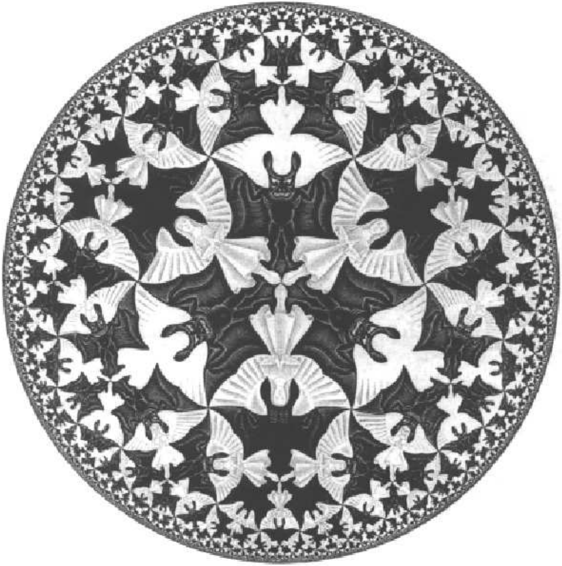


A rendition of *Circle Limit II*

Escher's *Circle Limit III*



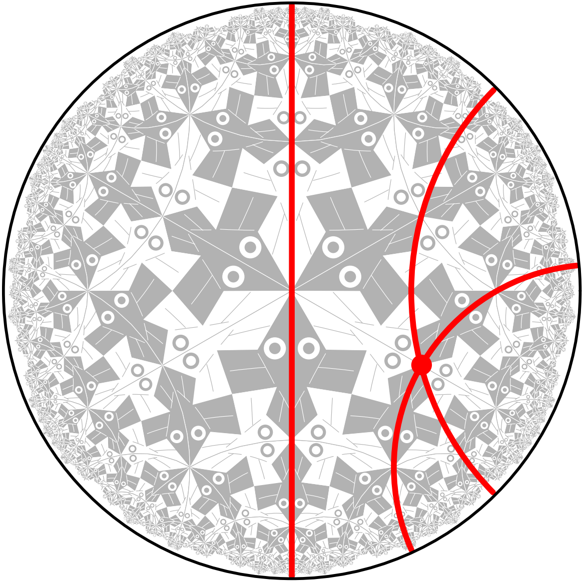
Escher's *Circle Limit IV*



Hyperbolic Geometry and Regular Tessellations

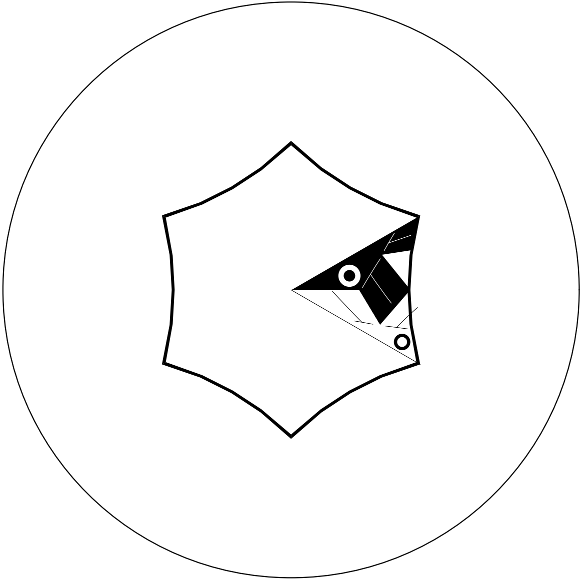
- In 1901, David Hilbert proved that, unlike the sphere, there was no isometric (distance-preserving) embedding of the hyperbolic plane into ordinary Euclidean 3-space.
- Thus we must use *models* of hyperbolic geometry in which Euclidean objects have hyperbolic meaning, and which must distort distance.
- One such model is the *Poincaré disk model*. The hyperbolic points in this model are represented by interior point of a Euclidean circle — the *bounding circle*. The hyperbolic lines are represented by (internal) circular arcs that are perpendicular to the bounding circle (with diameters as special cases).
- This model is appealing to artests since (1) angles have their Euclidean measure (i.e. it is conformal), so that motifs of a repeating pattern retain their approximate shape as they get smaller toward the edge of the bounding circle, and (2) it can display an entire pattern in a finite area.

Poincaré Disk Model of Hyperbolic Geometry



Repeating Patterns

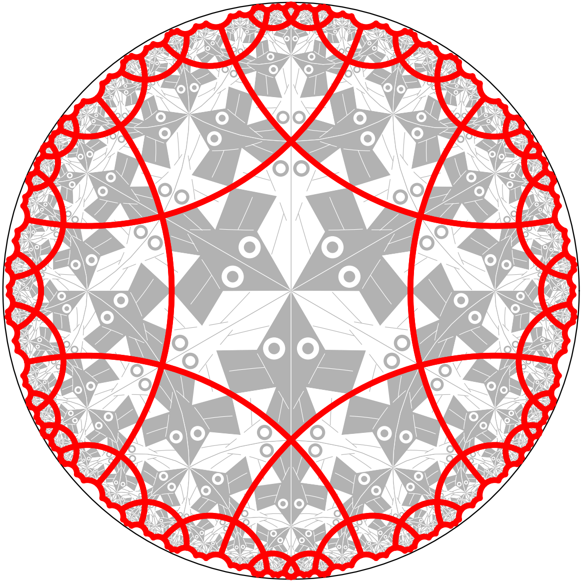
A **repeating pattern** is composed of congruent copies of the motif.



Regular Tessellations

- The *regular tessellation*, $\{p, q\}$, is an important kind of repeating pattern composed of regular p-sided polygons meeting q at a vertex.
- If $(p-2)(q-2) < 4$, $\{p, q\}$ is a spherical tessellation (assuming $p > 2$ and $q > 2$ to avoid special cases).
- If $(p-2)(q-2) = 4$, $\{p, q\}$ is a Euclidean tessellation.
- If $(p-2)(q-2) > 4$, $\{p, q\}$ is a hyperbolic tessellation. The next slide shows the $\{6, 4\}$ tessellation.
- Escher based his 4 "Circle Limit" patterns, and many of his spherical and Euclidean patterns on regular tessellations.

The Regular Tessellation $\{6, 4\}$



A Table of the Regular Tessellations

q \ p	3	4	5	6	7	8	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
8	*	*	*	*	*	*	$\cdots$
7	*	*	*	*	*	*	$\cdots$
6	□	*	*	*	*	*	$\cdots$
5	○	*	*	*	*	*	$\cdots$
4	○	□	*	*	*	*	$\cdots$
3	○	○	○	□	*	*	$\cdots$

□ - Euclidean tessellations

○ - spherical tessellations

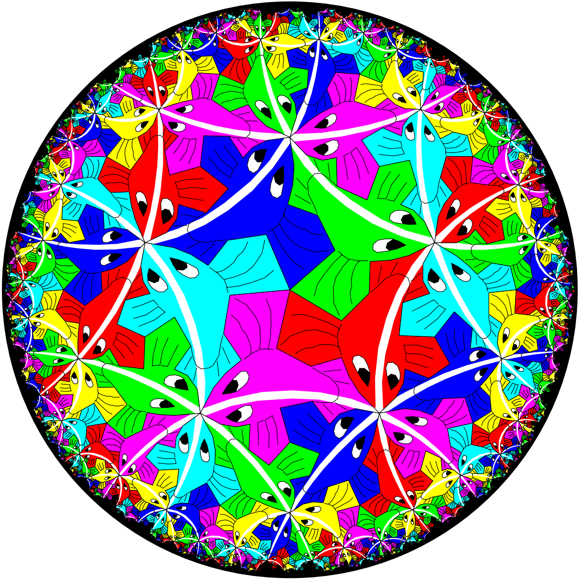
$*$ - hyperbolic tessellations

A Family of *Circle Limit III* Patterns

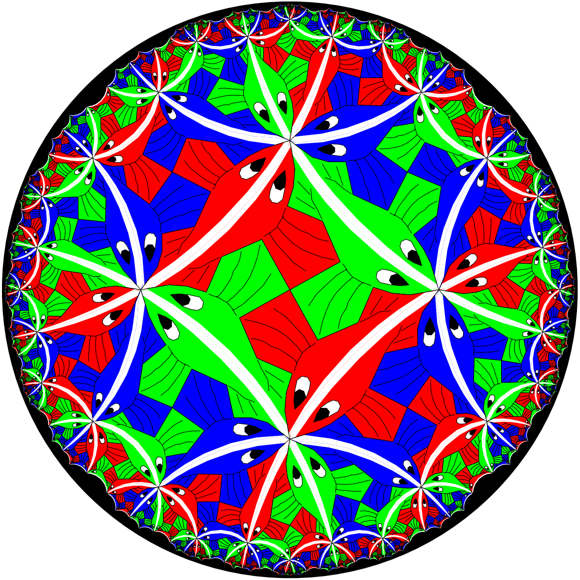
We use the symbolism **(p,q,r)** to denote a pattern of fish in which **p** meet at right fin tips, **q** meet at left fin tips, and **r** fish meet at their noses. Of course **p** and **q** must be at least three, and **r** must be odd so that the fish swim head-to-tail (as they do in *Circle Limit III*).

Escher's *Circle Limit III* pattern itself would be labeled (4,3,3) in this notation.

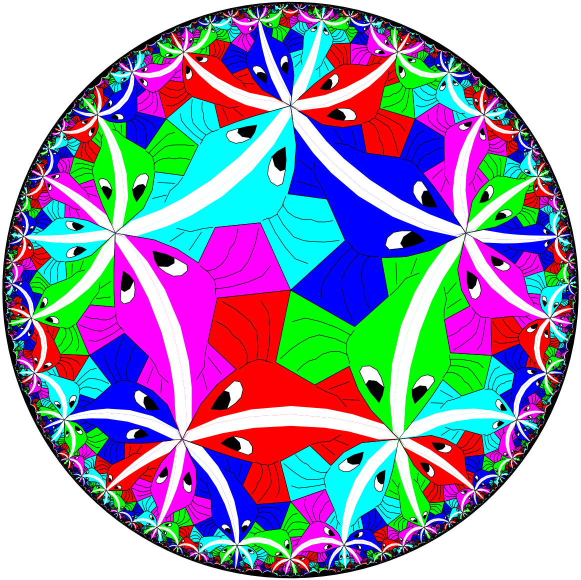
A (5,3,3) Pattern



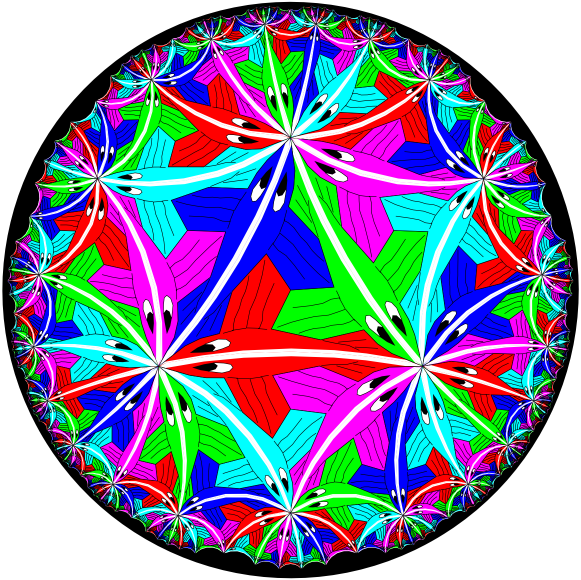
A (4,4,3) Pattern



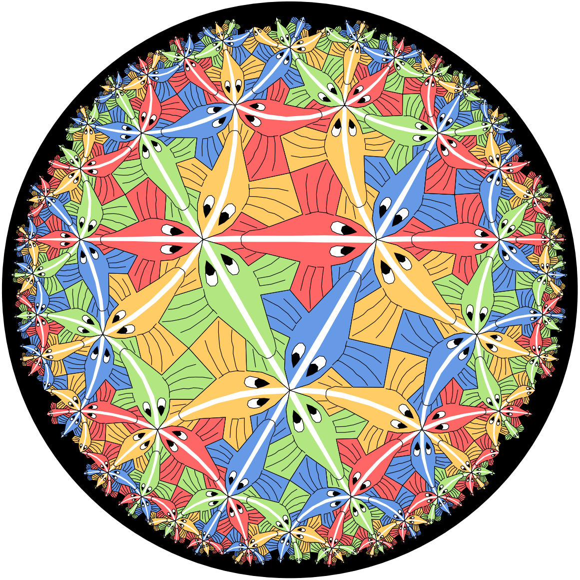
A (5,5,3) Pattern



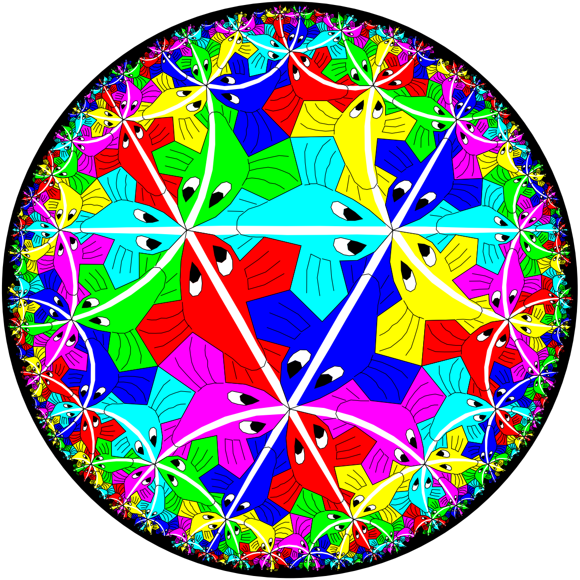
A (3,3,5) Pattern



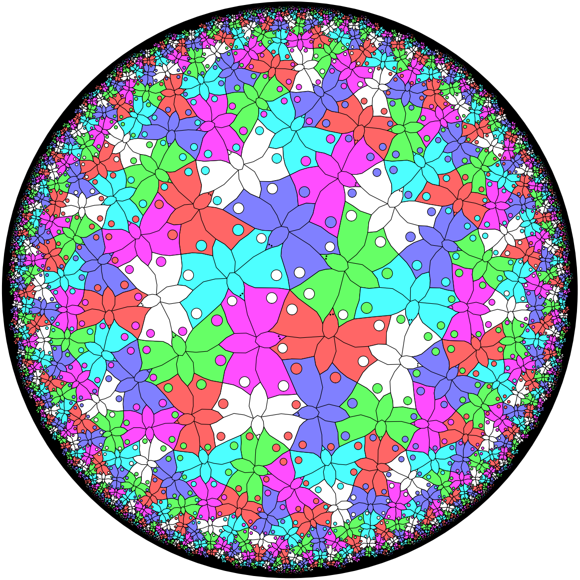
A (3,4,3) Pattern



A (3,5,3) Pattern



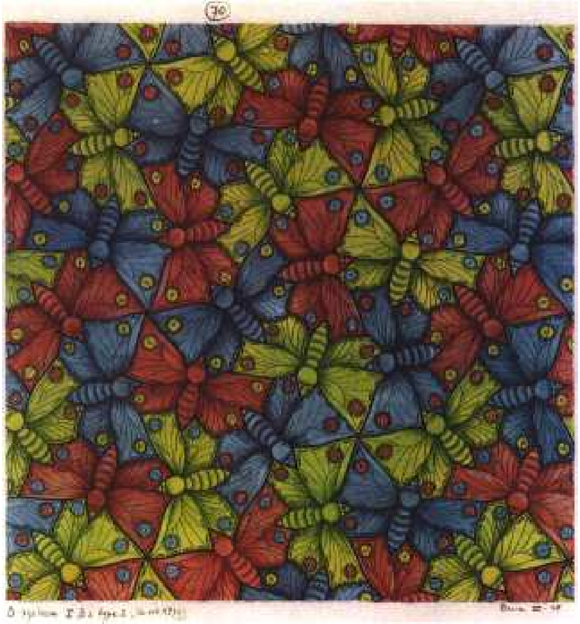
A Butterfly Pattern Based on the $\{5,4\}$ Tessellation



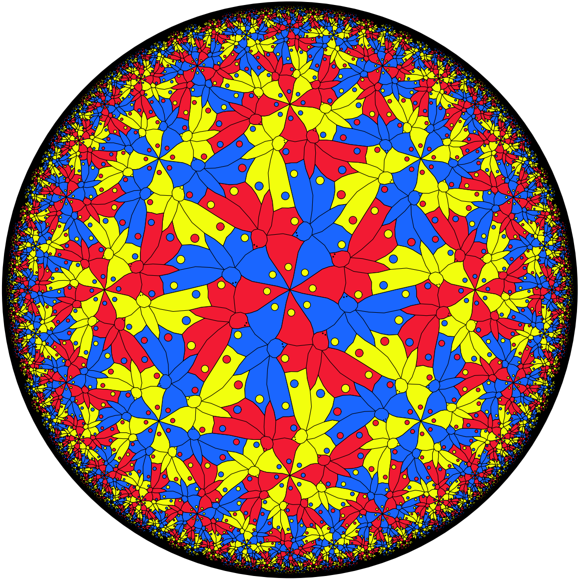
The Family of Butterfly Patterns

- Theoretically, we can create a butterfly pattern based on $\{p, q\}$ like the one above for any values of p and q provided $p \geq 3$ and $q \geq 3$.
- For these patterns, p butterflies meet at their left front wing tips and q butterflies meet at their right rear wings.
- Escher created only one member of this family of patterns, his Regular Division Drawing Number 70, based on the Euclidean hexagon tessellation $\{6, 3\}$. At least 3 colors are needed to satisfy the map-coloring principle at the meeting points of right rear wings.
- Following Escher, we add the restriction to our patterns that all circles on the butterfly wings around a p-fold meeting point of left wingtips be a different color from the butterflies meeting there.
- The hyperbolic butterfly pattern based on the $\{5, 4\}$ tessellation requires at least five colors for color symmetry since five is prime, and six colors if the circles on the wings are to be a different color.

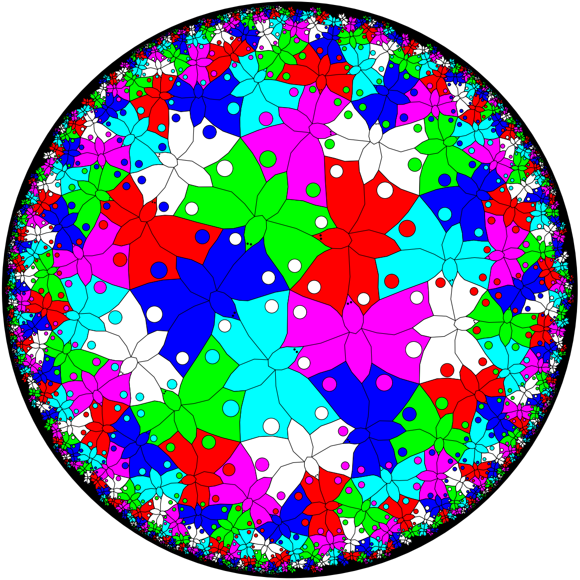
Escher's 3-colored butterfly pattern
Regular Division Drawing Number 70



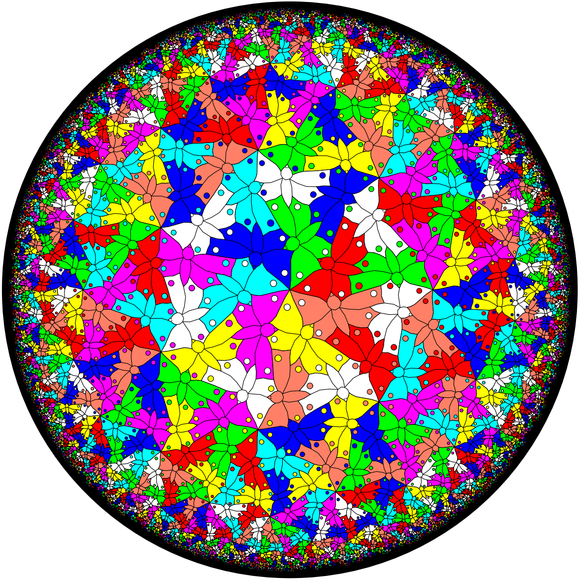
A 3-colored $(8, 3)$ butterfly pattern



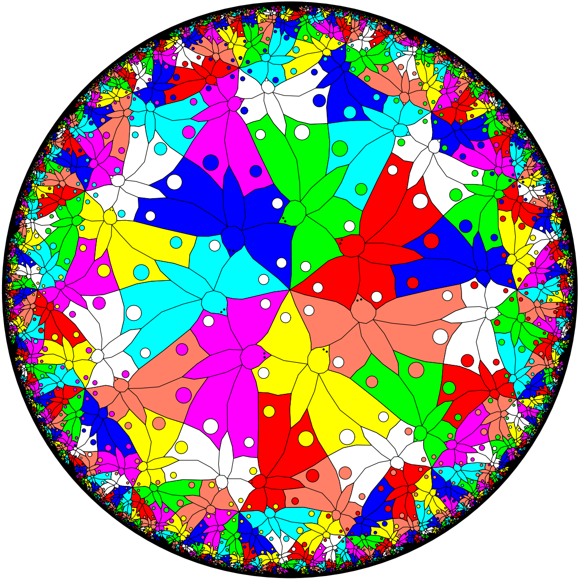
A 6-colored $(5, 5)$ butterfly pattern



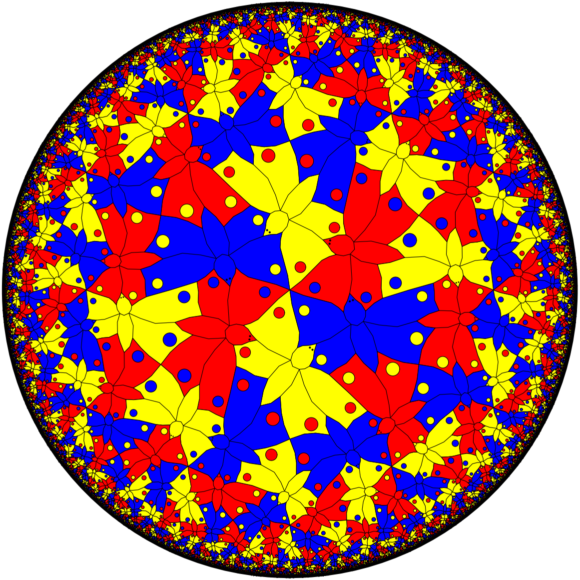
An 8-colored $(7, 3)$ butterfly pattern



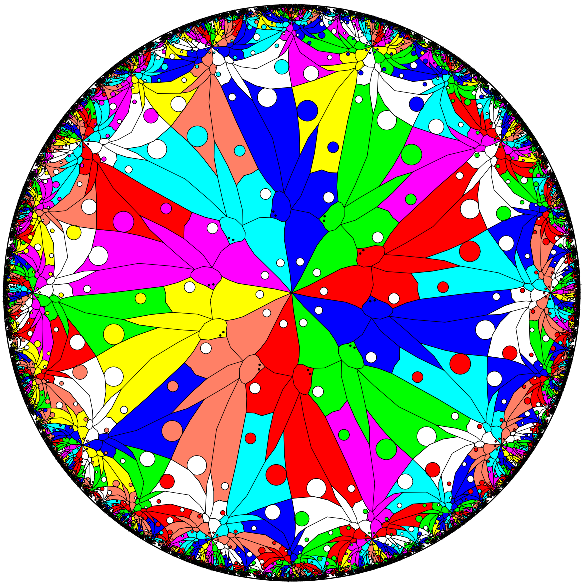
An 8-colored $(7, 4)$ butterfly pattern



A 3-colored $(6, 4)$ butterfly pattern
that violates the color of circles convention



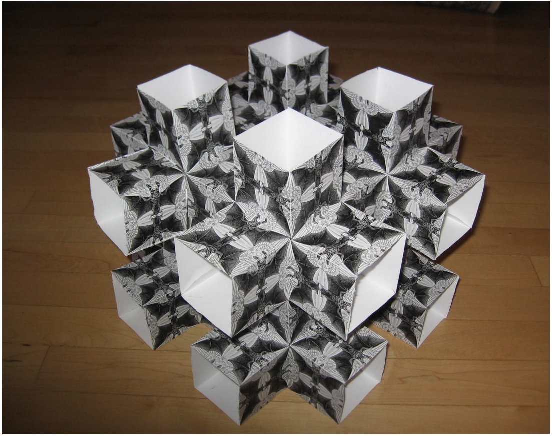
A $(10, 4)$ butterfly pattern showing distortion for large p



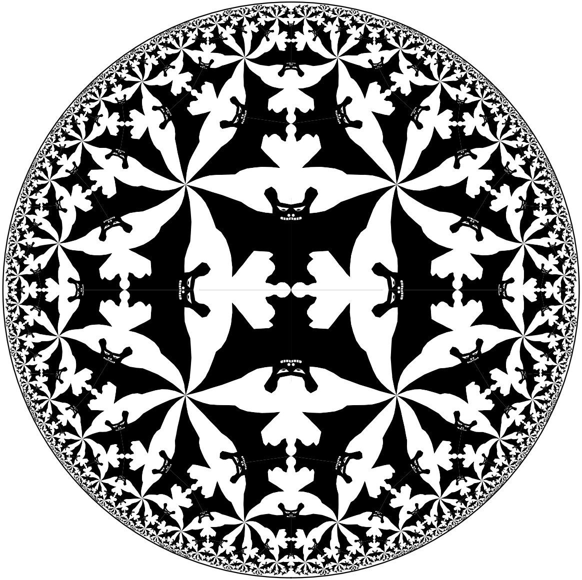
Triply Periodic Polyhedra

- A *triply periodic polyhedron* is a (non-closed) polyhedron that repeats in three different directions in Euclidean 3-space.
- We will consider the special case of *uniform* triply periodic polyhedra which have the same vertex figure at each vertex — i.e. there is a symmetry of the polyhedron that takes any vertex to any other vertex..
- We will mostly discuss a speciallization of uniform triply periodic polyhedra: *regular* triply periodic polyhedra which are "flag-transitive" — there is a symmetry of the polyhedron that takes any vertex, edge containing that vertex, and face containing that edge to any other such (vertex, edge, face) combination.
- In 1926 John Petrie and H.S.M. Coxeter proved that there are exactly three regular triply periodic polyhedra, which Coxeter denoted $\{4,6|4\}$, $\{6,4|4\}$, and $\{6,6|3\}$, where $\{p,q|r\}$ denotes a polyhedron made up of p-sided regular polygons meeting q at a vertex, and with regular r-sided holes.

Angels and Devils on the $\{4, 6|4\}$ polyhedron



The corresponding Angels and Devils pattern in the hyperbolic plane



Relation between periodic polyhedra and regular tessellations — a 2-Step Process

- (1) Some triply periodic polyhedra approximate TPMS's.

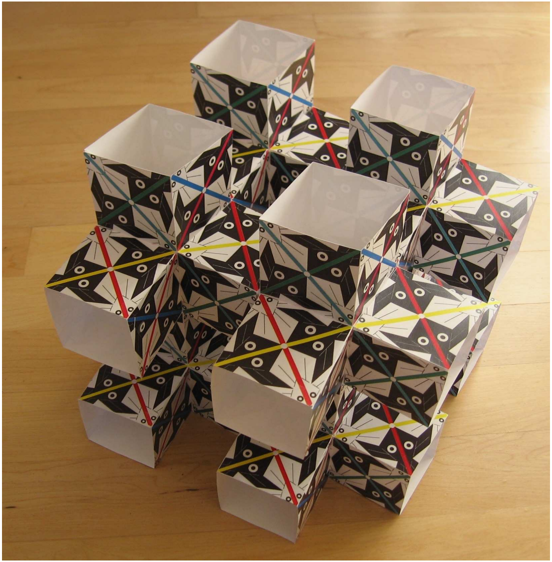
  As a bonus, some triply periodic polyhedra contain embedded Euclidean lines which are also lines embedded in the corresponding TPMS.

- (2) As a minimal surface, a TPMS has negative curvature (except for isolated points of zero curvature), and so its universal covering surface also has negative curvature and thus has the same large-scale geometry as the hyperbolic plane.

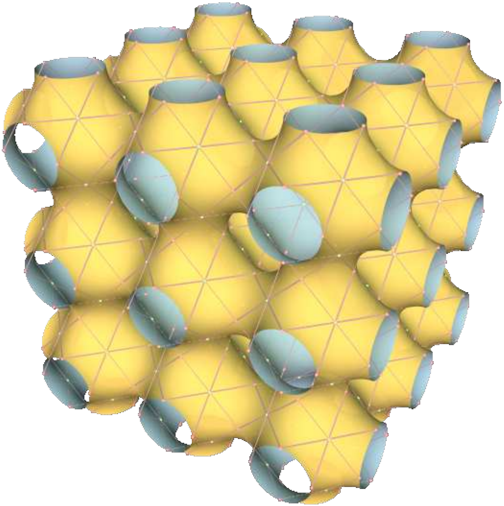
  So the polygons of the triply periodic polyhedron can be transferred to the polygons of a corresponding regular tessellation of the hyperbolic plane.

- We show this relationship in the next slides.

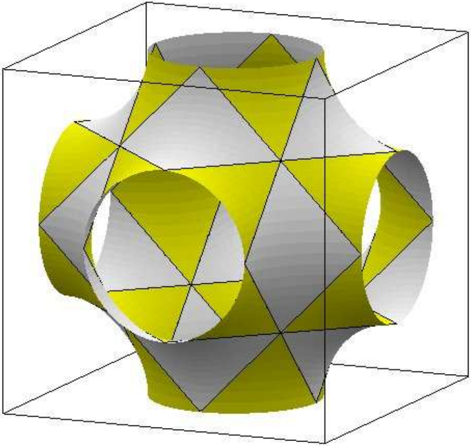
A pattern of fish on the $\{4, 6|4\}$ polyhedron — showing colored embedded lines



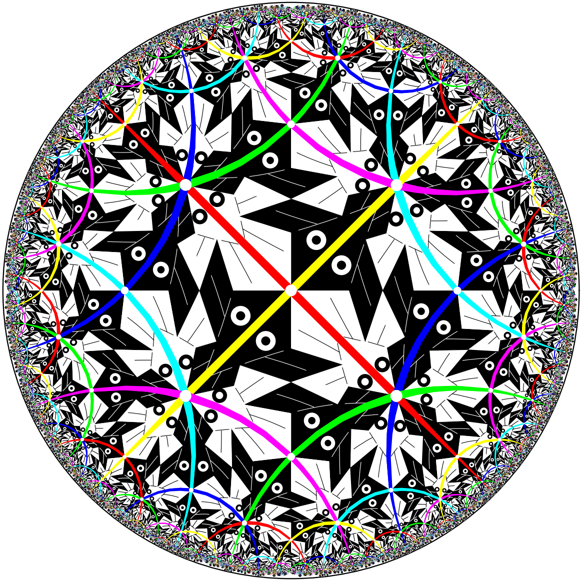
Schwarz's P-surface — approximated by the previous triply periodic polyhedron, and showing corresponding embedded lines



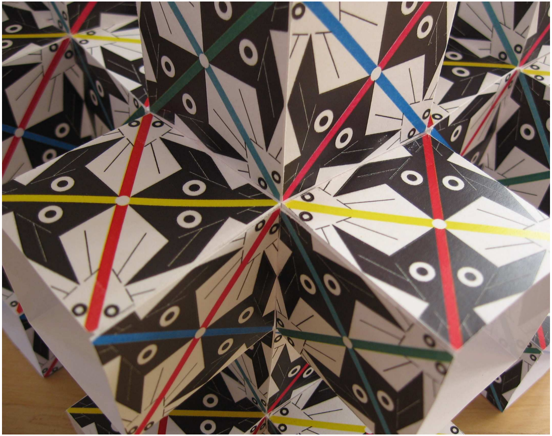
A close-up of Schwarz's P-surface showing corresponding embedded lines and "skew rhombi"



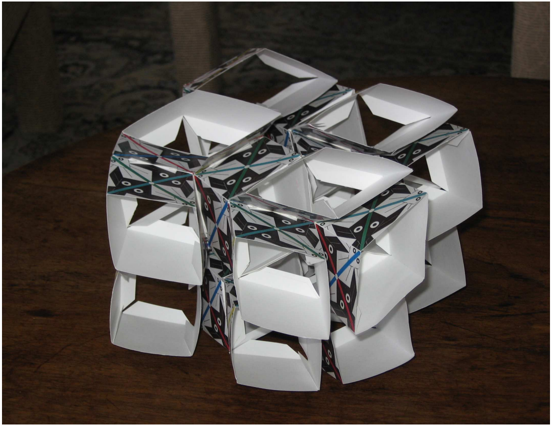
The pattern of fish "unfolded" onto a repeating pattern of the hyperbolic plane — showing the embedded lines as hyperbolic lines, which bound the "skew rhombi".



A close-up of a vertex of the $\{4,6|4\}$ polyhedron

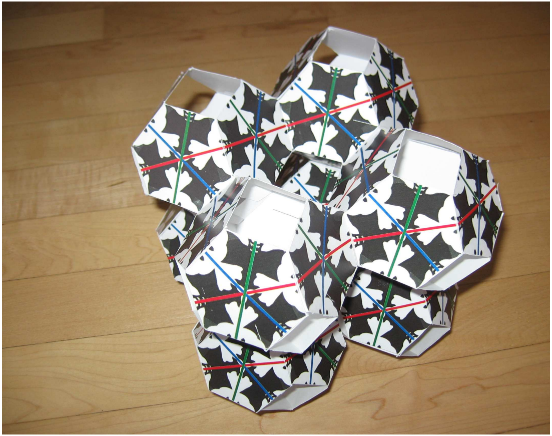


The squashed $\{4, 6|4\}$ polyhedron

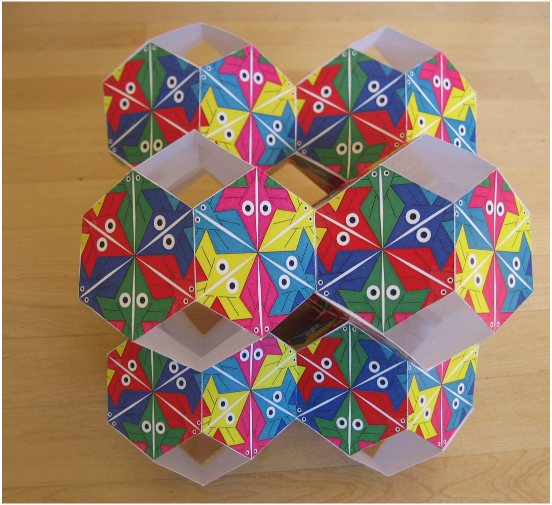


Patterns on the $\{6, 4|4\}$ Polyhedron

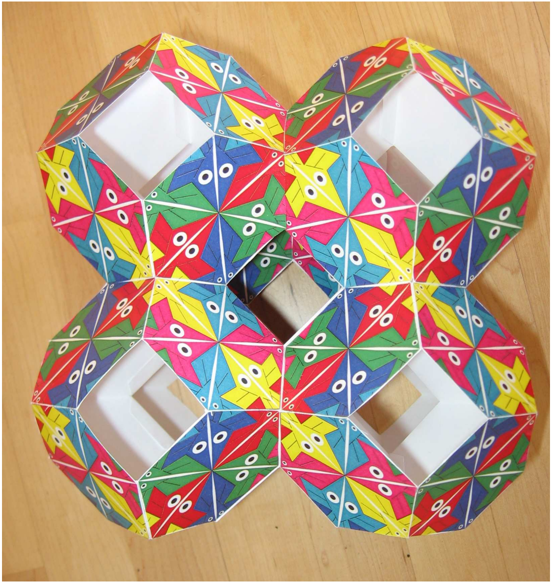
A pattern of angels and devils on the $\{6, 4|4\}$ polyhedron



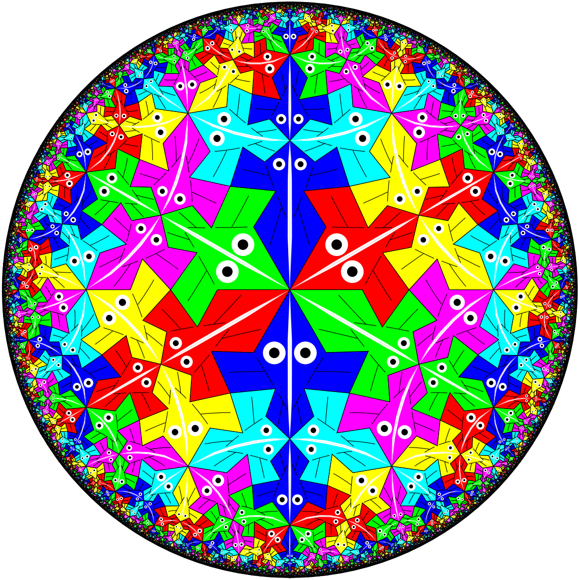
A Pattern of Fish on the $\{6, 4|4\}$ Polyhedron



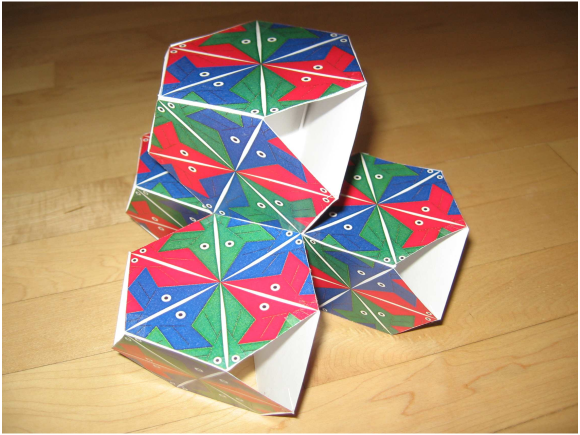
A top view of the fish on the $\{6, 4|4\}$ polyhedron — showing fish along embedded lines



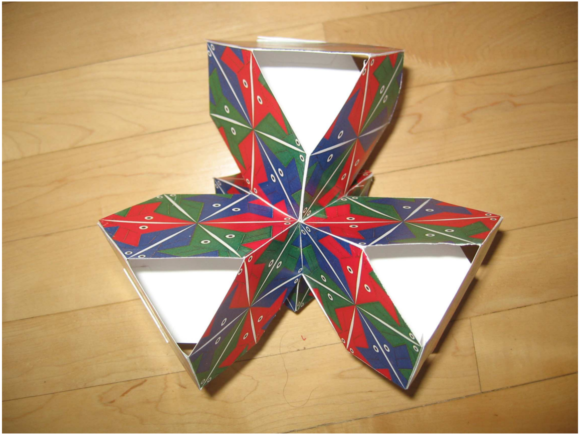
The corresponding hyperbolic pattern of fish — a version of Escher's Circle Limit I pattern with 6-color symmetry



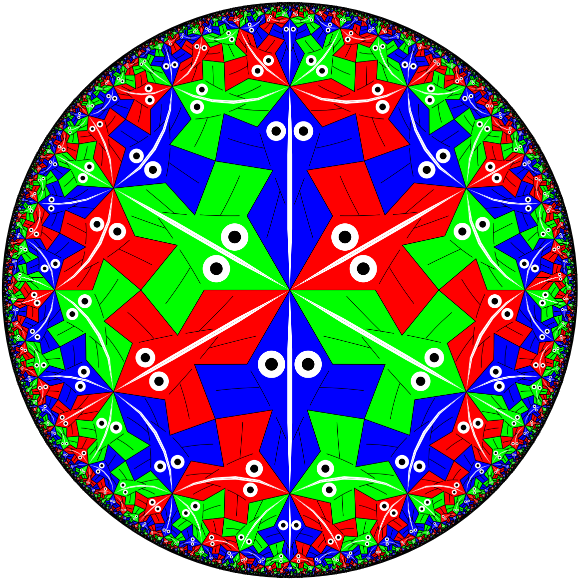
A Pattern of Fish on the $\{6, 6|3\}$ Polyhedron



A top view of the fish on the $\{6, 6|3\}$ polyhedron — showing a vertex



The corresponding hyperbolic pattern of fish — based on the $\{6, 6\}$ tessellation

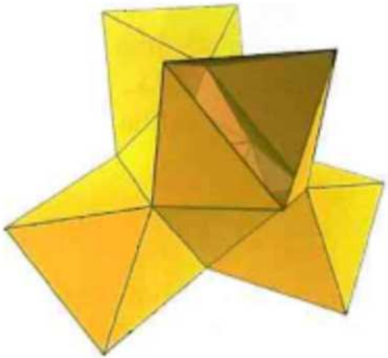


Patterns of Fish on a $\{3, 8\}$ Polyhedron

Using a uniform triply periodic $\{3, 8\}$ polyhedron, we show a pattern of fish inspired by Escher's hyperbolic print *Circle Limit III*, which is based on the regular $\{3, 8\}$ tessellation. This polyhedron is related to Schwarz's D-surface, a TRMS with the topology of a thickened diamond lattice, which has embedded lines. The red, green, and yellow fish swim along those lines (the blue fish swim in loops around the "waists"). We show:

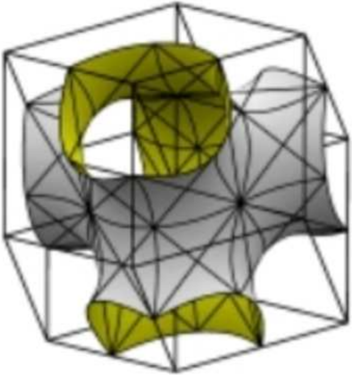
- A piece of the triply periodic polyhedron.
- A corresponding piece of the patterned polyhedron.
- A piece of Schwarz's D-surface showing embedded lines.
- Escher's *Circle Limit III* with the equilateral triangle tessellation superimposed.
- A large piece of the patterned polyhedron.
- A top view of the large piece.

A piece of the triply periodic polyhedron

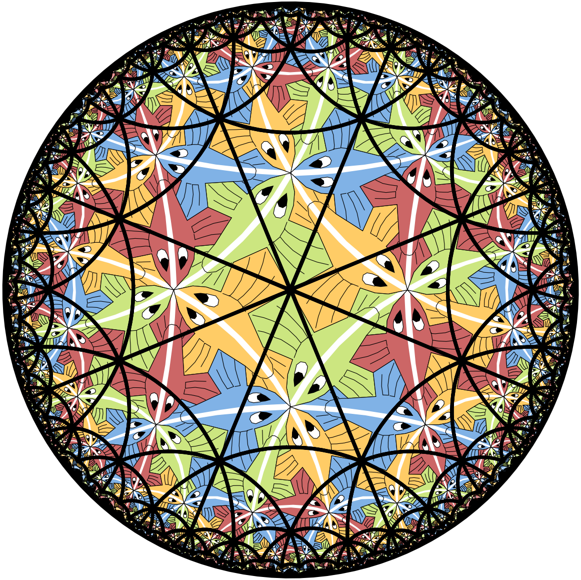


A corresponding piece of the patterned polyhedron

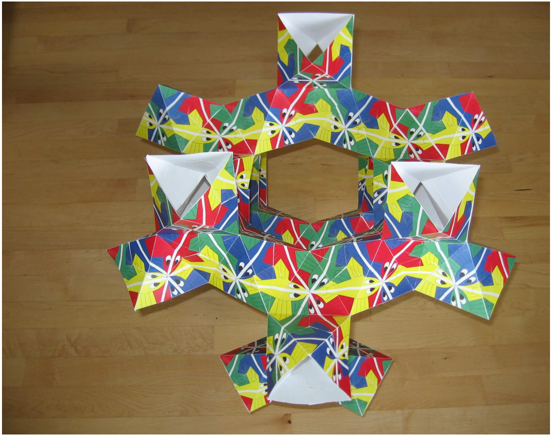
A piece of Schwarz's D-surface showing embedded lines



Escher's Circle Limit III with the equilateral triangle tessellation superimposed



A large piece of the patterned polyhedron



A top view of the large piece

Future Work

- Put other patterns on the regular triply periodic polyhedra, exploiting their embedded lines.
- Place patterns on non-regular, uniform triply periodic polyhedra.
- Put patterns on non-uniform triply periodic polyhedra — especially those that more closely approximate triply periodic minimal surfaces.
- Draw patterns on TPMS's — the gyroid, for example.

Thank You!

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